SECONDARY SCHOOL

Mathematics

For Class 9

R S Aggarwal
V Aggarwal

Secondary School Mathematics

FOR CLASS 9

(In accordance with the latest CBSE syllabus)

R S Aggarwal, MSc, PhD

Veena Aggarwal, BA, BEd

Bharati Bhawan

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Preface

We feel proud to present this edition of the book. It is based on the new CBSE syllabus. Apart from the CBSE course, this book can be used as a textbook for the courses of those state boards which follow the CBSE syllabus. The book has been thoroughly revised and reset. It now has a large number of questions from the NCERT textbook with full explanatory solutions. A large number of multiple-choice questions (MCQ) on all topics have also been included. This book will also be of immense help to the students who wish to sit for the board examination.

Though revised and reset, it continues to have the qualities which made it so popular among teachers and students in the past. We have emphasized the theoretical as well as the numerical aspects of the mathematics course. The underlying concepts have been gradually and systematically developed. The exposition is simple, yet rigorous. The language is precise and simple. In each chapter, all the results and concepts of a particular topic have been put together. These are followed by a large number of solved examples. Quite a large number of problems have been given as exercises.

We are thankful to the editorial and production staff of Bharati Bhawan for doing such a good job. We also wish to thank all teachers and students who sent suggestions for the improvement of the book. We hope that we shall continue to receive such invaluable feedback.

—Authors

Mathematics Syllabus

For Class 9

Unit I: Number Systems

1. Real Numbers **(18 Periods)**

Review of representation of natural numbers, integers, rational numbers on the number line. Representation of terminating/nonterminating recurring decimals on the number line through successive magnification. Rational numbers as recurring/terminating decimals. Operations on real numbers.

Examples of nonrecurring/nonterminating decimals. Existence of nonrational numbers (irrational numbers) such as $\sqrt{2}, \sqrt{3}$ and their representation on the number line. Explaining that every real number is represented by a unique point on the number line and conversely, every point on the number line represents a unique real number.

Definition of nth root of a real number. Existence of $\sqrt{x}$ for a given positive real number x and its representation on the number line with geometric proof.

Rationalisation (with precise meaning) of real numbers of the type (and their combinations) $\dfrac{1}{a+b\sqrt{x}}$ and $\dfrac{1}{\sqrt{x}+\sqrt{y}}$, where x and y are natural numbers, a and b are integers.

Recall of laws of exponents with integral powers. Rational exponents with positive real bases (to be done by particular cases, allowing learner to arrive at the general laws).

Unit II: Algebra

1. Polynomials **(23 Periods)**

Definition of a polynomial in one variable, its coefficients, with examples and counter-examples, its terms, zero polynomial. Degree of a polynomial. Constant, linear, quadratic, cubic polynomials; monomials, binomials, trinomials. Factors and multiples. Zeros of a polynomial. Motivate and state the Remainder Theorem with examples. Statement and proof of the Factor Theorem. Factorisation of $ax^2+bx+c, a \neq 0$, where a, b and c are real numbers, and of cubic polynomials using the Factor Theorem.

Recall of algebraic expressions and identities. Further verification of identities of the type

$$(x+y+z)^2 = x^2+y^2+z^2+2xy+2yz+2zx,$$
$$(x \pm y)^3 = x^3 \pm y^3 \pm 3xy(x \pm y),\ x^3 \pm y^3 = (x \pm y)(x^2 \mp xy + y^2)$$
$$x^3+y^3+z^3-3xyz = (x+y+z)(x^2+y^2+z^2-xy-yz-zx),$$

and their use in factorization of polynomials.

2. Linear Equations in Two Variables **(14 Periods)**

Recall of linear equations in one variable. Introduction to the equation in two variables. Focus on linear equations of the type $ax+by+c=0$. Prove that a linear equation in two variables has infinitely many solutions, and justify their being written as ordered pairs of real numbers, plotting them and showing that they lie on a line. Graph of linear equations in two variables. Examples, problems from real life, including problems on Ratio and Proportion, and with algebraic and graphical solutions being done simultaneously.

Unit III: Coordinate Geometry

1. Coordinate Geometry **(6 Periods)**

The Cartesian plane, coordinates of a point, names and terms associated with the coordinate plane, notations, plotting points in the plane.

Unit IV: Geometry

1. Introduction to Euclid's Geometry **(6 Periods)**

History—Geometry in India and Euclid's geometry. Euclid's method of formalizing observed phenomenon into rigorous mathematics with definitions, common/obvious notions, axioms/postulates, and theorems. The five postulates of Euclid. Equivalent versions of the fifth postulate. Showing the relationship between axiom and theorem, for example:

(Axiom) 1. Given two distinct points, there exists one and only one line through them.

(Theorem) 2. (Prove) Two distinct lines cannot have more than one point in common.

2. Lines and Angles **(13 Periods)**

1. **(Motivate)** If a ray stands on a line, then the sum of the two adjacent angles so formed is and the converse.
2. **(Prove)** If two lines intersect, vertically opposite angles are equal.

3. **(Motivate)** Results on corresponding angles, alternate angles, interior angles when a transversal intersects two parallel lines.
4. **(Motivate)** Lines, which are parallel to a given line, are parallel.
5. **(Prove)** The sum of the angles of a triangle is 180°.
6. **(Motivate)** If a side of a triangle is produced, the exterior angle so formed is equal to the sum of the two interior opposite angles.

3. Triangles **(20 Periods)**

1. **(Motivate)** Two triangles are congruent if any two sides and the included angle of one triangle are equal to any two sides and the included angle of the other triangle (SAS congruence).
2. **(Prove)** Two triangles are congruent if any two angles and the included side of one triangle are equal to any two angles and the included side of the other triangle (ASA congruence).
3. **(Motivate)** Two triangles are congruent if the three sides of one triangle are equal to three sides of the other triangle (SSS congruence).
4. **(Motivate)** Two right triangles are congruent if the hypotenuse and a side of one triangle are equal (respectively) to the hypotenuse and a side of the other triangle.
5. **(Prove)** The angles opposite to equal sides of a triangle are equal.
6. **(Motivate)** The sides opposite to equal angles of a triangle are equal.
7. **(Motivate)** Triangle inequalities and relation between 'angle and facing side' inequalities in triangles.

4. Quadrilaterals **(10 Periods)**

1. **(Prove)** The diagonal divides a parallelogram into two congruent triangles.
2. **(Motivate)** In a parallelogram opposite sides are equal, and conversely.
3. **(Motivate)** In a parallelogram opposite angles are equal, and conversely.
4. **(Motivate)** A quadrilateral is a parallelogram if a pair of its opposite sides are parallel and equal.
5. **(Motivate)** In a parallelogram, the diagonals bisect each other, and conversely.
6. **(Motivate)** In a triangle, the line segment joining the midpoints of any two sides is parallel to the third side and is half of it, and (motivate) its converse.

5. Area **(7 Periods)**

Review concept of area, recall area of a rectangle.

1. **(Prove)** Parallelograms on the same base and between the same parallels have the same area.
2. **(Motivate)** Triangles on the same (or equal base) base and between the same parallels are equal in area.

6. Circles **(15 Periods)**

Through examples, arrive at definitions of circle and related concepts—radius, circumference, diameter, chord, arc, secant, sector segment, subtended angle.

1. **(Prove)** Equal chords of a circle subtend equal angles at the centre, and (motivate) its converse.
2. **(Motivate)** The perpendicular from the centre of a circle to a chord bisects the chord and conversely, the line drawn through the centre of a circle to bisect a chord is perpendicular to the chord.
3. **(Motivate)** There is one and only one circle passing through three given noncollinear points.
4. **(Motivate)** Equal chords of a circle (or of congruent circles) are equidistant from the centre (or their respective centres), and conversely.
5. **(Prove)** The angle subtended by an arc at the centre is double the angle subtended by it at any point on the remaining part of the circle.
6. **(Motivate)** Angles in the same segment of a circle are equal.
7. **(Motivate)** If a line segment joining two points subtends equal angles at two other points lying on the same side of the line containing the segment, the four points lie on a circle.
8. **(Motivate)** The sum of either of the pair of the opposite angles of a cyclic quadrilateral is 180°, and its converse.

7. Constructions **(10 Periods)**

1. Construction of bisectors of line segments and angles of measure 60°, 90°, 45°, etc., equilateral triangles.
2. Construction of a triangle given its base, sum/difference of the other two sides and one base angle.
3. Construction of a triangle of given perimeter, and base angles.

Unit V: Mensuration

1. Areas **(4 Periods)**

Area of a triangle using Heron's formula (without proof) and its application in finding the area of a quadrilateral.

2. Surface Areas and Volumes **(12 Periods)**

Surface areas and volumes of cubes, cuboids, spheres (including hemispheres) and right circular cylinders/cones.

Unit VI: Statistics and Probability

1. Statistics **(13 Periods)**

Introduction to Statistics: Collection of data, presentation of data—tabular form, ungrouped/grouped, bar graphs, histograms (with varying base lengths), frequency polygons. Mean, median and mode of ungrouped data.

2. Probability **(9 Periods)**

History, repeated experiments and observed frequency approach to probability. Focus is on empirical probability. (A large amount of time to be devoted to group and to individual activities to motivate the concept; the experiments to be drawn from real-life situations, and from examples used in the chapter on statistics.)

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Weightage

MATHEMATICS

CLASS 9

Time: 3 Hours **Max. Marks: 80**

The weightage or the distribution of marks over different dimensions of the question paper shall be as follows:

Weightage to Content/Subject Units

S. No.	Unit	Marks
1.	Number Systems • *Real Numbers*	8
2.	Algebra • *Polynomials* • *Linear Equations in Two Variables*	17
3.	Coordinate Geometry • *Coordinate Geometry*	4
4.	Geometry • *Introduction to Euclid's Geometry* • *Lines and Angles* • *Triangles* • *Quadrilaterals* • *Area* • *Circles* • *Constructions*	28
5.	Mensuration • *Area of a Triangle using Heron's Formula* • *Surface Areas and Volumes*	13
6.	Statistics and Probability • *Statistics* • *Probability*	10
	Total	80

✿

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1 Number Systems

INTRODUCTION

We have learnt about various types of numbers in our earlier classes. Let us review them and learn more about numbers.

NATURAL NUMBERS *Counting numbers are called natural numbers.*

The collection of natural numbers is denoted by N and is written as

$N = \{1, 2, 3, 4, 5, 6, \ldots\}$.

REMARKS (i) The least natural number is 1.

(ii) There are infinitely many natural numbers.

WHOLE NUMBERS *All natural numbers together with* 0 *form the collection* W *of all whole numbers, written as*

$W = \{0, 1, 2, 3, 4, 5, \ldots\}$.

REMARKS (i) The least whole number is 0.

(ii) There are infinitely many whole numbers.

(iii) Every natural number is a whole number.

(iv) All whole numbers are not natural numbers, as 0 is a whole number which is not a natural number.

INTEGERS *All natural numbers,* 0 *and negatives of natural numbers form the collection of all integers.* It is represented by Z after the German word 'zahlen' meaning 'to count'. Thus, we write

$Z = \{\ldots, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, \ldots\}$.

REMARKS (i) 0 is neither negative nor positive.

(ii) There are infinitely many integers.

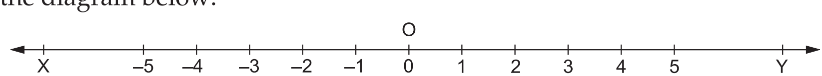
(iii) Every natural number is an integer.

(iv) Every whole number is an integer.

REPRESENTATION OF INTEGERS ON NUMBER LINE

A number line is a visual representation of numbers on a graduated straight line.

To represent integers on the number line, draw a line XY which extends endlessly in both the directions, as indicated by the arrowheads in the diagram below.



Take any point O on this line. Let this point represent the integer 0 (zero). Now, taking a fixed length, called unit length, set off equal distances to the right as well as to the left of O.

On the right-hand side of O, the points at distances of 1 unit, 2 units, 3 units, 4 units, 5 units, etc., from O denote respectively the positive integers 1, 2, 3, 4, 5, etc.

Similarly, on the left-hand side of O, the points at distances of 1 unit, 2 units, 3 units, 4 units, 5 units, etc., from O denote respectively the negative integers –1, –2, –3, –4, –5, etc.

Since the line can be extended endlessly on both sides of O, it follows that we can represent each and every integer by some point on this line.

For instance, starting from O and moving to its right, after 836 units, we get a point which represents the integer 836.

Similarly, starting from O and moving to its left, a point after 750 units, represents the integer '–750'.

Thus, *each and every integer can be represented by some point on the number line.*

RATIONAL NUMBERS *The numbers of the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$, are known as rational numbers.* The collection of rational numbers is denoted by Q and is written as

$$Q = \left\{\frac{p}{q} : p, q \text{ are integers}, q \neq 0\right\}.$$

'Rational' comes from the word 'ratio' and Q comes from the word 'quotient'.

Thus, $\frac{1}{4}, \frac{3}{2}, \frac{11}{79}, -\frac{2001}{2002}$, etc., are all rational numbers.

REMARKS
(i) There are infinitely many rational numbers.
(ii) There is no least or greatest rational number.
(iii) 0 is a rational number, since we can write, $0 = \frac{0}{1}$.
(iv) Every natural number is a rational number since we can write, $1 = \frac{1}{1}, 2 = \frac{2}{1}, 3 = \frac{3}{1}$, etc.
(v) Every integer is a rational number since an integer a can be written as $\frac{a}{1}$, e.g., $-31 = \frac{-31}{1}, 0 = \frac{0}{1}$ and $79 = \frac{79}{1}$.

Hence, *rational numbers include natural numbers, whole numbers and integers.*

EQUIVALENT RATIONAL NUMBERS Rational numbers do not have a unique representation in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

Thus, $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \ldots = \frac{15}{30} = \frac{16}{32} = \ldots = \frac{144}{288} = \ldots$.

These are known as equivalent rational numbers.

SIMPLEST FORM OF A RATIONAL NUMBER *A rational number $\frac{p}{q}$ is said to be in its simplest form, if p and q are integers having no common factor other than* 1 *(that is, p and q are co-primes) and $q \neq 0$.*

Thus, the simplest form of each of $\frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}$, etc., is $\frac{1}{2}\cdot$

Similarly, the simplest form of $\frac{6}{9}$ is $\frac{2}{3}$ and that of $\frac{95}{133}$ is $\frac{5}{7}\cdot$

EXAMPLE 1 *Write four rational numbers equivalent to* $\frac{4}{7}\cdot$

SOLUTION We have

$$\frac{4}{7} = \frac{4 \times 2}{7 \times 2} = \frac{4 \times 3}{7 \times 3} = \frac{4 \times 4}{7 \times 4} = \frac{4 \times 5}{7 \times 5}$$

$$\Rightarrow \quad \frac{4}{7} = \frac{8}{14} = \frac{12}{21} = \frac{16}{28} = \frac{20}{35}\cdot$$

Thus, four rational numbers equivalent to $\frac{4}{7}$ are

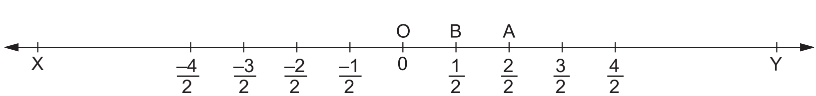
$$\frac{8}{14}, \frac{12}{21}, \frac{16}{28} \text{ and } \frac{20}{25}\cdot$$

REPRESENTATION OF RATIONAL NUMBERS ON REAL LINE

Draw a line XY which extends endlessly in both the directions. Take a point O on it and let it represent 0 (zero).

Taking a fixed length, called unit length, mark off OA = 1 unit.

The midpoint B of OA denotes the rational number $\frac{1}{2}\cdot$ Starting from O, set off equal distances each equal to $OB = \frac{1}{2}$ unit.

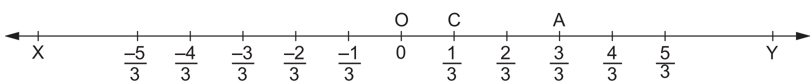


From the point O, on its right, the points at distances equal to OB, $2OB$, $3OB$, $4OB$, etc., denote respectively the rational numbers $\frac{1}{2}, \frac{2}{2}, \frac{3}{2}, \frac{4}{2}$, etc.

Similarly, from the point O, on its left, the points at distances equal to OB, $2OB$, $3OB$, $4OB$, etc., denote respectively the rational numbers $\frac{-1}{2}, \frac{-2}{2}, \frac{-3}{2}, \frac{-4}{2}$, etc.

Thus, each rational number with 2 as its denominator can be represented by some point on the number line.

Next, draw the line XY. Take a point O on it representing 0. Let $OA = 1$ unit. Divide OA into three equal parts with OC as the first part. Then, C represents the rational number $\frac{1}{3}$.



From the point O, set off equal distances, each equal to $OC = \frac{1}{3}$ unit on both sides of O.

The points at distances equal to OC, $2OC$, $3OC$, $4OC$, etc., from the point O on its right denote respectively the rational numbers $\frac{1}{3}, \frac{2}{3}, \frac{3}{3}, \frac{4}{3}$, etc.

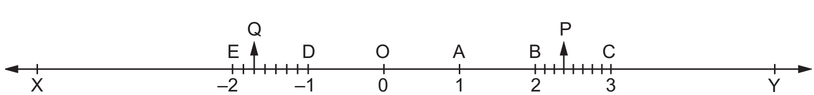
Similarly, the points at distances equal to OC, $2OC$, $3OC$, $4OC$, etc., from the point O on its left denote respectively the rational numbers $\frac{-1}{3}, \frac{-2}{3}, \frac{-3}{3}, \frac{-4}{3}$, etc.

Thus, each rational number with 3 as its denominator can be represented by some point on the number line (or, the real line).

Proceeding in this manner, we can represent each and every rational number by some point on the line.

EXAMPLE 2 *Represent (i)* $2\frac{3}{8}$ *and (ii)* $-1\frac{5}{7}$ *on real line.*

SOLUTION Draw a line XY and taking a fixed length as unit length, represent integers on this line.

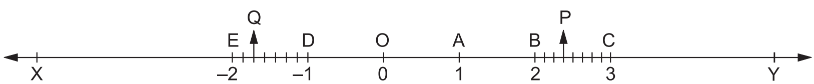


(i) On the right of O, take $OA = 1$ unit. Then, $OB = 2$ units.
Divide the 3rd unit BC into 8 equal parts.
BP represents $\frac{3}{8}$ of a unit. Therefore, P represents $2\frac{3}{8}$.

(ii) On the left of O, take $OD = 1$ unit.
Divide the 2nd unit DE into 7 equal parts.
DQ represents $\frac{5}{7}$ of a unit. Therefore, Q represents $-1\frac{5}{7}$.

EXAMPLE 3 *Represent (i)* $\frac{8}{5}$ *and (ii)* $-\frac{11}{7}$ *on the number line.*

SOLUTION Draw a line XY and taking a fixed length as unit length, represent integers on this line.



(i) $\frac{8}{5} = 1\frac{3}{5}$.
On the right of O, take $OA = 1$ unit.
Divide the 2nd unit AB into 5 equal parts.
AP represents $\frac{3}{5}$ of a unit. Therefore, P represents $1\frac{3}{5}$.

(ii) $-\frac{11}{7} = -1\frac{4}{7}$.
On the left of O, take $OD = 1$ unit.
Divide the 2nd unit DE into 7 equal parts.
Then, DQ represents $\frac{4}{7}$ of a unit. Therefore, Q represents $-\frac{11}{7}$.

FINDING RATIONAL NUMBERS BETWEEN TWO GIVEN RATIONAL NUMBERS

METHOD 1 Suppose we are required to find one rational number between two rational numbers x and y such that $x < y$.

Then, $\frac{1}{2}(x+y)$ is a rational number lying between x and y.

EXAMPLE 4 *Find a rational number lying between (i)* $\frac{1}{3}$ *and* $\frac{1}{2}$*; (ii)* $\frac{2}{3}$ *and* $-\frac{3}{4}$.

SOLUTION (i) Let $x = \frac{1}{3}$ and $y = \frac{1}{2}$.

$\therefore$ required rational number lying between x and y

$$= \frac{1}{2}(x+y) = \frac{1}{2}\left(\frac{1}{3}+\frac{1}{2}\right) = \frac{1}{2} \times \frac{5}{6} = \frac{5}{12}.$$

Hence, $\frac{5}{12}$ is a rational number lying between $\frac{1}{3}$ and $\frac{1}{2}\cdot$

(ii) Let $x=\frac{2}{3}$ and $y=-\frac{3}{4}\cdot$

$\therefore$ required rational number lying between $\frac{2}{3}$ and $\frac{-3}{4}$

$$=\frac{1}{2}(x+y)=\frac{1}{2}\left[\frac{2}{3}+\left(-\frac{3}{4}\right)\right]=\frac{1}{2}\left(\frac{2}{3}-\frac{3}{4}\right)$$

$$=\frac{1}{2}\times\left(-\frac{1}{12}\right)=-\frac{1}{24}\cdot$$

Hence, $-\frac{1}{24}$ is a rational number lying between $\frac{2}{3}$ and $-\frac{3}{4}\cdot$

EXAMPLE 5 *Find three rational numbers between* -2 *and* -3. [2014]

SOLUTION A rational number lying between -2 and -3 is $\frac{1}{2}[(-2)+(-3)]$, i.e., $-\frac{5}{2}\cdot$

Now, a rational number lying between -2 and $-\frac{5}{2}$ is

$$\frac{1}{2}\left[(-2)+\left(-\frac{5}{2}\right)\right],\ \text{i.e.,}\ \frac{1}{2}\times\left(-\frac{9}{2}\right),\ \text{i.e.,}\ -\frac{9}{4}\cdot$$

And, a rational number lying between $-\frac{5}{2}$ and -3 is

$$\frac{1}{2}\left[\left(-\frac{5}{2}\right)+(-3)\right],\ \text{i.e.,}\ \frac{1}{2}\times\left(-\frac{11}{2}\right),\ \text{i.e.,}\ -\frac{11}{4}\cdot$$

Thus, we have $-2>-\frac{9}{4}>-\frac{5}{2}>-\frac{11}{4}>3$.

Hence, three rational numbers between -2 and -3 are $-\frac{9}{4}, -\frac{5}{2}$ and $-\frac{11}{4}\cdot$

METHOD 2 Suppose we are required to find n rational numbers between two rational numbers, x and y with like denominators.

Then, we convert the given rational numbers into equivalent rational numbers by multiplying the numerator and denominator by a suitable number, usually $(n+1)$.

Now, the required rational numbers may be manually chosen.

EXAMPLE 6 *Find five rational numbers between* $\frac{3}{5}$ *and* $\frac{4}{5}$. [2015]

SOLUTION Let $n = 5$.

We convert $\frac{3}{5}$ and $\frac{4}{5}$ into equivalent rational numbers by multiplying the numerator and denominator by $(n+1)$, i.e., 6.

Thus, $\frac{3}{5} = \frac{3 \times 6}{5 \times 6} = \frac{18}{30}$ and $\frac{4}{5} = \frac{4 \times 6}{5 \times 6} = \frac{24}{30}$.

Clearly, we have

$$\frac{18}{30} < \frac{19}{30} < \frac{20}{30} < \frac{21}{30} < \frac{22}{30} < \frac{23}{30} < \frac{24}{30}$$

or $$\frac{3}{5} < \frac{19}{30} < \frac{2}{3} < \frac{7}{10} < \frac{11}{15} < \frac{23}{30} < \frac{4}{5}.$$

Hence, five rational numbers between $\frac{3}{5}$ and $\frac{4}{5}$ are $\frac{19}{20}$, $\frac{2}{3}$, $\frac{7}{10}$, $\frac{11}{15}$, and $\frac{23}{30}$.

EXAMPLE 7 *Find six rational numbers between* 3 *and* 4. [2015]

SOLUTION Let $n = 6$.

We convert 3 and 4 into equivalent rational numbers using $(n+1) = 7$ as multiplying factor.

Thus, $3 = \frac{3}{1} = \frac{3 \times 7}{1 \times 7} = \frac{21}{7}$ and $4 = \frac{4}{1} = \frac{4 \times 7}{1 \times 7} = \frac{28}{7}$.

Now, $$\frac{21}{7} < \frac{22}{7} < \frac{23}{7} < \frac{24}{7} < \frac{25}{7} < \frac{26}{7} < \frac{27}{7} < \frac{28}{7}$$

or $$3 < \frac{22}{7} < \frac{23}{7} < \frac{24}{7} < \frac{25}{7} < \frac{26}{7} < \frac{27}{7} < 4.$$

Hence, six rational numbers between 3 and 4 are $\frac{22}{7}$, $\frac{23}{7}$, $\frac{24}{7}$, $\frac{25}{7}$, $\frac{26}{7}$ and $\frac{27}{7}$.

EXAMPLE 8 *Insert* 10 *rational numbers between* $-\frac{5}{13}$ *and* $\frac{6}{13}$.

SOLUTION We have

$$-\frac{5}{13} < -\frac{4}{13} < -\frac{3}{13} < -\frac{2}{13} < -\frac{1}{13} < 0 < \frac{1}{13} < \frac{2}{13} < \frac{3}{13} < \frac{4}{13} < \frac{5}{13} < \frac{6}{13}.$$

Hence, 10 rational numbers between $-\frac{5}{13}$ and $\frac{6}{13}$ are

$$-\frac{4}{13}, -\frac{3}{13}, -\frac{2}{13}, -\frac{1}{13}, 0, \frac{1}{13}, \frac{2}{13}, \frac{3}{13}, \frac{4}{13} \text{ and } \frac{5}{13}.$$

EXAMPLE 9 *Insert* 100 *rational numbers between* $-\frac{4}{11}$ *and* $\frac{7}{11}$.

SOLUTION Clearly, we have

$$-\frac{4}{11}=\frac{(-4)\times 10}{11\times 10}=-\frac{40}{110} \text{ and } \frac{7}{11}=\frac{7\times 10}{11\times 10}=\frac{70}{110}.$$

$$\text{Now, } -\frac{40}{110}<-\frac{39}{110}<-\frac{38}{110}<\ldots<-\frac{1}{110}<0<\frac{1}{110}<\frac{2}{110}<\ldots<\frac{59}{110}<\frac{60}{110}.$$

Hence, 100 rational numbers between $-\frac{4}{11}$ and $\frac{7}{11}$ are

$$-\frac{39}{110}, -\frac{38}{110}, \ldots, -\frac{2}{110}, -\frac{1}{110}, 0, \frac{1}{110}, \frac{2}{110}, \ldots, \frac{60}{110}.$$

METHOD 3 Suppose we are required to find n rational numbers between two given rational numbers x and y (especially those with unlike denominators) such that $x < y$.

Let $d=\frac{(y-x)}{(n+1)}$.

Then, n rational numbers lying between x and y are $(x+d)$, $(x+2d)$, $(x+3d), \ldots, (x+nd)$.

REMARK *There are infinitely many rational numbers between any two given rational numbers.*

EXAMPLE 10 *Insert five rational numbers between* $-\frac{2}{3}$ *and* $\frac{3}{4}$.

SOLUTION Let $x=-\frac{2}{3}$ and $y=\frac{3}{4}$. Clearly, $x<y$. $n=6$.

$$\text{Let } d=\frac{y-x}{(n+1)}=\frac{\frac{3}{4}-\left(-\frac{2}{3}\right)}{6+1}=\frac{\left(\frac{3}{4}+\frac{2}{3}\right)}{7}=\frac{17}{12}\times\frac{1}{7}=\frac{17}{84}.$$

So, the five rational numbers between $-\frac{2}{3}$ and $\frac{3}{4}$ are

$(x+d)$, $(x+2d)$, $(x+3d)$, $(x+4d)$ and $(x+5d)$,

i.e. $\left(-\frac{2}{3}+\frac{17}{84}\right), \left(-\frac{2}{3}+\frac{34}{84}\right), \left(-\frac{2}{3}+\frac{51}{84}\right), \left(-\frac{2}{3}+\frac{68}{84}\right)$ and $\left(-\frac{2}{3}+\frac{85}{84}\right)$

i.e. $-\frac{39}{84}, -\frac{22}{84}, -\frac{5}{84}, \frac{12}{84}$ and $\frac{29}{84}$,

i.e. $-\frac{13}{28}, -\frac{11}{42}, -\frac{5}{84}, \frac{1}{7}$ and $\frac{29}{84}$.

EXAMPLE 11 *Find nine rational numbers between* 0 *and* 0.1.

SOLUTION Here $x = 0$, $y = 0.1$ and $n = 9$.

$$\therefore \quad d = \frac{(y-x)}{(n+1)} = \frac{(0.1-0)}{(9+1)} = \frac{0.1}{10} = 0.01.$$

Hence, the required numbers between 0 and 0.1 are

$$(x+d), (x+2d), (x+3d), (x+4d), (x+5d), (x+6d), (x+7d),$$
$$(x+8d), (x+9d),$$

i.e. 0.01, 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08 and 0.09,

i.e. $\frac{1}{100}, \frac{2}{100}, \frac{3}{100}, \frac{4}{100}, \frac{5}{100}, \frac{6}{100}, \frac{7}{100}, \frac{8}{100}$ and $\frac{9}{100}$,

i.e. $\frac{1}{100}, \frac{1}{50}, \frac{3}{100}, \frac{1}{25}, \frac{1}{20}, \frac{3}{50}, \frac{7}{100}, \frac{2}{25}$ and $\frac{9}{100}$.

Hence, nine rational numbers between 0 and 0.1 are

$$\frac{1}{100}, \frac{1}{50}, \frac{3}{100}, \frac{1}{25}, \frac{1}{20}, \frac{3}{50}, \frac{7}{100}, \frac{2}{25} \text{ and } \frac{9}{100}.$$

EXERCISE 1A

1. Is zero a rational number? Justify.

2. Represent each of the following rational numbers on the number line:

(i) $\frac{5}{7}$ (ii) $\frac{8}{3}$ (iii) $-\frac{23}{6}$ (iv) 1.3 (v) –2.4

3. Find a rational number between

(i) $\frac{3}{8}$ and $\frac{2}{5}$ (ii) 1.3 and 1.4 (iii) –1 and $\frac{1}{2}$

(iv) $-\frac{3}{4}$ and $-\frac{2}{5}$ (v) $\frac{1}{9}$ and $\frac{2}{9}$ [2015]

4. Find three rational numbers lying between $\frac{3}{5}$ and $\frac{7}{8}$.

How many rational numbers can be determined between these two numbers? [2011]

5. Find four rational numbers between $\frac{3}{7}$ and $\frac{5}{7}$. [2010]

6. Find six rational numbers between 2 and 3.

7. Find five rational numbers between $\frac{3}{5}$ and $\frac{2}{3}$.

8. Insert 16 rational numbers between 2.1 and 2.2.

9. State whether the following statements are true or false. Give reasons for your answer.

(i) Every natural number is a whole number.

(ii) Every whole number is a natural number.

(iii) Every integer is a whole number.

(iv) Every integer is a rational number.

(v) Every rational number is an integer.

(vi) Every rational number is a whole number.

ANSWERS (EXERCISE 1A)

1. Yes, because 0 can be written as $\frac{0}{1}$ which is of the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

3. (i) $\frac{31}{80}$ (ii) 1.35 (iii) $-\frac{1}{4}$ (iv) $-\frac{23}{40}$ (v) $\frac{1}{6}$

4. $\frac{107}{160}, \frac{59}{80}, \frac{129}{160}$

5. $\frac{10}{21}, \frac{11}{21}, \frac{12}{21}, \frac{13}{21}$

6. $\frac{15}{7}, \frac{16}{7}, \frac{17}{7}, \frac{18}{7}, \frac{19}{7}, \frac{20}{7}$

7. $\frac{11}{18}, \frac{28}{45}, \frac{19}{30}, \frac{29}{45}, \frac{59}{90}$

8. 2.105, 2.11, 2.115, 2.12, 2.125, 2.13, 2.135, 2.14, 2.145, 2.15, 2.155, 2.16, 2.165, 2.17, 2.175, 2.18

9. (i) True

(ii) False; 0 is a whole number which is not a natural number.

(iii) False; negative integers are not whole numbers.

(iv) True; every integer can be written in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$.

(v) False; fractional numbers are not integers.

(vi) False; fractional numbers are not whole numbers.

DECIMAL REPRESENTATION OF RATIONAL NUMBERS

Every rational number $\frac{p}{q}$ can be expressed as a decimal. On dividing p by q, two possibilities arise

(i) The remainder becomes zero and the division concludes after a finite number of steps. In this case, the decimal expansion obtained also terminates or ends.

(ii) The remainder never becomes zero and a repeating string of remainders is obtained. In this case, we get a digit or a block of digits repeating in the decimal expansion.

Thus, we have two types of decimal expressions:

1. TERMINATING DECIMAL A decimal that ends after a finite number of digits is called a terminating decimal.

Examples We have (i) $\frac{1}{4} = 0.25$, (ii) $\frac{5}{8} = 0.625$, (iii) $2\frac{3}{5} = \frac{13}{5} = 2.6$.

$$\begin{array}{r} 4\overline{)\,1.0\,}(.25 \\ -8 \\ \hline 20 \\ -20 \\ \hline \times \end{array} \qquad \begin{array}{r} 8\overline{)\,5.0\,}(.625 \\ -48 \\ \hline 20 \\ -16 \\ \hline 40 \\ -40 \\ \hline \times \end{array} \qquad \begin{array}{r} 5\overline{)\,13\,}(2.6 \\ -10 \\ \hline 30 \\ -30 \\ \hline \times \end{array}$$

Thus, each of the numbers $\frac{1}{4}, \frac{5}{8}$ and $2\frac{3}{5}$ can be expressed as a terminating decimal.

IMPORTANT RULE *A rational number $\frac{p}{q}$ is expressible as a terminating decimal only when prime factors of q are* 2 *and* 5 *only.*

Examples Each one of the numbers $\frac{1}{2}, \frac{3}{4}, \frac{7}{20}, \frac{13}{25}$ is a terminating decimal since the denominator of each has no prime factors other than 2 and 5.

EXAMPLE *Without actual division, find which of the following rational numbers are terminating decimals: (i) $\frac{5}{32}$ (ii) $\frac{11}{24}$ (iii) $\frac{27}{80}$.*

SOLUTION (i) Denominator of $\frac{5}{32}$ is 32.

And, $32 = 2^5$.

$\therefore$ 32 has no prime factors other than 2.

So, $\frac{5}{32}$ is a terminating decimal.

(ii) Denominator of $\frac{11}{24}$ is 24.

And, $24 = 2^3 \times 3$.

Thus, 24 has a prime factor 3, which is other than 2 and 5.

$\therefore$ $\frac{11}{24}$ is not a terminating decimal.

(iii) Denominator of $\frac{27}{80}$ is 80.

And, $80 = 2^4 \times 5$.

Thus, 80 has no prime factors other than 2 and 5.

$\therefore$ $\frac{27}{80}$ is a terminating decimal.

2. REPEATING (OR RECURRING) DECIMALS *A decimal in which a digit or a set of digits is repeated periodically, is called a repeating, or a recurring, decimal.*

In a recurring decimal, we place a bar over the first block of the repeating digits and omit the other repeating blocks.

Examples We have

(i) $\frac{2}{3} = 0.666\ldots = 0.\overline{6}$.

$$\begin{array}{r} 3\overline{)\,2.0\,}(.666\ldots \\ -18 \\ \hline 20 \\ -18 \\ \hline 20 \\ -18 \\ \hline 2 \end{array}$$

(ii) $\frac{3}{11} = 0.2727\ldots = 0.\overline{27}$.

$$\begin{array}{r} 11\overline{)\,3.0\,}(.2727\ldots \\ -22 \\ \hline 80 \\ -77 \\ \hline 30 \\ -22 \\ \hline 80 \\ -77 \\ \hline 3 \end{array}$$

(iii) $\frac{15}{7} = 2.142857142857\ldots$

$= 2.\overline{142857}.$

$$
\begin{array}{l}
7\overline{)\,15\,}(2.14285714\ldots \\
\underline{-14} \\
\quad 10 \\
\quad \underline{-7} \\
\quad\; 30 \\
\quad \underline{-28} \\
\qquad 20 \\
\qquad \underline{-14} \\
\qquad\; 60 \\
\qquad \underline{-56} \\
\qquad\quad 40 \\
\qquad\quad \underline{-35} \\
\qquad\quad\; 50 \\
\qquad\quad \underline{-49} \\
\qquad\qquad 10 \\
\qquad\qquad \underline{-7} \\
\qquad\qquad\; 30 \\
\qquad\qquad \underline{-28} \\
\qquad\qquad\quad 2
\end{array}
$$

(iv) $\frac{11}{6} = 1.8333\ldots$

$= 1.8\overline{3}.$

$$
\begin{array}{l}
6\overline{)\,11\,}(1.833\ldots \\
\underline{-6} \\
\quad 50 \\
\underline{-48} \\
\quad\; 20 \\
\quad \underline{-18} \\
\qquad 20 \\
\qquad \underline{-18} \\
\qquad\quad 2
\end{array}
$$

You must have noticed a repeating string of remainders in each of the above cases.

In (i), it is 2, 2, …; in (ii), it is 8, 3, 8, 3, … .

In (iii), it is 1, 3, 2, 6, 4, 5, 1, …; in (iv), it is 5, 2, 2, 2, … .

Kindly note that the number of entries in the repeating string of remainders is less than the divisor.

In $\frac{2}{3}$, only one number 2 repeats itself and the divisor is 3.

In $\frac{15}{7}$, a set of 6 digits, namely 132645, repeats itself and the divisor is 7.

Thus, *if the divisor is n then the maximum number of entries in the repeating block of digits in the decimal expansion of* $\frac{1}{n}$ *is* $(n-1)$.

LENGTH OF PERIOD Repeated number of decimal places in a rational number is called the length of its period.

Example $\frac{15}{7} = 2.\overline{142857}.$

So, the length of its period is 6.

SPECIAL CHARACTERISTICS OF RATIONAL NUMBERS

(i) Every rational number is expressible either as a terminating decimal or as a nonterminating recurring decimal.

(ii) A number whose decimal expansion is terminating or nonterminating recurring is rational.

SOLVED EXAMPLES

EXAMPLE 1 *Express* $3\frac{1}{8}$ *in decimal form.*

SOLUTION We have $3\frac{1}{8} = \frac{25}{8}$.

By actual division, we have

$$\begin{array}{l} 8\overline{)\,25.000\,}(3.125 \\ \underline{-24} \\ \quad 10 \\ \quad \underline{-8} \\ \quad\ \ 20 \\ \quad \underline{-16} \\ \qquad 40 \\ \quad\ \underline{-40} \\ \qquad \times \end{array}$$

$\therefore \quad \frac{25}{8} = 3.125.$

EXAMPLE 2 *Express* $\frac{2}{11}$ *in decimal form.*

SOLUTION By actual division, we have

$$\begin{array}{l} 11\overline{)\,2.0\,}(0.1818\ldots \\ \quad \underline{-0} \\ \quad 20 \\ \quad \underline{-11} \\ \qquad 90 \\ \quad\ \underline{-88} \\ \qquad\ 20 \\ \qquad \underline{-11} \\ \qquad\quad 90 \\ \qquad \underline{-88} \\ \qquad\quad\ 2 \end{array}$$

$\therefore \quad \frac{2}{11} = 0.1818\ldots = 0.\overline{18}.$

EXAMPLE 3 *Write* $\frac{3}{13}$ *in decimal form and say what kind of decimal representation it has.* [2010]

SOLUTION By actual division, we have

$$\begin{array}{l}
13\overline{)\,3.0\,}(0.23076923\\
\underline{-26}\\
\quad 40\\
\underline{\quad -39}\\
\qquad 100\\
\underline{\qquad -91}\\
\qquad\quad 90\\
\underline{\qquad\quad -78}\\
\qquad\quad 120\\
\underline{\qquad\quad -117}\\
\qquad\qquad\quad 30\\
\underline{\qquad\qquad -26}\\
\qquad\qquad\quad 40\\
\underline{\qquad\qquad\quad -39}\\
\qquad\qquad\qquad\quad 1
\end{array}$$

$$\therefore \quad \frac{3}{13} = 0.23076923\ldots = 0.\overline{230769}.$$

Clearly, $\frac{3}{13}$ has a nonterminating recurring decimal representation.

<u>EXAMPLE 4</u> *Find the decimal expansion of* $\frac{1}{7}\cdot$ *Can you predict what the decimal expansions of* $\frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}$ *are, without actually doing the long division? If so, how?*

SOLUTION By long division, we have

$$\begin{array}{l}
7\overline{)\,1.0\,}(.1428571428\ldots\\
\underline{\;-7}\\
\quad 30\\
\underline{\;-28}\\
\qquad 20\\
\underline{\quad -14}\\
\qquad\; 60\\
\underline{\qquad -56}\\
\qquad\quad 40\\
\underline{\qquad\; -35}\\
\qquad\qquad 50\\
\underline{\qquad\quad -49}\\
\qquad\qquad\; 10\\
\underline{\qquad\qquad -7}\\
\qquad\qquad\quad 30\\
\underline{\qquad\qquad\; -28}\\
\qquad\qquad\qquad 20\\
\underline{\qquad\qquad\quad -14}\\
\qquad\qquad\qquad\; 60\\
\underline{\qquad\qquad\qquad -56}\\
\qquad\qquad\qquad\quad 4
\end{array}$$

$$\therefore \quad \frac{1}{7} = 0.1428571428\ldots = 0.\overline{142857}.$$

Clearly, $\frac{2}{7} = 2 \times \frac{1}{7} = 2 \times 0.\overline{142857} = 0.\overline{285714}$;

$$\frac{3}{7} = 3 \times \frac{1}{7} = 3 \times 0.\overline{142857} = 0.\overline{428571};$$

$$\frac{4}{7} = 4 \times \frac{1}{7} = 4 \times 0.\overline{142857} = 0.\overline{571428};$$

$$\frac{5}{7} = 5 \times \frac{1}{7} = 5 \times 0.\overline{142857} = 0.\overline{714285};$$

$$\frac{6}{7} = 6 \times \frac{1}{7} = 6 \times 0.\overline{142857} = 0.\overline{857142}.$$

EXAMPLE 5 *What can the maximum number of digits be in the repeating block of digits in the decimal expansion of* $\frac{1}{17}$*? Perform the division to check your answer.*

SOLUTION By long division, we have

$$17\overline{)\,1.00\,}(0.05882352941176470588$$

$$\begin{array}{r}
-85 \\ \hline
150 \\
-136 \\ \hline
140 \\
-136 \\ \hline
40 \\
-34 \\ \hline
60 \\
-51 \\ \hline
90 \\
-85 \\ \hline
50 \\
-34 \\ \hline
160 \\
-153 \\ \hline
70 \\
-68 \\ \hline
20 \\
-17 \\ \hline
30 \\
-17 \\ \hline
130 \\
-119 \\ \hline
110 \\
-102 \\ \hline
8
\end{array}
\qquad\qquad
\begin{array}{r}
80 \\
-68 \\ \hline
120 \\
-119 \\ \hline
100 \\
-85 \\ \hline
150 \\
-136 \\ \hline
140 \\
-136 \\ \hline
4
\end{array}$$

Clearly, $\frac{1}{17} = 0.05882352941176470588\ldots$

$= 0.\overline{0588235294117647}$.

Length of period = 16.

Hence, the maximum number of digits in the repeating block of digits in the decimal expansion of $\frac{1}{17}$ is less than 17, i.e., 16.

EXAMPLE 6 *Express* $0.\overline{3}$ *in the form* $\frac{p}{q}$, *where p and q are integers and* $q \neq 0$.

[2014]

SOLUTION Let $x = 0.\overline{3}$.

Then, $x = 0.333\ldots$. ... (i)

Since the repeating block has only one digit 3, we multiply x by 10 to get

$10x = 3.333\ldots$. ... (ii)

Subtracting (i) from (ii), we get

$$9x = 3 \Leftrightarrow x = \frac{3}{9} = \frac{1}{3}\cdot$$

Hence, $0.\overline{3} = \frac{1}{3}\cdot$

EXAMPLE 7 *Express* $1.\overline{4}$ *as a fraction in simplest form.*

SOLUTION Let $x = 1.\overline{4}$.

Then, $x = 1.444\ldots$. ... (i)

Since the repeating block has only one digit 4, we multiply x by 10 to get

$10x = 14.444\ldots$. ... (ii)

Subtracting (i) from (ii), we get

$$9x = 13 \Leftrightarrow x = \frac{13}{9} = 1\frac{4}{9}\cdot$$

Hence, $1.\overline{4} = 1\frac{4}{9}\cdot$

EXAMPLE 8 *Express* 0.9999 ... *as a fraction in simplest form.*

SOLUTION Let $x = 0.9999\ldots$. ... (i)

Since the repeating block has only digit 9, we multiply x by 10 to get

$10x = 9.9999\ldots$. ... (ii)

Subtracting (i) from (ii), we get

$$9x = 9 \Leftrightarrow x = 1.$$

Hence, $0.9999\ldots = 1$.

EXAMPLE 9 *Express* $0.\overline{57}$ *in the form* $\frac{p}{q}$, *where p and q are integers and* $q \neq 0$. [2014]

SOLUTION Let $x = 0.\overline{57}$.

Then, $x = 0.575757 \ldots$ . ... (i)

Since the repeating block 57 has two digits, we multiply x by 100 to get

$100x = 57.5757 \ldots$ . ... (ii)

Subtracting (i) from (ii), we get

$$99x = 57 \Leftrightarrow x = \frac{57}{99} = \frac{19}{33}.$$

Hence, $0.\overline{57} = \frac{19}{33}$.

EXAMPLE 10 *Express* $0.\overline{001}$ *as a fraction in simplest form.* [2011]

SOLUTION Let $x = 0.\overline{001}$.

Then, $x = 0.001001001 \ldots$ . ... (i)

Since the repeating block 001 has 3 digits, we multiply x by 1000 to get

$1000x = 1.001001001 \ldots$ . ... (ii)

Subtracting (i) from (ii), we get

$$999x = 1 \Leftrightarrow x = \frac{1}{999}.$$

Hence, $0.\overline{001} = \frac{1}{999}$.

EXAMPLE 11 *Express* $2.4\overline{178}$ *in the form* $\frac{p}{q}$, *where p and q are integers and* $q \neq 0$. [2015]

SOLUTION Let $x = 2.4\overline{178}$.

Then, $x = 2.4178178178 \ldots$ . ... (i)

Since the repeating block 178 has 3 digits, we multiply x by 1000 to get

$1000x = 2417.8178178178 \ldots$ . ... (ii)

Subtracting (i) from (ii), we get

$$999x = 2415.4 \Rightarrow x = \frac{2415.4}{999} = \frac{24154}{9990} = \frac{12077}{4995}.$$

EXERCISE 1B

1. Without actual division, find which of the following rational numbers are terminating decimals.

(i) $\frac{13}{80}$ (ii) $\frac{7}{24}$ (iii) $\frac{5}{12}$ (iv) $\frac{31}{375}$ (v) $\frac{16}{125}$

2. Write each of the following in decimal form and say what kind of decimal expansion each has.

(i) $\frac{5}{8}$ (ii) $\frac{7}{25}$ (iii) $\frac{3}{11}$ (iv) $\frac{5}{13}$ (v) $\frac{11}{24}$

(vi) $\frac{261}{400}$ (vii) $\frac{231}{625}$ (viii) $2\frac{5}{12}$

3. Express each of the following decimals in the form $\frac{p}{q}$, where p, q are integers and $q \neq 0$.

(i) $0.\overline{2}$ [2014] (ii) $0.\overline{53}$ [2010] (iii) $2.\overline{93}$ [2010, '15]

(iv) $18.\overline{48}$ [2014] (v) $0.\overline{235}$ [2010] (vi) $0.00\overline{32}$

(vii) $1.3\overline{23}$ [2010] (viii) $0.3\overline{178}$ [2014] (ix) $32.12\overline{35}$ [2011]

(x) $0.40\overline{7}$ [2014]

4. Express $2.\overline{36} + 0.\overline{23}$ as a fraction in simplest form. [2011]

5. Express in the form of $\frac{p}{q}$: $0.\overline{38} + 1.\overline{27}$. [2015]

ANSWERS (EXERCISE 1B)

1. (i) and (v)

2. (i) 0.625; terminating

(ii) 0.28; terminating

(iii) $0.\overline{27}$; nonterminating recurring

(iv) $0.\overline{384615}$; nonterminating recurring

(v) $0.458\overline{3}$; nonterminating recurring

(vi) 0.6525; terminating

(vii) 0.3696; terminating

(viii) $2.41\overline{6}$; nonterminating recurring

3. (i) $\frac{2}{9}$ (ii) $\frac{53}{99}$ (iii) $\frac{97}{33}$ (iv) $\frac{610}{33}$ (v) $\frac{235}{999}$ (vi) $\frac{8}{2475}$

(vii) $\frac{131}{99}$ (viii) $\frac{635}{1998}$ (ix) $\frac{318023}{9900}$ (x) $\frac{367}{900}$

4. $\frac{257}{99}$ **5.** $\frac{164}{99}$

IRRATIONAL NUMBERS

IRRATIONAL NUMBER *A number which can neither be expressed as a terminating decimal nor as a repeating decimal, is called an irrational number.*

Thus, nonterminating, nonrepeating decimals are irrational numbers.

EXAMPLES OF IRRATIONAL NUMBERS

TYPE 1 (i) Clearly, 0.01001000100001 … is a nonterminating and nonrepeating decimal and therefore, it is irrational.

Similarly,

(ii) 0.02002000200002 … is irrational,
0.03003000300003 … is irrational, and so on.

(iii) 0.12112111211112 … is irrational,
0.13113111311113 … is irrational, and so on.

(iv) 0.54554555455554 … is irrational,
0.64664666466664 … is irrational, and so on.

TYPE 2 If m is a positive integer which is not a perfect square then $\sqrt{m}$ is irrational.

Thus $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}, \sqrt{10}, \sqrt{11}$, etc., are all irrational numbers.

TYPE 3 If m is a positive integer which is not a perfect cube then $\sqrt[3]{m}$ is irrational.

Thus, $\sqrt[3]{2}, \sqrt[3]{3}, \sqrt[3]{4}, \sqrt[3]{5}, \sqrt[3]{6}, \sqrt[3]{7}, \sqrt[3]{8}, \sqrt[3]{9}$, etc., are all irrational numbers.

The Number π: π is a number whose exact value is not $\frac{22}{7}$.

In fact π has a value which is nonterminating and nonrepeating.

So, π is irrational, while $\frac{22}{7}$ is rational.

PROPERTIES OF IRRATIONAL NUMBERS

1. *Irrational numbers satisfy the commutative, associative and distributive laws for addition and multiplication.*

2. (i) *Sum of two irrationals need not be an irrational.*

Example Each one of $(2+\sqrt{3})$ and $(4-\sqrt{3})$ is irrational.

But, $(2+\sqrt{3})+(4-\sqrt{3}) = 6$, which is rational.

(ii) *Difference of two irrationals need not be an irrational.*

Example Each one of $(5+\sqrt{2})$ and $(3+\sqrt{2})$ is irrational.

But, $(5+\sqrt{2})-(3+\sqrt{2}) = 2$, which is rational.

(iii) *Product of two irrationals need not be an irrational.*

Example $\sqrt{3}$ is irrational.

But, $\sqrt{3}\times\sqrt{3} = 3$, which is rational.

(iv) *Quotient of two irrationals need not be an irrational.*

Example Each one of $2\sqrt{3}$ and $\sqrt{3}$ is irrational.

But, $\frac{2\sqrt{3}}{\sqrt{3}} = 2$, which is rational.

3. (i) *Sum of a rational and an irrational is irrational.*

(ii) *Difference of a rational and an irrational is irrational.*

(iii) *Product of a rational and an irrational is irrational.*

(iv) *Quotient of a rational and an irrational is irrational.*

REMARK If a is rational and $\sqrt{b}$ is irrational then each one of $(a+\sqrt{b}), (a-\sqrt{b}), a\sqrt{b}$ and $\frac{a}{\sqrt{b}}$ is irrational.

Examples: Each one of $(4+\sqrt{3}), (8-\sqrt{5}), 5\sqrt{3}$ and $\frac{3}{\sqrt{2}}$ is irrational.

AN IMPORTANT RESULT

If a and b are two distinct positive rational numbers then

1. $\frac{a+b}{2}$ is a rational number lying between a and b.

2. $\sqrt{ab}$ is an irrational number lying between a and b.

SOLVED EXAMPLES

EXAMPLE 1 *Are the square roots of all positive integers irrational? If no, give two examples.*

SOLUTION The square root of every positive integer is not always irrational.

Examples (i) $\sqrt{4} = 2$, which is rational.

(ii) $\sqrt{9} = 3$, which is rational.

EXAMPLE 2 *Are the cube roots of all positive integers irrational? If no, give two examples.*

SOLUTION The cube root of a positive integer is not always irrational.

Examples (i) $\sqrt[3]{8} = 2$, which is rational.

(ii) $\sqrt[3]{27} = 3$, which is rational.

EXAMPLE 3 *Write three numbers whose decimal expansions are nonterminating and nonrepeating.*

SOLUTION We know that if m is a positive integer which is not a perfect square, then $\sqrt{m}$ is irrational.

Also, we know that the decimal expansion of an irrational number is nonterminating and nonrepeating.

So, the required numbers are $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$.

EXAMPLE 4 *Insert a rational and an irrational number between* 3 *and* 4.

SOLUTION A rational number between 3 and 4 is $\frac{3+4}{2} = 3.5$. An irrational number between 3 and 4, is $\sqrt{3 \times 4}$, i.e., $\sqrt{12}$.

EXAMPLE 5 *Find two irrational numbers between* $\frac{1}{7}$ *and* $\frac{2}{7}\cdot$

SOLUTION By actual division, we know that $\frac{1}{7} = 0.\overline{142857}$ and $\frac{2}{7} = 0.\overline{285714}$.

Now, consider the numbers

$$a = 0.16016001600016\ldots \text{ and } b = 0.21211211121111\ldots.$$

Clearly, a and b are irrational numbers such that $\frac{1}{7} < a < b < \frac{2}{7}\cdot$

Hence, a and b are the required numbers.

EXAMPLE 6 *Find two rational and two irrational numbers between* $\sqrt{2}$ *and* $\sqrt{3}$. [2014, '15]

SOLUTION We have

$$\sqrt{2} = 1.41421356\ldots \text{ and } \sqrt{3} = 1.732050807\ldots.$$

If we consider the numbers 1.5 and 1.6, we find that both of them are rational numbers such that $\sqrt{2} < 1.5 < 1.6 < \sqrt{3}$.

Hence, two rational numbers between $\sqrt{2}$ and $\sqrt{3}$ are 1.5 and 1.6.

Now, consider the numbers

$$a = 1.5050050005\ldots \text{ and } b = 1.6161161116\ldots.$$

Clearly, a and b are irrational numbers such that

$$\sqrt{2} < a < b < \sqrt{3}.$$

Hence, a and b are two irrational numbers between $\sqrt{2}$ and $\sqrt{3}$.

EXAMPLE 7 *Find two rational numbers in the form* $\frac{p}{q}$ *between* 0.343443444344443 … *and* 0.363663666366663 … . [2010]

SOLUTION Let $a = 0.343443444344443\ldots$ and $b = 0.363663666366663\ldots$.

Consider the numbers

$$c = 0.35 = \frac{35}{100} = \frac{7}{20} \quad \text{and} \quad d = 0.36 = \frac{36}{100} = \frac{9}{25}\cdot$$

Clearly, c and d are rational numbers such that $a < c < d < b$.

Hence, $\frac{7}{20}$ and $\frac{9}{25}$ are the required rational numbers.

EXERCISE 1C

1. What are irrational numbers? How do they differ from rational numbers? Give examples.

2. Classify the following numbers as rational or irrational. Give reasons to support your answer.

(i) $\sqrt{\frac{3}{81}}$ (ii) $\sqrt{361}$ (iii) $\sqrt{21}$ (iv) $\sqrt{1.44}$

(v) $\frac{2}{3}\sqrt{6}$ (vi) 4.1276 (vii) $\frac{22}{7}$

(viii) 1.232332333 … (ix) 3.040040004 …

(x) 2.356565656 … (xi) 6.834834 …

3. Let x be a rational number and y be an irrational number. Is $x+y$ necessarily an irrational number? Give an example in support of your answer. [2010, '14]

4. Let a be a rational number and b be an irrational number. Is ab necessarily an irrational number? Justify your answer with an example.

5. Is the product of two irrationals always irrational? Justify your answer.

6. Give an example of two irrational numbers whose [2015]

(i) difference is an irrational number.

(ii) difference is a rational number.

(iii) sum is an irrational number.

(iv) sum is a rational number.

(v) product is an irrational number.

(vi) product is a rational number.

(vii) quotient is an irrational number.

(viii) quotient is a rational number.

7. Examine whether the following numbers are rational or irrational.

(i) $3+\sqrt{3}$ (ii) $\sqrt{7}-2$ (iii) $\sqrt[3]{5}\times\sqrt[3]{25}$

(iv) $\sqrt{7}\times\sqrt{343}$ (v) $\sqrt{\frac{13}{117}}$ (vi) $\sqrt{8}\times\sqrt{2}$

8. Insert a rational and an irrational number between 2 and 2.5.

9. How many irrational numbers lie between $\sqrt{2}$ and $\sqrt{3}$? Find any three irrational numbers lying between $\sqrt{2}$ and $\sqrt{3}$. [2010]

10. Find two rational and two irrrational numbers between 0.5 and 0.55.

11. Find three different irrational numbers between the rational numbers $\frac{5}{7}$ and $\frac{9}{11}$.

12. Find two rational numbers of the form $\frac{p}{q}$ between the numbers 0.2121121112... and 0.2020020002... .

13. Find two irrational numbers between 0.16 and 0.17.

14. State, in each case, whether the given statement is true or false.

(i) The sum of two rational numbers is rational.

(ii) The sum of two irrational numbers is irrational.

(iii) The product of two rational numbers is rational.

(iv) The product of two irrational numbers is irrational.

(v) The sum of a rational number and an irrational number is irrational.

(vi) The product of a nonzero rational number and an irrational number is a rational number.

(vii) Every real number is rational.

(viii) Every real number is either rational or irrational.

(ix) π is irrational and $\frac{22}{7}$ is rational.

ANSWERS (EXERCISE 1C)

2. (i) Irrational (ii) Rational (iii) Irrational
(iv) Rational (v) Irrational (vi) Rational
(vii) Rational (viii) Irrational (ix) Irrational
(x) Rational (xi) Rational

3. Yes **4.** Yes **5.** No

6. (i) $(2-\sqrt{3})$ and $(2+\sqrt{3})$ (ii) $(2+\sqrt{3})$ and $(5+\sqrt{3})$
(iii) $(5+\sqrt{2})$ and $(\sqrt{3}-5)$ (iv) $(3+\sqrt{2})$ and $(3-\sqrt{2})$
(v) $(2+\sqrt{2})$ and $(3-\sqrt{2})$ (vi) $(4+\sqrt{3})$ and $(4-\sqrt{3})$
(vii) $\sqrt{18}$ and $\sqrt{3}$ (viii) $\sqrt{27}$ and $\sqrt{3}$

7. (i) Irrational (ii) Irrational (iii) Rational
(iv) Rational (v) Rational (vi) Rational

8. 2.25; $\sqrt{5}$

9. Infinite; 1.5050050005 ..., 1.6161161116 ..., 1.70770770777...

10. 0.51 and 0.53; 0.5151151115 ... and 0.5455455545...

11. 0.727222722227 ..., 0.7577577757 ..., 0.8080080008 ...

12. $\frac{51}{250}, \frac{103}{500}$

13. 0.16116111611116 ..., 0.1686686668666 ...

14. (i) True (ii) False (iii) True (iv) False (v) True (vi) False
(vii) False (viii) True (ix) True

REAL NUMBERS

REAL NUMBERS *A number whose square is non-negative, is called a real number.*

In fact, all rational and all irrational numbers form the collection of all real numbers.

Every real number is either rational or irrational.

Consider a real number.

(i) If it is an integer or it has a terminating or repeating decimal representation then it is rational.

(ii) If it has a nonterminating and nonrepeating decimal representation then it is irrational.

The totality of rationals and irrationals forms the collection of all real numbers.

COMPLETENESS PROPERTY On the number line, each point corresponds to a unique real number. And, every real number can be represented by a unique point on the real line.

DENSITY PROPERTY Between any two real numbers, there exist infinitely many real numbers.

ADDITION PROPERTIES OF REAL NUMBERS

(i) CLOSURE PROPERTY The sum of two real numbers is always a real number.

(ii) ASSOCIATIVE LAW $(a+b)+c=a+(b+c)$ for all real numbers a, b, c.

(iii) COMMUTATIVE LAW $a+b=b+a$ for all real numbers a and b.

(iv) EXISTENCE OF ADDITIVE IDENTITY Clearly, 0 is a real number such that $0+a=a+0=a$ for every real number a.
0 is called the *additive identity* for real numbers.

(v) EXISTENCE OF ADDITIVE INVERSE For each real number a, there exists a real number $(-a)$ such that $a+(-a)=(-a)+a=0$.
a and $(-a)$ are called the *additive inverse* (or *negative*) of each other.

MULTIPLICATION PROPERTIES OF REAL NUMBERS

(i) CLOSURE PROPERTY The product of two real numbers is always a real number.

(ii) ASSOCIATIVE LAW $(ab)c=a(bc)$ for all real numbers a, b, c.

(iii) COMMUTATIVE LAW $ab=ba$ for all real numbers a and b.

(iv) EXISTENCE OF MULTIPLICATIVE IDENTITY Clearly, 1 is a real number such that $1\cdot a=a\cdot 1=a$ for every real number a.
1 is called the *multiplicative identity* for real numbers.

(v) EXISTENCE OF MULTIPLICATIVE INVERSE For each nonzero real number a, there exists a real number $\left(\frac{1}{a}\right)$ such that $a \cdot \frac{1}{a} = \frac{1}{a} \cdot a = 1$.

a and $\frac{1}{a}$ are called the *multiplicative inverse* (or *reciprocal*) of each other.

(vi) DISTRIBUTIVE LAWS OF MULTIPLICATION OVER ADDITION We have

$a(b+c) = ab + ac$ and $(a+b)c = ac + bc$ for all real numbers a, b, c.

SOME MORE RESULTS ON REAL NUMBERS

For all positive real numbers a and b, we have

(i) $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$, (ii) $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$.

SOLVED EXAMPLES

EXAMPLE 1 *Add* $(3\sqrt{2} + 7\sqrt{3})$ *and* $(\sqrt{2} - 5\sqrt{3})$.

SOLUTION We have

$$\begin{aligned}(3\sqrt{2}+7\sqrt{3}) + (\sqrt{2}-5\sqrt{3}) &= (3\sqrt{2}+\sqrt{2}) + (7\sqrt{3}-5\sqrt{3})\\ &= (3+1)\sqrt{2} + (7-5)\sqrt{3}\\ &= (4\sqrt{2}+2\sqrt{3}).\end{aligned}$$

EXAMPLE 2 *Multiply* $5\sqrt{11}$ *by* $3\sqrt{11}$.

SOLUTION We have

$$5\sqrt{11} \times 3\sqrt{11} = 5 \times 3 \times \sqrt{11} \times \sqrt{11} = (5 \times 3 \times 11) = 165.$$

EXAMPLE 3 *Divide* $15\sqrt{15}$ *by* $3\sqrt{5}$.

SOLUTION We have

$$15\sqrt{15} \div 3\sqrt{5} = \frac{15\sqrt{15}}{3\sqrt{5}} = \frac{5 \times \sqrt{5} \times \sqrt{3}}{\sqrt{5}} = 5\sqrt{3}.$$

EXAMPLE 4 *Simplify*

(i) $(3+\sqrt{3})(2+\sqrt{2})$ *(ii)* $(6+\sqrt{6})(6-\sqrt{6})$

(iii) $(\sqrt{3}+\sqrt{2})^2$ *(iv)* $(\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2})$

SOLUTION We have

(i) $$\begin{aligned}(3+\sqrt{3})(2+\sqrt{2}) &= 3 \times 2 + 3 \times \sqrt{2} + \sqrt{3} \times 2 + \sqrt{3} \times \sqrt{2}\\ &= 6 + 3\sqrt{2} + 2\sqrt{3} + \sqrt{6}.\end{aligned}$$

(ii) $$\begin{aligned}(6+\sqrt{6})(6-\sqrt{6}) &= (6)^2 - (\sqrt{6})^2 \qquad [\because (a+b)(a-b) = a^2 - b^2]\\ &= (36-6) = 30.\end{aligned}$$

(iii) $(\sqrt{3}+\sqrt{2})^2 = (\sqrt{3})^2 + (\sqrt{2})^2 + 2 \times \sqrt{3} \times \sqrt{2}$

$$[\because (a+b)^2 = a^2 + b^2 + 2ab]$$

$$= 3 + 2 + 2\sqrt{6} = 5 + 2\sqrt{6}.$$

(iv) $(\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2}) = (\sqrt{5})^2 - (\sqrt{2})^2$

$$[\because (a-b)(a+b) = a^2 - b^2]$$

$$= 5 - 2 = 3.$$

EXAMPLE 5 *Simplify* $\dfrac{\sqrt{72}}{5\sqrt{72}+3\sqrt{288}-2\sqrt{648}}$. [2014]

SOLUTION We have

$$\frac{\sqrt{72}}{5\sqrt{72}+3\sqrt{288}-2\sqrt{648}} = \frac{\sqrt{72}}{5\sqrt{72}+3\sqrt{4\times 72}-2\sqrt{9\times 72}}$$

$$= \frac{\sqrt{72}}{5\sqrt{72}+3\cdot\sqrt{4}\cdot\sqrt{72}-2\cdot\sqrt{9}\cdot\sqrt{72}}$$

$$= \frac{\sqrt{72}}{5\sqrt{72}+3\times 2\times\sqrt{72}-2\times 3\times\sqrt{72}}$$

$$= \frac{\sqrt{72}}{5\sqrt{72}+6\sqrt{72}-6\sqrt{72}}$$

$$= \frac{\sqrt{72}}{(5+6-6)\sqrt{72}} = \frac{1}{5}.$$

EXAMPLE 6 *Simplify* $\dfrac{\sqrt{a^2-b^2}+a}{\sqrt{a^2+b^2}+b} \div \dfrac{\sqrt{a^2+b^2}-b}{a-\sqrt{a^2-b^2}}$.

SOLUTION We have

$$\frac{\sqrt{a^2-b^2}+a}{\sqrt{a^2+b^2}+b} \div \frac{\sqrt{a^2+b^2}-b}{a-\sqrt{a^2-b^2}} = \frac{a+\sqrt{a^2-b^2}}{\sqrt{a^2+b^2}+b} \times \frac{a-\sqrt{a^2-b^2}}{\sqrt{a^2+b^2}-b}$$

$$= \frac{(a)^2-(\sqrt{a^2-b^2})^2}{(\sqrt{a^2+b^2})^2-(b)^2}$$

$$= \frac{a^2-(a^2-b^2)}{(a^2+b^2)-b^2} = \frac{a^2-a^2+b^2}{a^2+b^2-b^2} = \frac{b^2}{a^2}.$$

EXERCISE 1D

1. Add

(i) $(2\sqrt{3}-5\sqrt{2})$ and $(\sqrt{3}+2\sqrt{2})$

(ii) $(2\sqrt{2}+5\sqrt{3}-7\sqrt{5})$ and $(3\sqrt{3}-\sqrt{2}+\sqrt{5})$

(iii) $\left(\frac{2}{3}\sqrt{7}-\frac{1}{2}\sqrt{2}+6\sqrt{11}\right)$ and $\left(\frac{1}{3}\sqrt{7}+\frac{3}{2}\sqrt{2}-\sqrt{11}\right)$

2. Multiply

(i) $3\sqrt{5}$ by $2\sqrt{5}$ (ii) $6\sqrt{15}$ by $4\sqrt{3}$ (iii) $2\sqrt{6}$ by $3\sqrt{3}$

(iv) $3\sqrt{8}$ by $3\sqrt{2}$ (v) $\sqrt{10}$ by $\sqrt{40}$ (vi) $3\sqrt{28}$ by $2\sqrt{7}$

3. Divide

(i) $16\sqrt{6}$ by $4\sqrt{2}$ (ii) $12\sqrt{15}$ by $4\sqrt{3}$ (iii) $18\sqrt{21}$ by $6\sqrt{7}$

4. Simplify

(i) $(3-\sqrt{11})(3+\sqrt{11})$ [2014] (ii) $(-3+\sqrt{5})(-3-\sqrt{5})$ [2014]

(iii) $(3-\sqrt{3})^2$ (iv) $(\sqrt{5}-\sqrt{3})^2$

(v) $(5+\sqrt{7})(2+\sqrt{5})$ (vi) $(\sqrt{5}-\sqrt{2})(\sqrt{2}-\sqrt{3})$

5. Simplify $(3+\sqrt{3})(2+\sqrt{2})^2$. [2010]

6. Examine whether the following numbers are rational or irrational:

(i) $(5-\sqrt{5})(5+\sqrt{5})$ (ii) $(\sqrt{3}+2)^2$

(iii) $\dfrac{2\sqrt{13}}{3\sqrt{52}-4\sqrt{117}}$ (iv) $\sqrt{8}+4\sqrt{32}-6\sqrt{2}$

7. On her birthday Reema distributed chocolates in an orphanage. The total number of chocolates she distributed is given by $(5+\sqrt{11})(5-\sqrt{11})$. [2014]

(i) Find the number of chocolates distributed by her.

(ii) Write the moral values depicted here by Reema.

8. Simplify

(i) $3\sqrt{45}-\sqrt{125}+\sqrt{200}-\sqrt{50}$ [2010]

(ii) $\dfrac{2\sqrt{30}}{\sqrt{6}}-\dfrac{3\sqrt{140}}{\sqrt{28}}+\dfrac{\sqrt{55}}{\sqrt{99}}$

(iii) $\sqrt{72}+\sqrt{800}-\sqrt{18}$ [2014]

ANSWERS (EXERCISE 1D)

1. (i) $3(\sqrt{3}-\sqrt{2})$ (ii) $\sqrt{2}+8\sqrt{3}-6\sqrt{5}$ (iii) $\sqrt{7}+\sqrt{2}+5\sqrt{11}$

2. (i) 30 (ii) $72\sqrt{5}$ (iii) $18\sqrt{2}$ (iv) 36 (v) 20

(vi) 84 **3.** (i) $4\sqrt{3}$ (ii) $3\sqrt{5}$ (iii) $3\sqrt{3}$

4. (i) –2 (ii) 4 (iii) $12-6\sqrt{3}$ (iv) $8-2\sqrt{15}$

(v) $10+5\sqrt{5}+2\sqrt{7}+\sqrt{35}$ (vi) $\sqrt{10}-\sqrt{15}-2+\sqrt{6}$

5. $18+12\sqrt{2}+6\sqrt{3}+4\sqrt{6}$

6. (i) Rational (ii) Irrational (iii) Rational (iv) Irrational

7. (i) 14

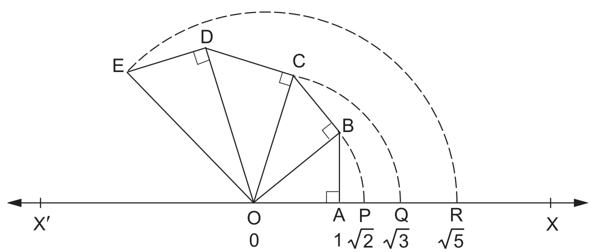
(ii) To help the poor and needy and to make the deprived children happy.

8. (i) $4\sqrt{5}+5\sqrt{2}$ (ii) $-\frac{2}{3}\sqrt{5}$ (iii) $23\sqrt{2}$

REPRESENTING IRRATIONAL NUMBERS ON THE NUMBER LINE

EXAMPLE 1 *Represent each of the numbers $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$ on the real line.*

SOLUTION Let $X'OX$ be a horizontal line, taken as the x-axis and let O be the origin. Let O represent 0.



Take $OA = 1$ unit and draw $AB \perp OA$ such that $AB = 1$ unit. Join OB. Then,

$$OB = \sqrt{OA^2 + AB^2} = \sqrt{1^2 + 1^2} = \sqrt{2} \text{ units.}$$

With O as centre and OB as radius, draw an arc, meeting OX at P.

Then, $OP = OB = \sqrt{2}$ units.

Thus, the point P represents $\sqrt{2}$ on the real line.

Now, draw $BC \perp OB$ such that $BC = 1$ unit.

Join OC. Then,

$$OC = \sqrt{OB^2 + BC^2} = \sqrt{(\sqrt{2})^2 + 1^2} = \sqrt{3} \text{ units.}$$

With O as centre and OC as radius, draw an arc, meeting OX at Q. Then,

$$OQ = OC = \sqrt{3} \text{ units.}$$

Thus, the point Q represents $\sqrt{3}$ on the real line.

Now, draw $CD \perp OC$ such that $CD = 1$ unit.

Join OD. Then,

$$OD = \sqrt{OC^2 + CD^2} = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{4} = 2 \text{ units.}$$

Now, draw $DE \perp OD$ such that $DE = 1$ unit.

Join OE. Then,

$$OE = \sqrt{OD^2 + DE^2} = \sqrt{2^2 + 1^2} = \sqrt{5} \text{ units.}$$

With O as centre and OE as radius, draw an arc, meeting OX at R. Then, $OR = OE = \sqrt{5}$ units.

Thus, the point R represents $\sqrt{5}$ on the real line.

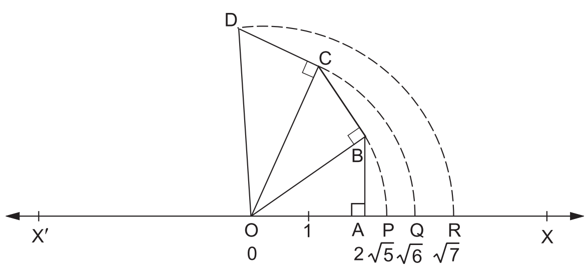
Hence, the points P, Q, R represent the numbers $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$ respectively.

REMARK In the same way, we can locate $\sqrt{n}$ for any positive integer n, after $\sqrt{n-1}$ has been located.

EXAMPLE 2 *Represent each of the numbers $\sqrt{5}$, $\sqrt{6}$ and $\sqrt{7}$ on the real line.*

SOLUTION Draw a horizontal line $X'OX$, taken as the x-axis.

Take O as the origin to represent 0.



Let $OA = 2$ units and let $AB \perp OA$ such that $AB = 1$ unit.

Join OB. Then,

$$OB = \sqrt{OA^2 + AB^2} = \sqrt{2^2 + 1^2} = \sqrt{5}.$$

With O as centre and OB as radius, draw an arc, meeting OX at P.

Then, $OP = OB = \sqrt{5}$.

Thus, P represents $\sqrt{5}$ or the real line.

Now, draw $BC \perp OB$ and set off $BC = 1$ unit.

Join OC. Then,

$$OC = \sqrt{OB^2 + BC^2} = \sqrt{(\sqrt{5})^2 + 1^2} = \sqrt{6}.$$

With O as centre and OC as radius, draw an arc, meeting OX at Q.

Then, $OQ = OC = \sqrt{6}$.

Thus, Q represents $\sqrt{6}$ on the real line.

Now, draw $CD \perp OC$ and set off $CD = 1$ unit.

Join OD. Then,

$$OD = \sqrt{OC^2 + CD^2} = \sqrt{(\sqrt{6})^2 + 1^2} = \sqrt{7}.$$

With O as centre and OD as radius, draw an arc, meeting OX at R. Then,

$$OR = OD = \sqrt{7}.$$

Thus, the points P, Q, R represent the real numbers $\sqrt{5}$, $\sqrt{6}$ and $\sqrt{7}$ respectively.

EXISTENCE OF $\sqrt{x}$ FOR A GIVEN POSITIVE REAL NUMBER x

Let x be a given positive real number.

We shall prove the existence of $\sqrt{x}$ in the following three steps.

STEP 1 *Finding $\sqrt{x}$ geometrically.*

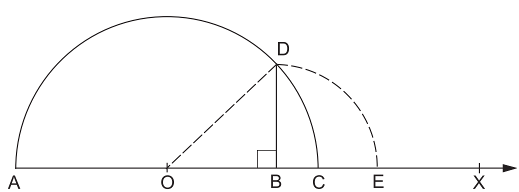
Draw a line segment $AB = x$ units and extend it to C such that $BC = 1$ unit.

Find the midpoint O of AC.

With O as centre and OA as radius, draw a semicircle.

Now, draw $BD \perp AC$, intersecting the semicircle at D.

We shall show that $BD = \sqrt{x}$.



STEP 2 *Proving that $BD = \sqrt{x}$.*

We have, $AB = x$ units and $BC = 1$ unit. So, $AC = (x+1)$ units.

Now, $OD = OC$ (radii of the same semicircle)

$$= \frac{1}{2}AC = \frac{1}{2}(x+1) \text{ units.}$$

And, $OB = (OC - BC) = \left(\frac{x+1}{2} - 1\right)$ units $= \frac{1}{2}(x-1)$ units.

$$\therefore \quad BD^2 = (OD^2 - OB^2) = \frac{1}{4}(x+1)^2 - \frac{1}{4}(x-1)^2$$

$$= \frac{1}{4} \cdot \{(x+1)^2 - (x-1)^2\} = \left(\frac{1}{4} \times 4x\right) = x$$

$\Rightarrow \quad BD = \sqrt{x}$.

Thus, we have proved that $BD = \sqrt{x}$.

STEP 3 *Representing $\sqrt{x}$ on the real line.*

Take BC produced as the number line with the origin at B.

Now, $BC = 1$ unit.

So, B and C denote 0 and 1 respectively.

With B as centre and BD as radius, draw an arc, meeting BC produced at E. Then, $BE = BD = \sqrt{x}$.

Hence, the point E represents $\sqrt{x}$.

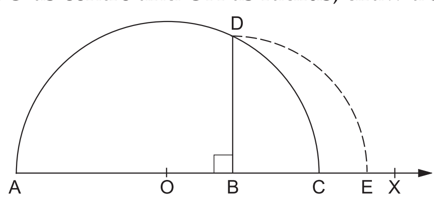
Thus, for every positive real number x, $\sqrt{x}$ is also a positive real number and hence $\sqrt{x}$ exists.

EXAMPLE 3 *Find the value of $\sqrt{4.3}$ geometrically.*

SOLUTION Draw a line segment $AB = 4.3$ units and extend it to C such that $BC = 1$ unit.

Find the midpoint O of AC.

With O as centre and OA as radius, draw a semicircle.



Now, draw $BD \perp AC$, intersecting the semicircle at D.

Then, $BD = \sqrt{4.3}$ units.

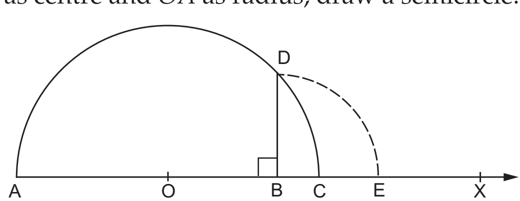
With B as centre and BD as radius, draw an arc, meeting AC produced at E. Then, $BE = BD = \sqrt{4.3}$ units.

EXAMPLE 4 *Represent $\sqrt{8.47}$ on the real line.*

SOLUTION Draw a line segment $AB = 8.47$ units and extend it to C such that $BC = 1$ unit.

Find the midpoint O of AC.

With O as centre and OA as radius, draw a semicircle.



Now, draw $BD \perp AC$, intersecting the semicircle at D.

Then, $BD = \sqrt{8.47}$ units.

With B as centre and BD as radius, draw an arc, meeting AC produced at E. Then, $BE = BD = \sqrt{8.47}$ units.

REPRESENTING REAL NUMBERS ON THE NUMBER LINE (SUCCESSIVE MAGNIFICATION)

EXAMPLE 5 *Locate* 3.895 *on the number line.*

SOLUTION We proceed stepwise as follows.

STEP 1 3.895 lies between 3 and 4.

We divide this into 10 equal divisions and mark them as 3.1, 3.2, 3.3, …, and so on as shown in Fig. (ii) given below.

STEP 2 3.895 lies between 3.8 and 3.9.

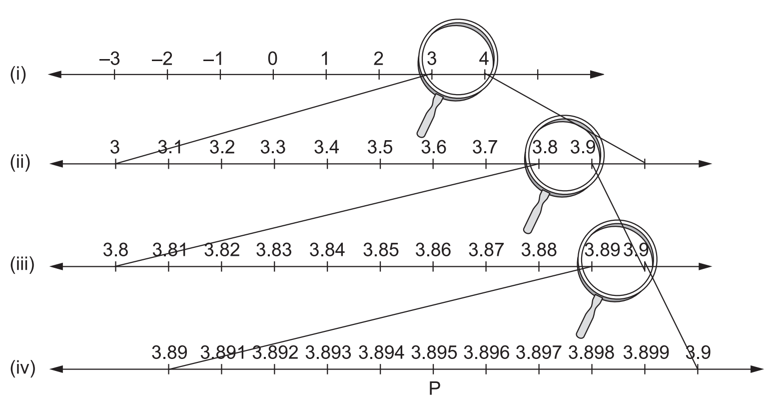
If we look at this part of the number line closely, as through a lens, it would appear as shown in Fig. (iii).

We divide this into 10 equal divisions and label them as 3.81, 3.82, 3.83, …, and so on.

STEP 3 3.895 lies between 3.89 and 3.9.

If we look at this part of the number line closely, it would appear as shown in Fig. (iv).

We divide this into 10 equal divisions and label them as 3.891, 3.892, 3.893, …, and so on. Mark 3.895 as P.



Clearly, point P on the number line represents 3.895.

EXAMPLE 6 *Visualize the representation of* $2.3\overline{2}$ *on the number line up to 4 decimal places.*

SOLUTION $2.3\overline{2} = 2.3222$ (up to 4 decimal places).

We proceed stepwise as follows:

STEP 1 2.3222 lies between 2 and 3.

We divide this portion of the number line into 10 equal divisions and mark them as 2.1, 2.2, 2.3, …, and so on, as shown in Fig. (ii).

STEP 2 2.3222 lies between 2.3 and 2.4.

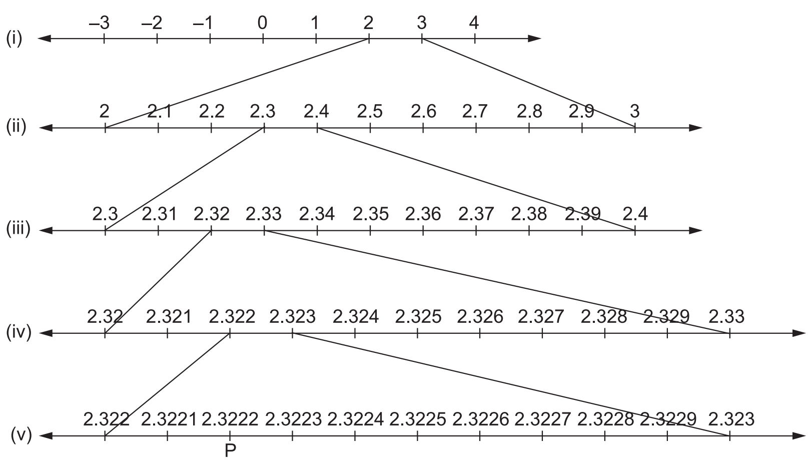
If we look at this part of the number line closely, as through a lens, it would appear as shown in Fig. (iii).

We divide this into 10 equal division and label them as 2.31, 2.32, 2.33, …, and so on.

STEP 3 2.3222 lies between 2.32 and 2.33.

If we look at this part of the number line closely, it would appear as shown in Fig. (iv).

We divide this into 10 equal divisions and label them as 2.321, 2.322, 2.323, 2.324, …, and so on.



STEP 4 2.3222 lies between 2.322 and 2.323.

If we look at this part of the number line closely, it would appear as shown in Fig. (v).

We divide this into 10 equal divisions and label them as 2.3221, 2.3222, 2.3223, …, and so on. Mark the division representing 2.3222 as P.

Clearly, point P on the number line represents $2.3\overline{2}$ up to 4 decimal places.

EXERCISE 1E

1. Represent $\sqrt{5}$ on the number line. [2010, '11, '14]

2. Locate $\sqrt{3}$ on the number line. [2012, '15]

3. Locate $\sqrt{10}$ on the number line. [2010]

4. Locate $\sqrt{8}$ on the number line. [2011]

5. Represent $\sqrt{4.7}$ geometrically on the number line.

6. Represent $\sqrt{10.5}$ on the number line. [2014]

7. Represent $\sqrt{7.28}$ geometrically on the number line.

8. Represent $(1+\sqrt{9.5})$ on the number line. [2015]

9. Visualize the representation of 3.765 on the number line using successive magnification. [2015]

10. Visualize the representation of $4.\overline{67}$ on the number line up to 4 decimal places. [2011]

RATIONALISATION

RATIONALISATION *Suppose we are given a number whose denominator is irrational. Then, the process of converting it into an equivalent expression whose denominator is a rational number by multiplying its numerator and denominator by a suitable number, is called rationalisation.*

EXAMPLE 1 *Simplify* $\frac{3}{\sqrt{5}}$ *by rationalising the denominator.*

SOLUTION On multiplying the numerator and denominator of the given number by $\sqrt{5}$, we get

$$\frac{3}{\sqrt{5}}=\frac{3}{\sqrt{5}}\times\frac{\sqrt{5}}{\sqrt{5}}=\frac{3\sqrt{5}}{5}.$$

RATIONALISING FACTOR (RF)

If the product of two irrational numbers is rational then each one is called the rationalising factor of the other.

If a and b are integers and x, y are natural numbers then

(i) $(a+\sqrt{b})$ and $(a-\sqrt{b})$ are RF of each other,

as $(a+\sqrt{b})(a-\sqrt{b})=(a^2-b)$, which is rational.

(ii) $(a+b\sqrt{x})$ and $(a-b\sqrt{x})$ are RF of each other,

as $(a+b\sqrt{x})(a-b\sqrt{x})=(a^2-b^2x)$, which is rational.

(iii) $(\sqrt{x}+\sqrt{y})$ and $(\sqrt{x}-\sqrt{y})$ are RF of each other,

as $(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})=(x-y)$, which is rational.

For rationalising the denominator of a number, we multiply its numerator and denominator by its rationalising factor.

Thus, the rationalising factor of $\frac{1}{3+\sqrt{2}}$ is $(3-\sqrt{2})$ since

$$\frac{1}{3+\sqrt{2}}\times\frac{3-\sqrt{2}}{3-\sqrt{2}}=\frac{(3-\sqrt{2})}{(3+\sqrt{2})(3-\sqrt{2})}=\frac{(3-\sqrt{2})}{(3)^2-(\sqrt{2})^2}$$

$$=\frac{3-\sqrt{2}}{9-2}=\frac{3-\sqrt{2}}{7}\text{, which is a rational number.}$$

Similarly, the rationalising factor $\frac{2}{7+4\sqrt{3}}$ is $(7-4\sqrt{3})$, the rationalising factor of $\frac{1}{\sqrt{7}-\sqrt{6}}$ is $(\sqrt{7}+\sqrt{6})$, and so on.

SOLVED EXAMPLES

EXAMPLE 1 *Rationalise the denominator of* $\frac{1}{(3+\sqrt{2})}$.

SOLUTION We have

$$\frac{1}{(3+\sqrt{2})}=\frac{1}{(3+\sqrt{2})}\times\frac{(3-\sqrt{2})}{(3-\sqrt{2})}=\frac{(3-\sqrt{2})}{(3)^2-(\sqrt{2})^2}$$

$$=\frac{(3-\sqrt{2})}{9-2}=\frac{(3-\sqrt{2})}{7}.$$

EXAMPLE 2 *Rationalise the denominator of* $\frac{1}{(8+5\sqrt{2})}$.

SOLUTION We have

$$\frac{1}{(8+5\sqrt{2})}=\frac{1}{(8+5\sqrt{2})}\times\frac{(8-5\sqrt{2})}{(8-5\sqrt{2})}=\frac{(8-5\sqrt{2})}{(8)^2-(5\sqrt{2})^2}$$

$$=\frac{(8-5\sqrt{2})}{64-50}=\frac{(8-5\sqrt{2})}{14}.$$

EXAMPLE 3 *Rationalise the denominator of* $\frac{6}{(\sqrt{5}+\sqrt{2})}$.

SOLUTION We have

$$\frac{6}{(\sqrt{5}+\sqrt{2})}=\frac{6}{(\sqrt{5}+\sqrt{2})}\times\frac{(\sqrt{5}-\sqrt{2})}{(\sqrt{5}-\sqrt{2})}=\frac{6(\sqrt{5}-\sqrt{2})}{(\sqrt{5})^2-(\sqrt{2})^2}$$

$$= \frac{6(\sqrt{5}-\sqrt{2})}{5-2} = \frac{6(\sqrt{5}-\sqrt{2})}{3} = 2(\sqrt{5}-\sqrt{2}).$$

EXAMPLE 4 *Rationalise the denominator of* $\frac{2}{(\sqrt{3}-\sqrt{5})}$. [2011]

SOLUTION We have

$$\frac{2}{(\sqrt{3}-\sqrt{5})} = \frac{2}{(\sqrt{3}-\sqrt{5})} \times \frac{(\sqrt{3}+\sqrt{5})}{(\sqrt{3}+\sqrt{5})} = \frac{2(\sqrt{3}+\sqrt{5})}{(\sqrt{3})^2-(\sqrt{5})^2}$$

$$= \frac{2(\sqrt{3}+\sqrt{5})}{3-5} = \frac{2(\sqrt{3}+\sqrt{5})}{-2} = -(\sqrt{3}+\sqrt{5}).$$

EXAMPLE 5 *Simplify by rationalising the denominator:* $\frac{6-4\sqrt{3}}{6+4\sqrt{3}}$. [2010, '14]

SOLUTION We have

$$\frac{6-4\sqrt{3}}{6+4\sqrt{3}} = \frac{(6-4\sqrt{3})}{(6+4\sqrt{3})} \times \frac{(6-4\sqrt{3})}{(6-4\sqrt{3})} = \frac{(6-4\sqrt{3})^2}{(6)^2-(4\sqrt{3})^2}$$

$$= \frac{6^2+(4\sqrt{3})^2-2\times 6\times 4\sqrt{3}}{36-48} = \frac{36+48-48\sqrt{3}}{-12}$$

$$= \frac{84-48\sqrt{3}}{-12} = \frac{48\sqrt{3}-84}{12} = (4\sqrt{3}-7).$$

EXAMPLE 6 *Find the values of a and b if* $\frac{3+\sqrt{2}}{3-\sqrt{2}} = a+b\sqrt{2}$. [2014, '15]

SOLUTION We have

$$\frac{3+\sqrt{2}}{3-\sqrt{2}} = \frac{(3+\sqrt{2})}{(3-\sqrt{2})} \times \frac{(3+\sqrt{2})}{(3+\sqrt{2})} = \frac{(3+\sqrt{2})^2}{(3)^2-(\sqrt{2})^2}$$

$$= \frac{3^2+(\sqrt{2})^2+2\times 3\times\sqrt{2}}{9-2} = \frac{9+2+6\sqrt{2}}{7} = \frac{11+6\sqrt{2}}{7}.$$

$$\therefore \quad \frac{3+\sqrt{2}}{3-\sqrt{2}} = a+b\sqrt{2} \Leftrightarrow \frac{11}{7}+\frac{6}{7}\sqrt{2} = a+b\sqrt{2}$$

$$\Leftrightarrow a = \frac{11}{7} \text{ and } b = \frac{6}{7}.$$

EXAMPLE 7 *If a and b are rational numbers and* $\frac{\sqrt{11}-\sqrt{7}}{\sqrt{11}+\sqrt{7}} = a-b\sqrt{77}$, *find the values of a and b.*

SOLUTION We have

$$\frac{\sqrt{11}-\sqrt{7}}{\sqrt{11}+\sqrt{7}} = \frac{(\sqrt{11}-\sqrt{7})}{(\sqrt{11}+\sqrt{7})} \times \frac{(\sqrt{11}-\sqrt{7})}{(\sqrt{11}-\sqrt{7})}$$

$$= \frac{(\sqrt{11}-\sqrt{7})^2}{(\sqrt{11})^2-(\sqrt{7})^2} = \frac{(11+7-2\times\sqrt{11}\times\sqrt{7})}{11-7}$$

$$= \frac{(18-2\sqrt{77})}{4} = \left(\frac{18}{4}-\frac{2}{4}\cdot\sqrt{77}\right) = \left(\frac{9}{2}-\frac{1}{2}\cdot\sqrt{77}\right).$$

$$\therefore \quad \frac{\sqrt{11}-\sqrt{7}}{\sqrt{11}+\sqrt{7}} = a-b\sqrt{77} \Leftrightarrow \frac{9}{2}-\frac{1}{2}\cdot\sqrt{77} = a-b\sqrt{77}$$

$$\Leftrightarrow a = \frac{9}{2} \text{ and } b = \frac{1}{2}.$$

Hence, $a = \frac{9}{2}$ and $b = \frac{1}{2}$.

EXAMPLE 8 *If a and b are rational numbers and* $\frac{4+3\sqrt{5}}{4-3\sqrt{5}} = a+b\sqrt{5}$, *find the values of a and b.*

SOLUTION We have

$$\frac{4+3\sqrt{5}}{4-3\sqrt{5}} = \frac{(4+3\sqrt{5})}{(4-3\sqrt{5})} \times \frac{(4+3\sqrt{5})}{4+3\sqrt{5}}$$

$$= \frac{(4+3\sqrt{5})^2}{(4)^2-(3\sqrt{5})^2} = \frac{(4+3\sqrt{5})^2}{16-45} = \frac{(4+3\sqrt{5})^2}{-29}$$

$$= \frac{(4)^2+(3\sqrt{5})^2+2\times4\times3\sqrt{5}\}}{-29}$$

$$= \frac{(16+45+24\sqrt{5})}{-29}$$

$$= \frac{(61+24\sqrt{5})}{-29} = \left(-\frac{61}{29}\right)+\left(-\frac{24}{29}\right)\sqrt{5}.$$

$$\therefore \quad \frac{4+3\sqrt{5}}{4-3\sqrt{5}} = a+b\sqrt{5} \Leftrightarrow \left(-\frac{61}{29}\right)+\left(-\frac{24}{29}\right)\sqrt{5} = a+b\sqrt{5}.$$

$$\therefore \quad a = -\frac{61}{29} \text{ and } b = -\frac{24}{29}.$$

EXAMPLE 9 *Evaluate after rationalising the denominator of* $\frac{25}{\sqrt{40}-\sqrt{80}}$, *it being given that* $\sqrt{5} = 2.236$ *and* $\sqrt{10} = 3.162$. [2011]

SOLUTION We have

$$\frac{25}{\sqrt{40}-\sqrt{80}} = \frac{25}{2\sqrt{10}-4\sqrt{5}} = \frac{25}{2(\sqrt{10}-2\sqrt{5})} \times \frac{(\sqrt{10}+2\sqrt{5})}{(\sqrt{10}+2\sqrt{5})}$$

$$= \frac{25(\sqrt{10}+2\sqrt{5})}{2(\sqrt{10}-2\sqrt{5})(\sqrt{10}+2\sqrt{5})} = \frac{25(\sqrt{10}+2\sqrt{5})}{2[(10)^2-(2\sqrt{5})^2]}$$

$$= \frac{25(\sqrt{10}+2\sqrt{5})}{2(100-20)} = \frac{25(\sqrt{10}+2\sqrt{5})}{160}$$

$$= \frac{5(\sqrt{10}+2\sqrt{5})}{32} = \frac{5(3.162+2\times 2.236)}{32}$$

$$= \frac{5\times 7.634}{32} = \frac{38.17}{32} = 1.1928 \approx 1.193.$$

EXAMPLE 10 *Find the values of a and b if*

$$\frac{7+3\sqrt{5}}{3+\sqrt{5}} - \frac{7-3\sqrt{5}}{3-\sqrt{5}} = a + \sqrt{5}\,b.$$ (2010, '11, '15)

SOLUTION We have

$$\frac{7+3\sqrt{5}}{3+\sqrt{5}} - \frac{7-3\sqrt{5}}{3-\sqrt{5}} = \frac{(7+3\sqrt{5})}{(3+\sqrt{5})}\times\frac{(3-\sqrt{5})}{(3-\sqrt{5})} - \frac{(7-3\sqrt{5})}{(3-\sqrt{5})}\times\frac{(3+\sqrt{5})}{(3+\sqrt{5})}$$

$$= \frac{(21-7\sqrt{5}+9\sqrt{5}-15)}{(3+\sqrt{5})(3-\sqrt{5})} - \frac{(21+7\sqrt{5}-9\sqrt{5}-15)}{(3-\sqrt{5})(3+\sqrt{5})}$$

$$= \frac{(6+2\sqrt{5})}{(3)^2-(\sqrt{5})^2} - \frac{(6-2\sqrt{5})}{(3)^2-(\sqrt{5})^2}$$

$$= \frac{(6+2\sqrt{5})}{(9-5)} - \frac{(6-2\sqrt{5})}{(9-5)} = \frac{6+2\sqrt{5}}{4} - \frac{6-2\sqrt{5}}{4}$$

$$= \frac{6+2\sqrt{5}-6+2\sqrt{5}}{4} = \frac{4\sqrt{5}}{4} = \sqrt{5}.$$

$$\therefore \quad \frac{7+3\sqrt{5}}{3+\sqrt{5}} - \frac{7-3\sqrt{5}}{3-\sqrt{5}} = a+\sqrt{5}\,b \;\Rightarrow\; \sqrt{5} = a+\sqrt{5}\,b$$

$$\Rightarrow\; a = 0 \text{ and } b = 1.$$

EXAMPLE 11 *Show that*

$$\frac{1}{3-\sqrt{8}} - \frac{1}{\sqrt{8}-\sqrt{7}} + \frac{1}{\sqrt{7}-\sqrt{6}} - \frac{1}{\sqrt{6}-\sqrt{5}} + \frac{1}{\sqrt{5}-2} = 5.$$ [2011, '15]

SOLUTION We have

$$\frac{1}{3-\sqrt{8}} - \frac{1}{\sqrt{8}-\sqrt{7}} + \frac{1}{\sqrt{7}-\sqrt{6}} - \frac{1}{\sqrt{6}-\sqrt{5}} + \frac{1}{\sqrt{5}-2}$$

$$= \frac{1}{(3-\sqrt{8})}\times\frac{(3+\sqrt{8})}{(3+\sqrt{8})} - \frac{1}{(\sqrt{8}-\sqrt{7})}\times\frac{(\sqrt{8}+\sqrt{7})}{(\sqrt{8}+\sqrt{7})}$$

$$+\frac{1}{(\sqrt{7}-\sqrt{6})}\times\frac{(\sqrt{7}+\sqrt{6})}{(\sqrt{7}+\sqrt{6})} - \frac{1}{(\sqrt{6}-\sqrt{5})}\times\frac{(\sqrt{6}+\sqrt{5})}{(\sqrt{6}+\sqrt{5})}$$

$$+\frac{1}{(\sqrt{5}-2)}\times\frac{(\sqrt{5}+2)}{(\sqrt{5}+2)}$$

$$= \frac{(3+\sqrt{8})}{(3)^2-(\sqrt{8})^2} - \frac{(\sqrt{8}+\sqrt{7})}{(\sqrt{8})^2-(\sqrt{7})^2} + \frac{(\sqrt{7}+\sqrt{6})}{(\sqrt{7})^2-(\sqrt{6})^2} - \frac{(\sqrt{6}+\sqrt{5})}{(\sqrt{6})^2-(\sqrt{5})^2} + \frac{(\sqrt{5}+2)}{(\sqrt{5})^2-(2)^2}$$

$$= \frac{(3+\sqrt{8})}{(9-8)} - \frac{(\sqrt{8}+\sqrt{7})}{(8-7)} + \frac{(\sqrt{7}+\sqrt{6})}{(7-6)} - \frac{(\sqrt{6}+\sqrt{5})}{(6-5)} + \frac{(\sqrt{5}+2)}{(5-4)}$$

$$= (3+\sqrt{8}) - (\sqrt{8}+\sqrt{7}) + (\sqrt{7}+\sqrt{6}) - (\sqrt{6}+\sqrt{5}) + (\sqrt{5}+2)$$

$$= 3+\sqrt{8}-\sqrt{8}-\sqrt{7}+\sqrt{7}+\sqrt{6}-\sqrt{6}-\sqrt{5}+\sqrt{5}+2$$

$$= 3+2 = 5.$$

EXAMPLE 12 *If* $\sqrt{2} = 1.414$, $\sqrt{3} = 1.732$, $\sqrt{5} = 2.236$ *and* $\sqrt{6} = 2.449$, *find the value of* $\frac{2+\sqrt{3}}{2-\sqrt{3}} + \frac{2-\sqrt{3}}{2+\sqrt{3}} + \frac{\sqrt{3}-1}{\sqrt{3}+1}$. [CBSE Sample Paper]

SOLUTION We have

$$\frac{2+\sqrt{3}}{2-\sqrt{3}} + \frac{2-\sqrt{3}}{2+\sqrt{3}} + \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

$$= \frac{(2+\sqrt{3})}{(2-\sqrt{3})} \times \frac{(2+\sqrt{3})}{(2+\sqrt{3})} + \frac{(2-\sqrt{3})}{(2+\sqrt{3})} \times \frac{(2-\sqrt{3})}{(2-\sqrt{3})} + \frac{(\sqrt{3}-1)}{(\sqrt{3}+1)} \times \frac{(\sqrt{3}-1)}{(\sqrt{3}-1)}$$

$$= \frac{(2+\sqrt{3})^2}{(2)^2-(\sqrt{3})^2} + \frac{(2-\sqrt{3})^2}{(2)^2-(\sqrt{3})^2} + \frac{(\sqrt{3}-1)^2}{(\sqrt{3})^2-(1)^2}$$

$$= \frac{2^2+(\sqrt{3})^2+2\times 2\times\sqrt{3}}{(4-3)} + \frac{2^2+(\sqrt{3})^2-2\times 2\times\sqrt{3}}{(4-3)} + \frac{(\sqrt{3})^2+(1)^2-2\times\sqrt{3}\times 1}{(3-1)}$$

$$= (4+3+4\sqrt{3}) + (4+3-4\sqrt{3}) + \frac{3+1-2\sqrt{3}}{2}$$

$$= 7+4\sqrt{3}+7-4\sqrt{3}+\frac{2(2-\sqrt{3})}{2} = 14+2-\sqrt{3} = 16-\sqrt{3}$$

$$= 16-1.732 = 14.268.$$

EXAMPLE 13 *If* $x = 3+2\sqrt{2}$, *find the value of* $\left(x^2+\frac{1}{x^2}\right)$. [2015]

SOLUTION Given, $x = 3+2\sqrt{2}$.

$$\therefore \quad \frac{1}{x} = \frac{1}{(3+2\sqrt{2})} = \frac{1}{(3+2\sqrt{2})} \times \frac{(3-2\sqrt{2})}{(3-2\sqrt{2})}$$

$$= \frac{(3-2\sqrt{2})}{(3)^2-(2\sqrt{2})^2} = \frac{(3-2\sqrt{2})}{(9-8)} = 3-2\sqrt{2}.$$

$$\therefore \quad x+\frac{1}{x}=(3+2\sqrt{2})+(3-2\sqrt{2})=6$$

$$\Rightarrow \quad \left(x+\frac{1}{x}\right)^2=6^2=36$$

$$\Rightarrow \quad x^2+\frac{1}{x^2}+2\times x\times\frac{1}{x}=36$$

$$\Rightarrow \quad \left(x^2+\frac{1}{x^2}\right)+2=36 \Rightarrow \left(x^2+\frac{1}{x^2}\right)=36-2=34.$$

Hence, $\left(x^2+\frac{1}{x^2}\right)=34.$

EXAMPLE 14 *If* $x=\frac{5-\sqrt{21}}{2}$, *prove that*

$$\left(x^3+\frac{1}{x^3}\right)-5\left(x^2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)=0.$$ [2011]

SOLUTION Given, $x=\frac{5-\sqrt{21}}{2}$.

$$\therefore \quad \frac{1}{x}=\frac{2}{5-\sqrt{21}}=\frac{2}{(5-\sqrt{21})}\times\frac{(5+\sqrt{21})}{(5+\sqrt{21})}=\frac{2(5+\sqrt{21})}{(5)^2-(\sqrt{21})^2}$$

$$=\frac{2(5+\sqrt{21})}{(25-21)}=\frac{2(5+\sqrt{21})}{4}=\frac{5+\sqrt{21}}{2}.$$

$$\therefore \quad x+\frac{1}{x}=\left(\frac{5-\sqrt{21}}{2}\right)+\left(\frac{5+\sqrt{21}}{2}\right)=\frac{5-\sqrt{21}+5+\sqrt{21}}{2}=\frac{10}{2}=5.$$

$$\therefore \quad \left(x+\frac{1}{x}\right)^2=5^2=25$$

$$\Rightarrow \quad x^2+\frac{1}{x^2}+2\times x\times\frac{1}{x}=25$$

$$\Rightarrow \quad \left(x^2+\frac{1}{x^2}\right)+2=25 \Rightarrow \left(x^2+\frac{1}{x^2}\right)=25-2=23.$$

And, $x+\frac{1}{x}=5 \Rightarrow \left(x+\frac{1}{x}\right)^3=(5)^3=125$

$$\Rightarrow x^3+\frac{1}{x^3}+3\times x\times\frac{1}{x}\left(x+\frac{1}{x}\right)=125$$

$$\Rightarrow \left(x^3+\frac{1}{x^3}\right)+3\times 5=125$$

$$\Rightarrow \left(x^3+\frac{1}{x^3}\right)=125-15=110.$$

$$\therefore \quad \left(x^3+\frac{1}{x^3}\right)-5\left(x^2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)$$

$$= 110-5\times 23+5 = 110-115+5 = 115-115 = 0.$$

EXAMPLE 15 *If* $x=\dfrac{\sqrt{2}+1}{\sqrt{2}-1}$ *and* $y=\dfrac{\sqrt{2}-1}{\sqrt{2}+1}$ *find the value of* x^2+y^2+xy. [2015]

SOLUTION We have

$$x=\frac{\sqrt{2}+1}{\sqrt{2}-1}=\frac{(\sqrt{2}+1)}{(\sqrt{2}-1)}\times\frac{(\sqrt{2}+1)}{(\sqrt{2}+1)}=\frac{(\sqrt{2})^2+(1)^2+2\times\sqrt{2}\times 1}{(\sqrt{2})^2-(1)^2}$$

$$=\frac{2+1+2\sqrt{2}}{(2-1)}=3+2\sqrt{2}$$

$$\text{and, } y=\frac{\sqrt{2}-1}{\sqrt{2}+1}=\frac{(\sqrt{2}-1)}{(\sqrt{2}+1)}\times\frac{(\sqrt{2}-1)}{(\sqrt{2}-1)}=\frac{(\sqrt{2})^2+(1)^2-2\times\sqrt{2}\times 1}{(\sqrt{2})^2-(1)^2}$$

$$=\frac{2+1-2\sqrt{2}}{(2-1)}=3-2\sqrt{2}.$$

$$\therefore \quad x^2+y^2+xy=(x+y)^2-xy$$

$$=[(3+\sqrt{2})+(3-\sqrt{2})]^2-(3+2\sqrt{2})(3-2\sqrt{2})$$

$$=(6)^2-[(3)^2-(2\sqrt{2})^2]=36-(9-8)$$

$$=36-1=35.$$

EXAMPLE 16 *Express* $\dfrac{3}{\sqrt{3}-\sqrt{2}+\sqrt{5}}$ *with rational denominator.* [2010]

SOLUTION We have

$$\frac{3}{(\sqrt{3}-\sqrt{2})+\sqrt{5}}=\frac{3}{(\sqrt{3}-\sqrt{2})+\sqrt{5}}\times\frac{(\sqrt{3}-\sqrt{2})-\sqrt{5}}{(\sqrt{3}-\sqrt{2})-\sqrt{5}}$$

$$=\frac{3(\sqrt{3}-\sqrt{2}-\sqrt{5})}{(\sqrt{3}-\sqrt{2})^2-(\sqrt{5})^2}=\frac{3(\sqrt{3}-\sqrt{2}-\sqrt{5})}{[(\sqrt{3})^2+(\sqrt{2})^2-2\times\sqrt{3}\times\sqrt{2}]-5}$$

$$=\frac{3(\sqrt{3}-\sqrt{2}-\sqrt{5})}{(3+2-2\sqrt{6})-5}=\frac{3(\sqrt{3}-\sqrt{2}-\sqrt{5})}{-2\sqrt{6}}\times\frac{\sqrt{6}}{\sqrt{6}}$$

$$=\frac{3(\sqrt{18}-\sqrt{12}-\sqrt{30})}{-2\times 6}=\frac{3(3\sqrt{2}-2\sqrt{3}-\sqrt{30})}{-12}$$

$$=\frac{3\sqrt{2}-2\sqrt{3}-\sqrt{30}}{-4}=\frac{2\sqrt{3}+\sqrt{30}-3\sqrt{2}}{4}.$$

EXAMPLE 17 *If* $x=\dfrac{1}{3-\sqrt{8}}$, *find the value of* x^3-2x^2-7x+5.

SOLUTION We have

$$x=\frac{1}{3-\sqrt{8}}=\frac{1}{(3-\sqrt{8})}\times\frac{(3+\sqrt{8})}{(3+\sqrt{8})}=\frac{(3+\sqrt{8})}{(3)^2-(\sqrt{8})^2}$$

$$=\frac{(3+\sqrt{8})}{9-8}=(3+\sqrt{8}).$$

$$x=3+\sqrt{8}\ \Rightarrow\ x-3=\sqrt{8}$$

$$\Rightarrow\ (x-3)^2=(\sqrt{8})^2=8$$

$$\Rightarrow\ x^2+9-6x=8\ \Rightarrow\ x^2-6x+1=0$$

$$\therefore\quad x^3-2x^2-7x+5=x(x^2-6x+1)+4(x^2-6x+1)+16x+1$$

$$=x\times 0+4\times 0+16(3+\sqrt{8})+1$$

$$=48+16\sqrt{8}+1=49+32\sqrt{2}.$$

EXERCISE 1F

1. Write the rationalising factor of the denominator in $\frac{1}{\sqrt{2}+\sqrt{3}}$. [2014]

2. Rationalise the denominator of each of the following.

(i) $\frac{1}{\sqrt{7}}$ (ii) $\frac{\sqrt{5}}{2\sqrt{3}}$ (iii) $\frac{1}{2+\sqrt{3}}$

(iv) $\frac{1}{\sqrt{5}-2}$ (v) $\frac{1}{5+3\sqrt{2}}$ (vi) $\frac{1}{\sqrt{7}-\sqrt{6}}$ [2010]

(vii) $\frac{4}{\sqrt{11}-\sqrt{7}}$ [2010] (viii) $\frac{1+\sqrt{2}}{2-\sqrt{2}}$ [2014] (ix) $\frac{3-2\sqrt{2}}{3+2\sqrt{2}}$

3. It being given that $\sqrt{2}=1.414$, $\sqrt{3}=1.732$, $\sqrt{5}=2.236$ and $\sqrt{10}=3.162$, find the value to three places of decimals, of each of the following.

(i) $\frac{2}{\sqrt{5}}$ (ii) $\frac{2-\sqrt{3}}{\sqrt{3}}$ (iii) $\frac{\sqrt{10}-\sqrt{5}}{\sqrt{2}}$ [2010]

4. Find rational numbers a and b such that

(i) $\frac{\sqrt{2}-1}{\sqrt{2}+1}=a+b\sqrt{2}$ [2012] (ii) $\frac{2-\sqrt{5}}{2+\sqrt{5}}=a\sqrt{5}+b$ [2014]

(iii) $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}=a+b\sqrt{6}$ [2010, '11] (iv) $\frac{5+2\sqrt{3}}{7+4\sqrt{3}}=a+b\sqrt{3}$ [2014]

5. It being given that $\sqrt{3}=1.732$, $\sqrt{5}=2.236$, $\sqrt{6}=2.449$ and $\sqrt{10}=3.162$, find to three places of decimal, the value of each of the following.

(i) $\frac{1}{\sqrt{6}+\sqrt{5}}$ [2014] (ii) $\frac{6}{\sqrt{5}+\sqrt{3}}$ [2011] (iii) $\frac{1}{4\sqrt{3}-3\sqrt{5}}$ [CBSE Sample Paper]

(iv) $\dfrac{3+\sqrt{5}}{3-\sqrt{5}}$ [2010] (v) $\dfrac{1+2\sqrt{3}}{2-\sqrt{3}}$ (vi) $\dfrac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}}$ [2012]

6. Simplify by rationalising the denominator.

(i) $\dfrac{7\sqrt{3}-5\sqrt{2}}{\sqrt{48}+\sqrt{18}}$ (ii) $\dfrac{2\sqrt{6}-\sqrt{5}}{3\sqrt{5}-2\sqrt{6}}$ [2014]

7. Simplify

(i) $\dfrac{4+\sqrt{5}}{4-\sqrt{5}}+\dfrac{4-\sqrt{5}}{4+\sqrt{5}}$ [2011]

(ii) $\dfrac{1}{\sqrt{3}+\sqrt{2}}-\dfrac{2}{\sqrt{5}-\sqrt{3}}-\dfrac{3}{\sqrt{2}-\sqrt{5}}$ [2014]

(iii) $\dfrac{2+\sqrt{3}}{2-\sqrt{3}}+\dfrac{2-\sqrt{3}}{2+\sqrt{3}}+\dfrac{\sqrt{3}-1}{\sqrt{3}+1}$

(iv) $\dfrac{2\sqrt{6}}{\sqrt{2}+\sqrt{3}}+\dfrac{6\sqrt{2}}{\sqrt{6}+\sqrt{3}}-\dfrac{8\sqrt{3}}{\sqrt{6}+\sqrt{2}}$ [2010, '11]

8. Prove that

(i) $\dfrac{1}{3+\sqrt{7}}+\dfrac{1}{\sqrt{7}+\sqrt{5}}+\dfrac{1}{\sqrt{5}+\sqrt{3}}+\dfrac{1}{\sqrt{3}+1}=1$ [2011]

(ii) $\dfrac{1}{1+\sqrt{2}}+\dfrac{1}{\sqrt{2}+\sqrt{3}}+\dfrac{1}{\sqrt{3}+\sqrt{4}}+\dfrac{1}{\sqrt{4}+\sqrt{5}}+\dfrac{1}{\sqrt{5}+\sqrt{6}}+\dfrac{1}{\sqrt{6}+\sqrt{7}}$

$+\dfrac{1}{\sqrt{7}+\sqrt{8}}+\dfrac{1}{\sqrt{8}+\sqrt{9}}=2$

9. Find the values of a and b if

$$\frac{7+3\sqrt{5}}{3+\sqrt{5}}-\frac{7-3\sqrt{5}}{3-\sqrt{5}}=a+b\sqrt{5}.$$ [2014]

10. Simplify $\dfrac{\sqrt{13}-\sqrt{11}}{\sqrt{13}+\sqrt{11}}+\dfrac{\sqrt{13}+\sqrt{11}}{\sqrt{13}-\sqrt{11}}\cdot$ [2015]

11. If $x=3+2\sqrt{2}$, check whether $x+\dfrac{1}{x}$ is rational or irrational. [2010]

12. If $x=2-\sqrt{3}$, find the value of $\left(x-\dfrac{1}{x}\right)^3\cdot$

13. If $x=9-4\sqrt{5}$, find the value of $x^2+\dfrac{1}{x^2}\cdot$ [2011]

14. If $x=\dfrac{5-\sqrt{21}}{2}$, find the value of $x+\dfrac{1}{x}\cdot$ [2014]

15. If $a=3-2\sqrt{2}$, find the value of $a^2-\dfrac{1}{a^2}\cdot$ [2010]

16. If $x = \sqrt{13} + 2\sqrt{3}$, find the value of $x - \frac{1}{x}\cdot$

17. If $x = 2 + \sqrt{3}$, find the value of $x^3 + \frac{1}{x^3}\cdot$ [2015]

18. If $x = \frac{5-\sqrt{3}}{5+\sqrt{3}}$ and $y = \frac{5+\sqrt{3}}{5-\sqrt{3}}$, show that $x - y = -\frac{10\sqrt{3}}{11}\cdot$ [2015]

19. If $a = \frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}}$ and $b = \frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}$, show that $3a^2 + 4ab - 3b^2 = 4 + \frac{56}{3}\sqrt{10}$.

20. If $a = \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$ and $b = \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$, find the value of $a^2 + b^2 - 5ab$. [2011]

21. If $p = \frac{3-\sqrt{5}}{3+\sqrt{5}}$ and $q = \frac{3+\sqrt{5}}{3-\sqrt{5}}$, find the value of $p^2 + q^2$. [2010]

22. Rationalise the denominator of each of the following.

(i) $\frac{1}{\sqrt{7}+\sqrt{6}-\sqrt{13}}$ [2011] (ii) $\frac{3}{\sqrt{3}+\sqrt{5}-\sqrt{2}}$ [2015]

(iii) $\frac{4}{2+\sqrt{3}+\sqrt{7}}$ [2015]

23. Given, $\sqrt{2} = 1.414$ and $\sqrt{6} = 2.449$, find the value of $\frac{1}{\sqrt{3}-\sqrt{2}-1}$ correct to 3 places of decimal. [2014]

24. If $x = \frac{1}{2-\sqrt{3}}$, find the value of $x^3 - 2x^2 - 7x + 5$.

25. Evaluate $\frac{15}{\sqrt{10}+\sqrt{20}+\sqrt{40}-\sqrt{5}-\sqrt{80}}$, it being given that $\sqrt{5} = 2.236$ and $\sqrt{10} = 3.162$.

HINT $$\frac{15}{\sqrt{10}+\sqrt{20}+\sqrt{40}-\sqrt{5}-\sqrt{80}} = \frac{15}{\sqrt{10}+2\sqrt{5}+2\sqrt{10}-\sqrt{5}-4\sqrt{5}}$$
$$= \frac{15}{3\sqrt{10}-3\sqrt{5}} = \frac{5}{\sqrt{10}-\sqrt{5}}\cdot$$

ANSWERS (EXERCISE 1F)

1. $\sqrt{2} - \sqrt{3}$

2. (i) $\frac{\sqrt{7}}{7}$ (ii) $\frac{\sqrt{15}}{6}$ (iii) $2 - \sqrt{3}$ (iv) $\sqrt{5} + 2$

(v) $\frac{5-3\sqrt{2}}{7}$ (vi) $\sqrt{7} + \sqrt{6}$ (vii) $\sqrt{11} + \sqrt{7}$ (viii) $\frac{4+3\sqrt{2}}{2}$

(ix) $17 - 12\sqrt{2}$

3. (i) 0.894 (ii) 0.155 (iii) 0.655

4. (i) $a = 3, b = -2$ (ii) $a = 4, b = -9$ (iii) $a = 5, b = 2$ (iv) $a = 11, b = -6$

5. (i) 0.213 (ii) 1.512 (iii) 4.545 (iv) 6.854 (v) 16.660 (vi) 4.441

6. (i) $\frac{114 - 41\sqrt{6}}{30}$ (ii) $\frac{4\sqrt{30} + 9}{21}$

7. (i) $\frac{42}{11}$ (ii) $2\sqrt{2}$ (iii) $16 - \sqrt{3}$ (iv) 0

9. $a = 1, b = 1$ **10.** 24 **11.** Rational **12.** $-24\sqrt{3}$ **13.** 322

14. 5 **15.** $-24\sqrt{2}$ **16.** $4\sqrt{3}$ **17.** 52 **20.** 93 **21.** 47

22. (i) $\frac{7\sqrt{6} + 6\sqrt{7} + \sqrt{546}}{84}$ (ii) $\frac{2\sqrt{3} - 3\sqrt{2} - \sqrt{30}}{4}$ (iii) $\frac{2\sqrt{3} - \sqrt{21} + 3}{3}$

23. -1.466 **24.** 3 **25.** 5.398

LAWS OF EXPONENTS

Let $a > 0, b > 0$ be real numbers and let m and n be rational numbers. Then, we have

(i) $a^m \times a^n = a^{m+n}$ (ii) $\frac{a^m}{a^n} = a^{m-n}$

(iii) $(a^m)^n = a^{mn}$ (iv) $a^m \times b^m = (ab)^m$

(v) $(ab)^m = a^m b^m$ (vi) $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.

(vii) $a^{-n} = \frac{1}{a^n}$ (viii) $a^0 = 1$

EXAMPLE 1 *Simplify*

(i) $3^{3/4} \times 3^{1/4}$ *(ii)* $\frac{5^{1/3}}{5^{1/6}}$ *(iii)* $3^{2/3} \times 7^{2/3}$

SOLUTION Using the laws of exponents, we have

(i) $3^{3/4} \times 3^{1/4} = 3^{(3/4+1/4)} = 3^1 = 3$ $[\because\ a^m \times a^n = a^{(m+n)}]$.

(ii) $\frac{5^{1/3}}{5^{1/6}} = 5^{(1/3-1/6)} = 5^{1/6}$ $\left[\because\ \frac{a^m}{a^n} = a^{(m-n)}\right]$.

(iii) $3^{2/3} \times 7^{2/3} = (3 \times 7)^{2/3} = (21)^{2/3}$ $[\because\ a^m \times b^m = (ab)^m]$.

EXAMPLE 2 *Simplify*

(i) $(36)^{1/2}$ *(ii)* $(64)^{1/3}$ *(iii)* $(32)^{1/5}$

SOLUTION We have

(i) $(36)^{1/2} = (6^2)^{1/2} = 6^{\left(2 \times \frac{1}{2}\right)} = 6^1 = 6.$

(ii) $(64)^{1/3} = (4^3)^{1/3} = 4^{\left(3 \times \frac{1}{3}\right)} = 4^1 = 4.$

(iii) $(32)^{1/5} = (2^5)^{1/5} = 2^{\left(5 \times \frac{1}{5}\right)} = 2^1 = 2.$

EXAMPLE 3 *Simplify*

(i) $(16)^{3/2}$ *(ii)* $(27)^{2/3}$ *(iii)* $(125)^{-1/3}$

SOLUTION We have

(i) $(16)^{3/2} = (4^2)^{3/2} = 4^{\left(2 \times \frac{3}{2}\right)} = 4^3 = 64.$

(ii) $(27)^{2/3} = (3^3)^{2/3} = 3^{\left(3 \times \frac{2}{3}\right)} = 3^2 = 9.$

(iii) $(125)^{-1/3} = \dfrac{1}{(125)^{1/3}} = \dfrac{1}{(5^3)^{1/3}} = \dfrac{1}{5^{\left(3 \times \frac{1}{3}\right)}} = \dfrac{1}{5^1} = \dfrac{1}{5}.$

IMPORTANT RESULTS

RESULT 1 Let $a > 0$ be a real number and n be a positive integer. Then, nth root of a is defined as $\sqrt[n]{a} = b$, if $b^n = a$ and $b > 0$.

In the language of exponents, we define $\sqrt[n]{a} = a^{1/n}$.

Thus, we have $\sqrt[3]{2} = 2^{\frac{1}{3}}$; $\sqrt[4]{5} = 5^{\frac{1}{4}}$, etc.

RESULT 2 Let $a > 0$ be a real number and $\dfrac{m}{n}$ be a rational number, $n > 0$. Then,

$$a^{\frac{m}{n}} = (a^m)^{\frac{1}{n}} = \left(a^{1/n}\right)^m \Rightarrow a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m.$$

Thus, $4^{\frac{3}{2}} = (4^3)^{\frac{1}{2}} = (64)^{\frac{1}{2}} = \sqrt{64} = 8.$

And, $4^{\frac{3}{2}} = (4^{1/2})^3 = (\sqrt{4})^3 = 2^3 = 8.$

SOLVED EXAMPLES

EXAMPLE 1 *Evaluate*

(i) $(32)^{\frac{1}{5}} + (-7)^0 + (64)^{\frac{1}{2}}$ [2010]

(ii) $(0.00032)^{-\frac{2}{5}}$ [2014]

(iii) $\left(\dfrac{64}{125}\right)^{-\frac{2}{3}}$ [2011]

SOLUTION (i) $(32)^{\frac{1}{5}} + (-7)^0 + (64)^{\frac{1}{2}}$

$= (2^5)^{\frac{1}{5}} + 1 + \sqrt{64}$ $\quad [\because\ a^0 = 1]$

$= 2^{\left(5 \times \frac{1}{5}\right)} + 1 + 8 = 2 + 1 + 8 = 11.$

(ii) $(0.00032)^{-\frac{2}{5}} = \dfrac{1}{(0.00032)^{\frac{2}{5}}}$ $\quad \left[\because\ a^{-n} = \dfrac{1}{a^n}\right]$

$= \dfrac{1}{[(0.2)^5]^{\frac{2}{5}}} = \dfrac{1}{(0.2)^{\left(5 \times \frac{2}{5}\right)}} = \dfrac{1}{(0.2)^2}$

$= \dfrac{1}{0.04} = \dfrac{100}{4} = 25.$

(iii) $\left(\dfrac{64}{125}\right)^{-\frac{2}{3}} = \left(\dfrac{4^3}{5^3}\right)^{-\frac{2}{3}} = \left[\left(\dfrac{4}{5}\right)^3\right]^{-\frac{2}{3}}$

$= \left(\dfrac{4}{5}\right)^{\left[3 \times \left(-\frac{2}{3}\right)\right]} = \left(\dfrac{4}{5}\right)^{-2} = \left(\dfrac{5}{4}\right)^2 = \dfrac{25}{16}.$

<u>EXAMPLE 2</u> *Evaluate*

(i) $\sqrt[3]{(343)^{-2}}$ [2010] *(ii)* $\sqrt[5]{(32)^{-3}}$

SOLUTION (i) $\sqrt[3]{(343)^{-2}} = [(343)^{-2}]^{1/3} = (343)^{(-2) \times \frac{1}{3}}$

$= (7^3)^{-\frac{2}{3}} = 7^{\left[3 \times \left(-\frac{2}{3}\right)\right]} = 7^{-2} = \dfrac{1}{7^2} = \dfrac{1}{49}.$

(ii) $\sqrt[5]{(32)^{-3}} = [(32)^{-3}]^{\frac{1}{5}} = (32)^{(-3) \times \frac{1}{5}} = (2^5)^{-\frac{3}{5}}$

$= 2^{\left[5 \times \left(-\frac{3}{5}\right)\right]} = 2^{-3} = \dfrac{1}{2^3} = \dfrac{1}{8}.$

<u>EXAMPLE 3</u> *Assuming that x, y, z are positive real numbers, simplify each of the following.*

(i) $\sqrt{x^3 y^{-2}}$ *(ii)* $(x^{-2/3} y^{-1/2})^2$ *(iii)* $\left(\dfrac{x^{-4}}{y^{-10}}\right)^{\frac{5}{4}}$

(iv) $\sqrt[4]{81x^8 y^4 z^{16}} \div \sqrt[3]{27x^3 y^6 z^9}$ [2015]

(v) $\sqrt[5]{x^4 \sqrt[4]{x^3 \sqrt[3]{x^2 \sqrt{x}}}}$ [2014]

(vi) $(\sqrt{x})^{-\frac{2}{3}} \sqrt{y^4} \div \sqrt{xy^{-\frac{1}{2}}} \sqrt{x^{-2} y^3}$ [2014]

SOLUTION (i) $\sqrt{x^3y^{-2}} = \sqrt{\dfrac{x^3}{y^2}} = \left(\dfrac{x^3}{y^2}\right)^{\frac{1}{2}} = \dfrac{(x^3)^{\frac{1}{2}}}{(y^2)^{\frac{1}{2}}} = \dfrac{x^{\left(3\times\frac{1}{2}\right)}}{y^{\left(2\times\frac{1}{2}\right)}} = \dfrac{x^{\frac{3}{2}}}{y}.$

(ii) $\left(x^{-\frac{2}{3}}y^{-\frac{1}{2}}\right)^2 = \left(x^{-\frac{2}{3}}\right)^2\left(y^{-\frac{1}{2}}\right)^2 = x^{\left[\left(-\frac{2}{3}\right)\times 2\right]}\cdot y^{\left[\left(-\frac{1}{2}\right)\times 2\right]}$

$$= x^{-\frac{4}{3}}\cdot y^{-1} = \frac{1}{x^{\frac{4}{3}}y}.$$

(iii) $\left(\dfrac{x^{-4}}{y^{-10}}\right)^{\frac{5}{4}} = \left(\dfrac{y^{10}}{x^4}\right)^{\frac{5}{4}} = \dfrac{(y^{10})^{\frac{5}{4}}}{(x^4)^{\frac{5}{4}}} = \dfrac{y^{\left(10\times\frac{5}{4}\right)}}{x^{\left(4\times\frac{5}{4}\right)}} = \dfrac{y^{\frac{25}{2}}}{x^5}.$

(iv) $\sqrt[4]{81x^8y^4z^{16}} \div \sqrt[3]{27x^3y^6z^9}$

$$= \frac{(81x^8y^4z^{16})^{\frac{1}{4}}}{(27x^3y^6z^9)^{\frac{1}{3}}} = \frac{(81)^{\frac{1}{4}}(x^8)^{\frac{1}{4}}\cdot(y^4)^{\frac{1}{4}}\cdot(z^{16})^{\frac{1}{4}}}{(27)^{\frac{1}{3}}\cdot(x^3)^{\frac{1}{3}}\cdot(y^6)^{\frac{1}{3}}\cdot(z^9)^{\frac{1}{3}}}$$

$$= \frac{(3^4)^{\frac{1}{4}}\cdot x^{\left(8\times\frac{1}{4}\right)}\cdot y^{\left(4\times\frac{1}{4}\right)}\cdot z^{\left(16\times\frac{1}{4}\right)}}{(3^3)^{\frac{1}{3}}\cdot x^{\left(3\times\frac{1}{3}\right)}\cdot y^{\left(6\times\frac{1}{3}\right)}\cdot z^{\left(9\times\frac{1}{3}\right)}} = \frac{3^{\left(4\times\frac{1}{4}\right)}\cdot x^2\cdot yz^4}{3^{\left(3\times\frac{1}{3}\right)}\cdot xy^2z^3}$$

$$= \frac{x^{(2-1)}\cdot z^{(4-3)}}{y^{(2-1)}} = \frac{xz}{y}.$$

(v) $\sqrt[5]{x^4\sqrt[4]{x^3\sqrt[3]{x^2\sqrt{x}}}} = \sqrt[5]{x^4\sqrt[4]{x^3\sqrt[3]{x^2\cdot x^{1/2}}}} = \sqrt[5]{x^4\sqrt[4]{x^3\sqrt[3]{x^{\left(2+\frac{1}{2}\right)}}}}$

$$= \sqrt[5]{x^4\sqrt[4]{x^3\sqrt[3]{x^{5/2}}}} = \sqrt[5]{x^4\sqrt[4]{x^3\cdot(x^{5/2})^{1/3}}}$$

$$= \sqrt[5]{x^4\sqrt[4]{x^3\cdot x^{5/6}}} = \sqrt[5]{x^4\sqrt[4]{x^{\left(3+\frac{5}{6}\right)}}} = \sqrt[5]{x^4\sqrt[4]{x^{\left(\frac{23}{6}\right)}}}$$

$$= \sqrt[5]{x^4\cdot\left(x^{\frac{23}{6}}\right)^{\frac{1}{4}}} = \sqrt[5]{x^4\cdot x^{\frac{23}{24}}} = \sqrt[5]{x^{\left(4+\frac{23}{24}\right)}} = \left(x^{\frac{119}{24}}\right)^{\frac{1}{5}}$$

$$= x^{\left(\frac{119}{24}\times\frac{1}{5}\right)} = x^{\frac{119}{120}}.$$

(vi) $(\sqrt{x})^{-\frac{2}{3}}\sqrt{y^4} \div \sqrt{xy^{-\frac{1}{2}}}\sqrt{x^{-2}y^3}$

$$= \frac{\left(x^{\frac{1}{2}}\right)^{-\frac{2}{3}}\cdot(y^4)^{\frac{1}{2}}}{\left(xy^{-\frac{1}{2}}\right)^{\frac{1}{2}}(x^{-2}y^3)^{\frac{1}{2}}} = \frac{x^{\left[\left(\frac{1}{2}\right)\times\left(-\frac{2}{3}\right)\right]}y^{\left(4\times\frac{1}{2}\right)}}{x^{\frac{1}{2}}\cdot\left(y^{-\frac{1}{2}}\right)^{\frac{1}{2}}\cdot(x^{-2})^{\frac{1}{2}}\cdot(y^3)^{\frac{1}{2}}}$$

$$= \frac{x^{-\frac{1}{3}}\cdot y^2}{x^{\frac{1}{2}}\cdot y^{\left[\left(-\frac{1}{2}\right)\times\frac{1}{2}\right]}\cdot x^{\left[(-2)\times\frac{1}{2}\right]}\cdot y^{\left(3\times\frac{1}{2}\right)}}$$

$$= \frac{x^{-\frac{1}{3}} \cdot y^2}{x^{\frac{1}{2}} \cdot y^{-\frac{1}{4}} \cdot x^{-1} \cdot y^{\frac{3}{2}}} = x^{\left(-\frac{1}{3}-\frac{1}{2}+1\right)} \cdot y^{\left(2+\frac{1}{4}-\frac{3}{2}\right)}$$

$$= x^{\frac{1}{6}} \cdot y^{\frac{3}{4}}.$$

EXAMPLE 4 *Simplify* $\left(\frac{81}{16}\right)^{-\frac{3}{4}} \times \left[\left(\frac{25}{9}\right)^{-\frac{3}{2}} \div \left(\frac{5}{2}\right)^{-3}\right]$. [2011]

SOLUTION $\left(\frac{81}{16}\right)^{-\frac{3}{4}} \times \left[\left(\frac{25}{9}\right)^{-\frac{3}{2}} \div \left(\frac{5}{2}\right)^{-3}\right] = \left(\frac{3^4}{2^4}\right)^{-\frac{3}{4}} \times \left[\left(\frac{5^2}{3^2}\right)^{-\frac{3}{2}} \div \left(\frac{5}{2}\right)^{-3}\right]$

$$= \left[\left(\frac{3}{2}\right)^4\right]^{-\frac{3}{4}} \times \left[\left\{\left(\frac{5}{3}\right)^2\right\}^{-\frac{3}{2}} \div \left(\frac{5}{2}\right)^{-3}\right] = \left(\frac{3}{2}\right)^{4\times\left(-\frac{3}{4}\right)} \times \left[\left(\frac{5}{3}\right)^{2\times\left(-\frac{3}{2}\right)} \div \left(\frac{5}{2}\right)^{-3}\right]$$

$$= \left(\frac{3}{2}\right)^{-3} \times \left[\left(\frac{5}{3}\right)^{-3} \div \left(\frac{5}{2}\right)^{-3}\right] = \left(\frac{2}{3}\right)^3 \times \left[\left(\frac{3}{5}\right)^3 \div \left(\frac{2}{5}\right)^3\right]$$

$$= \frac{2^3}{3^3} \times \left[\frac{3^3}{5^3} \div \frac{2^3}{5^3}\right] = \frac{2^3}{3^3} \times \left[\frac{3^3}{5^3} \times \frac{5^3}{2^3}\right] = \frac{2^3}{3^3} \times \frac{3^3}{2^3} = 1.$$

EXAMPLE 5 *Prove that* $\frac{2^{x-1}+2^x}{2^{x+1}-2^x} = \frac{3}{2}\cdot$

SOLUTION We have

$$\frac{2^{x-1}+2^x}{2^{x+1}-2^x} = \frac{2^{x-1}(1+2)}{2^x(2-1)} = \frac{3\cdot 2^{x-1}}{2^x} = 3\cdot 2^{x-1-x}$$

$$= 3\cdot 2^{-1} = \frac{3}{2}\cdot$$

EXAMPLE 6 *Prove that* $\frac{2^{30}+2^{29}+2^{28}}{2^{31}+2^{30}-2^{29}} = \frac{7}{10}\cdot$ [2011]

SOLUTION We have

$$\frac{2^{30}+2^{29}+2^{28}}{2^{31}+2^{30}-2^{29}} = \frac{2^{28}(2^2+2+1)}{2^{29}(2^2+2-1)} = \frac{2^{28}(4+2+1)}{2^{29}(4+2-1)}$$

$$= \frac{7}{5} \times \frac{1}{2^{(29-28)}} = \frac{7}{5\times 2} = \frac{7}{10}\cdot$$

EXAMPLE 7 *Prove that*

(i) $\frac{a^{-1}}{a^{-1}+b^{-1}} + \frac{a^{-1}}{a^{-1}-b^{-1}} = \frac{2b^2}{b^2-a^2}$ [2015]

(ii) $\frac{1}{1+x^{a-b}} + \frac{1}{1+x^{b-a}} = 1$ [2010]

SOLUTION We have

$$\text{(i)}\ \frac{a^{-1}}{a^{-1}+b^{-1}}+\frac{a^{-1}}{a^{-1}-b^{-1}}=\frac{\frac{1}{a}}{\frac{1}{a}+\frac{1}{b}}+\frac{\frac{1}{a}}{\frac{1}{a}-\frac{1}{b}}=\frac{\frac{1}{a}}{\left(\frac{b+a}{ab}\right)}+\frac{\frac{1}{a}}{\left(\frac{b-a}{ab}\right)}$$

$$=\frac{1}{a}\cdot\frac{ab}{(b+a)}+\frac{1}{a}\cdot\frac{ab}{(b-a)}=\frac{b}{b+a}+\frac{b}{b-a}$$

$$=\frac{b(b-a)+b(b+a)}{(b+a)(b-a)}=\frac{b^2-ab+b^2+ab}{b^2-a^2}$$

$$=\frac{2b^2}{b^2-a^2}.$$

$$\text{(ii)}\ \frac{1}{1+x^{a-b}}+\frac{1}{1+x^{b-a}}=\frac{1}{1+x^a\cdot x^{-b}}+\frac{1}{1+x^b\cdot x^{-a}}$$

$$=\frac{1}{1+\frac{x^a}{x^b}}+\frac{1}{1+\frac{x^b}{x^a}}=\frac{1}{\left(\frac{x^a+x^b}{x^b}\right)}+\frac{1}{\left(\frac{x^a+x^b}{x^a}\right)}$$

$$=\frac{x^b}{x^a+x^b}+\frac{x^a}{x^a+x^b}=\frac{x^b+x^a}{x^a+x^b}=1.$$

EXAMPLE 8 *Simplify*

$$\text{(i)}\ \left(\frac{x^l}{x^m}\right)^{\frac{1}{lm}}\cdot\left(\frac{x^m}{x^n}\right)^{\frac{1}{mn}}\cdot\left(\frac{x^n}{x^l}\right)^{\frac{1}{nl}}$$ [2011]

$$\text{(ii)}\ \left(\frac{x^p}{x^q}\right)^{p+q}\cdot\left(\frac{x^q}{x^r}\right)^{q+r}\cdot\left(\frac{x^r}{x^p}\right)^{r+p}$$ [2011]

SOLUTION We have

$$\text{(i)}\ \left(\frac{x^l}{x^m}\right)^{\frac{1}{lm}}\cdot\left(\frac{x^m}{x^n}\right)^{\frac{1}{mn}}\cdot\left(\frac{x^n}{x^l}\right)^{\frac{1}{nl}}$$

$$=(x^{l-m})^{\frac{1}{lm}}\cdot(x^{m-n})^{\frac{1}{mn}}\cdot(x^{n-l})^{\frac{1}{nl}}\qquad\left[\because\ \frac{a^p}{a^q}=a^{p-q}\right]$$

$$=x^{\frac{l-m}{lm}}\cdot x^{\frac{m-n}{mn}}\cdot x^{\frac{n-l}{nl}}$$

$$=x^{\frac{l-m}{lm}+\frac{m-n}{mn}+\frac{n-l}{nl}}$$

$$=x^{\frac{n(l-m)+l(m-n)+m(n-l)}{lmn}}=x^{\frac{nl-nm+lm-ln+mn-ml}{lmn}}$$

$$=x^0=1.$$

(ii) $\left(\frac{x^p}{x^q}\right)^{p+q} \cdot \left(\frac{x^q}{x^r}\right)^{q+r} \cdot \left(\frac{x^r}{x^p}\right)^{r+p}$

$$= (x^{p-q})^{p+q} \cdot (x^{q-r})^{q+r} \cdot (x^{r-p})^{r+p}$$
$$= x^{(p-q)(p+q)} \cdot x^{(q-r)(q+r)} \cdot x^{(r-p)(r+p)}$$
$$= x^{(p^2-q^2)} \cdot x^{(q^2-r^2)} \cdot x^{(r^2-p^2)}$$
$$= x^{(p^2-q^2)+(q^2-r^2)+(r^2-p^2)} = x^0 = 1.$$

EXAMPLE 9 *(i) If* $x^{\frac{1}{12}} = 49^{\frac{1}{24}}$*, find the value of x.* [2014]

(ii) If $(125)^x = \frac{25}{5^x}$*, find x.* [2014]

SOLUTION We have

(i) $x^{\frac{1}{12}} = 49^{\frac{1}{24}} \Leftrightarrow \left(x^{\frac{1}{12}}\right)^{12} = \left(49^{\frac{1}{24}}\right)^{12}$

$$\Leftrightarrow x^{\left(\frac{1}{12}\times 12\right)} = 49^{\left(\frac{1}{24}\times 12\right)}$$
$$\Leftrightarrow x = 49^{\frac{1}{2}} = \sqrt{49} = 7.$$

(ii) $(125)^x = \frac{25}{5^x} \Leftrightarrow (5^3)^x = \frac{5^2}{5^x}$

$$\Leftrightarrow 5^{3x} = 5^{2-x}$$
$$\Leftrightarrow 3x = 2 - x \Rightarrow 4x = 2 \Rightarrow x = \frac{2}{4} = \frac{1}{2}.$$

EXAMPLE 10 *If* $5^{x-2} \times 3^{2x-3} = 135$*, find the value of x.*

SOLUTION We have

$$5^{x-2} \times 3^{2x-3} = 135 \Rightarrow 5^{x-2} \times 3^{2x-3} = 5 \times 3^3$$
$$\Rightarrow x - 2 = 1 \text{ and } 2x - 3 = 3$$
$$\Rightarrow x = 3.$$

EXAMPLE 11 *Write the following in the descending order of magnitude.* [2010]

$$\sqrt[3]{2}, \sqrt{3}, \sqrt[6]{5}.$$

SOLUTION We have $\sqrt[3]{2} = 2^{\frac{1}{3}}$; $\sqrt{3} = 3^{\frac{1}{2}}$; $\sqrt[6]{5} = 5^{\frac{1}{6}}$.

LCM of 3, 2 and 6 = 6.

$$\therefore \quad 2^{\frac{1}{3}} = 2^{\left(\frac{1}{3}\times\frac{2}{2}\right)} = 2^{\frac{2}{6}} = (2^2)^{\frac{1}{6}} = 4^{\frac{1}{6}};$$

$$3^{\frac{1}{2}} = 3^{\left(\frac{1}{2}\times\frac{3}{3}\right)} = 3^{\frac{3}{6}} = (3^3)^{\frac{1}{6}} = (27)^{\frac{1}{6}}.$$

Clearly, $(27)^{\frac{1}{6}} > 5^{\frac{1}{6}} > 4^{\frac{1}{6}}$.

So, $3^{\frac{1}{2}} > 5^{\frac{1}{6}} > 2^{\frac{1}{3}}$ or $\sqrt{3} > \sqrt[6]{5} > \sqrt[3]{2}$.

Hence, the correct descending order is $\sqrt{3}$, $\sqrt[6]{5}$ and $\sqrt[3]{2}$.

EXERCISE 1G

1. Simplify

(i) $2^{\frac{2}{3}} \times 2^{\frac{1}{3}}$ [2015] (ii) $2^{\frac{2}{3}} \times 2^{\frac{1}{5}}$ [2014]

(iii) $7^{\frac{5}{6}} \times 7^{\frac{2}{3}}$ (iv) $(1296)^{\frac{1}{4}} \times (1296)^{\frac{1}{2}}$

2. Simplify

(i) $\dfrac{6^{1/4}}{6^{1/5}}$ (ii) $\dfrac{8^{1/2}}{8^{2/3}}$ (iii) $\dfrac{5^{6/7}}{5^{2/3}}$

3. Simplify

(i) $3^{\frac{1}{4}} \times 5^{\frac{1}{4}}$ (ii) $2^{\frac{5}{8}} \times 3^{\frac{5}{8}}$ (iii) $6^{\frac{1}{2}} \times 7^{\frac{1}{2}}$

4. Simplify

(i) $(3^4)^{\frac{1}{4}}$ (ii) $(3^{1/3})^4$ (iii) $\left(\dfrac{1}{3^4}\right)^{\frac{1}{2}}$

5. Evaluate

(i) $(125)^{\frac{1}{3}}$ (ii) $(64)^{\frac{1}{6}}$ (iii) $(25)^{\frac{3}{2}}$

(iv) $(81)^{\frac{3}{4}}$ (v) $(64)^{-\frac{1}{2}}$ (vi) $(8)^{-\frac{1}{3}}$

6. If $a = 2$, $b = 3$, find the values of

(i) $(a^b + b^a)^{-1}$ (ii) $(a^a + b^b)^{-1}$ [2014]

7. Simplify

(i) $\left(\dfrac{81}{49}\right)^{-\frac{3}{2}}$ [2011] (ii) $(14641)^{0.25}$ [2015]

(iii) $\left(\dfrac{32}{243}\right)^{-\frac{4}{5}}$ [2011] (iv) $\left(\dfrac{7776}{243}\right)^{-\frac{3}{5}}$ [2014]

8. Evaluate

(i) $\dfrac{4}{(216)^{-\frac{2}{3}}}+\dfrac{1}{(256)^{-\frac{3}{4}}}+\dfrac{2}{(243)^{-\frac{1}{5}}}$ [2015]

(ii) $\left(\dfrac{64}{125}\right)^{-\frac{2}{3}}+\left(\dfrac{256}{625}\right)^{-\frac{1}{4}}+\left(\dfrac{3}{7}\right)^{0}$

(iii) $\left(\dfrac{81}{16}\right)^{-\frac{3}{4}}\left[\left(\dfrac{25}{9}\right)^{-\frac{3}{2}}\div\left(\dfrac{5}{2}\right)^{-3}\right]$ [2010]

(iv) $\dfrac{(25)^{\frac{5}{2}}\times(729)^{\frac{1}{3}}}{(125)^{\frac{2}{3}}\times(27)^{\frac{2}{3}}\times 8^{\frac{4}{3}}}$

9. Evaluate

(i) $(1^3+2^3+3^3)^{\frac{1}{2}}$ [2015]

(ii) $\left[5\left(8^{\frac{1}{3}}+27^{\frac{1}{3}}\right)^{3}\right]^{\frac{1}{4}}$ [2011, '15]

(iii) $\dfrac{2^0+7^0}{5^0}$ [2015]

(iv) $\left[(16)^{\frac{1}{2}}\right]^{\frac{1}{2}}$ [2014]

10. Prove that

(i) $\left[8^{-\frac{2}{3}}\times 2^{\frac{1}{2}}\times 25^{-\frac{5}{4}}\right]\div\left[32^{-\frac{2}{5}}\times 125^{-\frac{5}{6}}\right]=\sqrt{2}$ [2015]

(ii) $\left(\dfrac{64}{125}\right)^{-\frac{2}{3}}+\dfrac{1}{\left(\dfrac{256}{625}\right)^{\frac{1}{4}}}+\dfrac{\sqrt{25}}{\sqrt[3]{64}}=\dfrac{65}{16}$ [2011]

(iii) $\left[7\left\{(81)^{\frac{1}{4}}+(256)^{\frac{1}{4}}\right\}^{\frac{1}{4}}\right]^{4}=16807$ [2015]

11. Simplify $\sqrt[4]{\sqrt[3]{x^2}}$ and express the result in the exponential form of x. [2011]

12. Simplify the product $\sqrt[3]{2}\cdot\sqrt[4]{2}\cdot\sqrt[12]{32}$.

13. Simplify

(i) $\left(\dfrac{15^{1/3}}{9^{1/4}}\right)^{-6}$ [2010] (ii) $\left(\dfrac{12^{1/5}}{27^{1/5}}\right)^{5/2}$ [2010] (iii) $\left(\dfrac{15^{1/4}}{3^{1/2}}\right)^{-2}$ [2011]

14. Find the value of x in each of the following.

(i) $\sqrt[5]{5x+2}=2$ [2014]

(ii) $\sqrt[3]{3x-2}=4$ [2014]

(iii) $\left(\dfrac{3}{4}\right)^{3}\left(\dfrac{4}{3}\right)^{-7}=\left(\dfrac{3}{4}\right)^{2x}$ [2010]

(iv) $5^{x-3}\times 3^{2x-8}=225$ [2010]

(v) $\dfrac{3^{3x}\cdot 3^{2x}}{3^{x}}=\sqrt[4]{3^{20}}$ [2015]

15. Prove that

(i) $\sqrt{x^{-1}y}\cdot\sqrt{y^{-1}z}\cdot\sqrt{z^{-1}x}=1.$

(ii) $\left(x^{\frac{1}{a-b}}\right)^{\frac{1}{a-c}}\cdot\left(x^{\frac{1}{b-c}}\right)^{\frac{1}{b-a}}\cdot\left(x^{\frac{1}{c-a}}\right)^{\frac{1}{c-b}}=1$ [2015]

(iii) $\dfrac{x^{a(b-c)}}{x^{b(a-c)}}\div\left(\dfrac{x^b}{x^a}\right)^c=1$ [2010]

(iv) $\dfrac{(x^{a+b})^2(x^{b+c})^2(x^{c+a})^2}{(x^a x^b x^c)^4}=1$

16. If x is a positive real number and exponents are rational numbers, simplify [2011]

$$\left(\frac{x^b}{x^c}\right)^{b+c-a}\cdot\left(\frac{x^c}{x^a}\right)^{c+a-b}\cdot\left(\frac{x^a}{x^b}\right)^{a+b-c}.$$

17. If $\dfrac{9^n\times 3^2\times(3^{-n/2})^{-2}-(27)^n}{3^{3m}\times 2^3}=\dfrac{1}{27}$, prove that $m-n=1$.

18. Write the following in ascending order of magnitude. [2015]

$\sqrt[6]{6},\ \sqrt[3]{7},\ \sqrt[4]{8}.$

ANSWERS (EXERCISE 1G)

1. (i) 2 (ii) $2^{\frac{13}{15}}$ (iii) $7^{\frac{3}{2}}$ (iv) 216 **2.** (i) $6^{\frac{1}{20}}$ (ii) $8^{-\frac{1}{6}}$ (iii) $5^{\frac{4}{21}}$

3. (i) $(15)^{\frac{1}{4}}$ (ii) $(6)^{\frac{5}{8}}$ (iii) $(42)^{\frac{1}{2}}$ **4.** (i) 3 (ii) $3^{\frac{4}{3}}$ (iii) $\frac{1}{9}$

5. (i) 5 (ii) 2 (iii) 125 (iv) 27 (v) $\frac{1}{8}$ (vi) $\frac{1}{2}$

6. (i) $\frac{1}{17}$ (ii) $\frac{1}{13}$

7. (i) $\frac{343}{729}$ (ii) 11 (iii) $\frac{81}{16}$ (iv) $\frac{1}{8}$

8. (i) 214 (ii) $\frac{61}{16}$ (iii) 1 (iv) $\frac{125}{16}$

9. (i) 6 (ii) 5 (iii) 2 (iv) 2

11. $x^{\frac{1}{6}}$ **12.** 2 **13.** (i) $\frac{27}{225}$ (ii) $\frac{2}{3}$ (iii) $\frac{3}{(15)^{1/2}}$

14. (i) $x=6$ (ii) $x=22$ (iii) $x=5$ (iv) $x=5$ (v) $x=\frac{5}{4}$

16. 1 **18.** $\sqrt[6]{6}<\sqrt[4]{8}<\sqrt[3]{7}$

HINTS TO SOME SELECTED QUESTIONS

14. (i) $\sqrt[5]{5x+2} = 2 \Rightarrow 5x+2 = 2^5 = 32.$

(ii) $\sqrt[3]{3x-2} = 4 \Rightarrow 3x-2 = 4^3 = 64.$

(iii) $\left(\frac{3}{4}\right)^3\left(\frac{3}{4}\right)^7 = \left(\frac{3}{4}\right)^{2x} \Rightarrow \left(\frac{3}{4}\right)^{3+7} = \left(\frac{3}{4}\right)^{2x} \Rightarrow 2x = 10.$

(iv) $5^{x-3} \times 3^{2x-8} = 5^2 \times 3^2 \Rightarrow x-3 = 2$ and $2x-8 = 2 \Rightarrow x = 5.$

(v) $\frac{3^{3x+2x}}{3^x} = (3^{20})^{\frac{1}{4}} \Rightarrow 3^{5x-x} = 3^{\left(20 \times \frac{1}{4}\right)} \Rightarrow 3^{4x} = 3^5 \Rightarrow 4x = 5 \Rightarrow x = \frac{5}{4}.$

15. (i) $\sqrt{x^{-1}y} \cdot \sqrt{y^{-1}z} \cdot \sqrt{z^{-1}x} = \sqrt{\frac{y}{x}} \cdot \sqrt{\frac{z}{y}} \cdot \sqrt{\frac{x}{z}} = \sqrt{\frac{y}{x} \times \frac{z}{y} \times \frac{x}{z}} = 1.$

(ii) $\left(x^{\frac{1}{a-b}}\right)^{\frac{1}{a-c}} \cdot \left(x^{\frac{1}{b-c}}\right)^{\frac{1}{b-a}} \cdot \left(x^{\frac{1}{c-a}}\right)^{\frac{1}{c-b}}$

$= x^{\frac{1}{(a-b)(a-c)}} \cdot x^{\frac{1}{(b-c)(b-a)}} \cdot x^{\frac{1}{(c-a)(c-b)}}$

$= x^{\frac{1}{(a-b)(a-c)} + \frac{1}{(b-c)(b-a)} + \frac{1}{(c-a)(c-b)}}$

$= x^{\frac{-(b-c)-(c-a)-(a-b)}{(a-b)(b-c)(c-a)}}$

$= x^{\frac{-b+c-c+a-a+b}{(a-b)(b-c)(c-a)}} = x^0 = 1.$

(iii) $\frac{x^{a(b-c)}}{x^{b(a-c)}} \div \left(\frac{x^b}{x^a}\right)^c = \frac{x^{(ab-ac)}}{x^{(ba-bc)}} \div x^{(b-a)c}$

$= x^{(ab-ac)-(ab-bc)} \div x^{(bc-ac)}$

$= x^{(ab-ac)-(ab-bc)-(bc-ac)}$

$= x^{ab-ac-ab+bc-bc+ac} = x^0 = 1.$

(iv) $\frac{(x^{a+b})^2(x^{b+c})^2(x^{c+a})^2}{(x^a x^b x^c)^4} = \frac{x^{(2a+2b)} \cdot x^{(2b+2c)} \cdot x^{(2c+2a)}}{x^{4a} \cdot x^{4b} \cdot x^{4c}}$

$= \frac{x^{(2a+2b+2b+2c+2c+2a)}}{x^{(4a+4b+4c)}}$

$= \frac{x^{(4a+4b+4c)}}{x^{(4a+4b+4c)}} = 1.$

16. $\left(\frac{x^b}{x^c}\right)^{b+c-a} \cdot \left(\frac{x^c}{x^a}\right)^{c+a-b} \cdot \left(\frac{x^a}{x^b}\right)^{a+b-c}$

$= x^{(b-c)(b+c-a)} \cdot x^{(c-a)(c+a-b)} \cdot x^{(a-b)(a+b-c)}$

$= x^{(b^2+bc-ba-bc-c^2+ac)} \cdot x^{(c^2+ac-bc-ac-a^2+ab)} \cdot x^{(a^2+ab-ac-ab-b^2+bc)}$

$= x^{(b^2-ab-c^2+ac)+(c^2-bc-a^2+ab)+(a^2-ac-b^2+bc)} = x^0 = 1.$

17. $\frac{9^n \times 3^2 \times \left(3^{-\frac{n}{2}}\right)^{-2} - (27)^n}{3^{3m} \times 2^3} = \frac{1}{27}$

$\Leftrightarrow \quad \frac{(3^2)^n \times 3^2 \times 3^{\left(-\frac{n}{2}\right)\times(-2)} - (3^3)^n}{3^{3m} \times 8} = \frac{1}{27}$

$$\Leftrightarrow \quad \frac{3^{2n} \times 3^2 \times 3^n - 3^{3n}}{3^{3m} \times 8} = \frac{1}{27} \quad \Leftrightarrow \quad \frac{3^{3n+2} - 3^{3n}}{3^{3m} \times 8} = \frac{1}{3^3}$$

$$\Leftrightarrow \quad \frac{3^{3n}(3^2 - 1)}{3^{3m} \times 8} = 3^{-3} \quad \Leftrightarrow \quad \frac{3^{3n} \times 8}{3^{3m} \times 8} = 3^{-3} \quad \Rightarrow \quad 3^{(3n-3m)} = 3^{-3}$$

$$\Leftrightarrow \quad 3n - 3m = -3 \quad \Leftrightarrow \quad n - m = -1 \quad \Leftrightarrow \quad m - n = 1.$$

MULTIPLE-CHOICE QUESTIONS (MCQ)

Choose the correct answer in each of the following questions.

1. Which of the following is a rational number? [2010]

(a) $1+\sqrt{3}$ (b) π (c) $2\sqrt{3}$ (d) 0

2. A rational number between –3 and 3 is [2010]

(a) 0 (b) -4.3

(c) –3.4 (d) 1.101100110001 …

3. Two rational numbers between $\frac{2}{3}$ and $\frac{5}{3}$ are [2010]

(a) $\frac{1}{6}$ and $\frac{2}{6}$ (b) $\frac{1}{2}$ and $\frac{2}{1}$ (c) $\frac{5}{6}$ and $\frac{7}{6}$ (d) $\frac{2}{3}$ and $\frac{4}{3}$

4. Every point on a number line represents [2010]

(a) a rational number (b) a natural number

(c) an irrational number (d) a unique number

5. Which of the following is a rational number?

(a) $\sqrt{2}$ (b) $\sqrt{23}$

(c) $\sqrt{225}$ (d) 0.1010010001…

6. Every rational number is

(a) a natural number (b) a whole number

(c) an integer (d) a real number

7. Between any two rational numbers there

(a) is no rational number

(b) is exactly one rational number

(c) are infinitely many rational numbers

(d) is no irrational number

8. The decimal representation of a rational number is

(a) always terminating

(b) either terminating or repeating

(c) either terminating or nonrepeating

(d) neither terminating nor repeating

9. The decimal representation of an irrational number is
(a) always terminating
(b) either terminating or repeating
(c) either terminating or nonrepeating
(d) neither terminating nor repeating

10. The decimal expansion that a rational number cannot have is
(a) 0.25 (b) $0.25\overline{28}$
(c) $0.\overline{2528}$ (d) 0.5030030003…

11. Which of the following is an irrational number? [2011]
(b) 3.14 (b) 3.141414 …
(c) 3.14444 … (d) 3.141141114 …

12. A rational number equivalent to $\frac{7}{19}$ is [2011]
(a) $\frac{17}{119}$ (b) $\frac{14}{57}$ (c) $\frac{21}{38}$ (d) $\frac{21}{57}$

13. Choose the rational number which does not lie between $-\frac{2}{3}$ and $-\frac{1}{5}$. [2011]
(a) $-\frac{3}{10}$ (b) $\frac{3}{10}$ (c) $-\frac{1}{4}$ (d) $-\frac{7}{20}$

14. π is (2010, '11)
(a) a rational number (b) an integer
(c) an irrational number (d) a whole number

15. The decimal expansion of $\sqrt{2}$ is
(a) finite decimal (b) 1.4121
(c) nonterminating recurring (d) nonterminating, nonrecurring

16. Which of the following is an irrational number?
(a) $\sqrt{23}$ (b) $\sqrt{225}$
(c) 0.3799 (d) $7.\overline{478}$

17. How many digits are there in the repeating block of digits in the decimal expansion of $\frac{17}{7}$?
(a) 16 (b) 6 (c) 26 (d) 7

18. Which of the following numbers is irrational?
(a) $\sqrt{\frac{4}{9}}$ (b) $\frac{\sqrt{1250}}{\sqrt{8}}$ (c) $\sqrt{8}$ (d) $\frac{\sqrt{24}}{\sqrt{6}}$

19. The product of two irrational numbers is

(a) always irrational

(b) always rational

(c) always an integer

(d) sometimes rational and sometimes irrational

20. Which of the following is a true statement?

(a) The sum of two irrational numbers is an irrational number.

(b) The product of two irrational numbers is an irrational number.

(c) Every real number is always rational.

(d) Every real number is either rational or irrational.

21. Which of the following is a true statement?

(a) π and $\frac{22}{7}$ are both rationals.

(b) π and $\frac{22}{7}$ are both irrationals.

(c) π is rational and $\frac{22}{7}$ is irrational.

(d) π is irrational and $\frac{22}{7}$ is rational.

22. A rational number lying between $\sqrt{2}$ and $\sqrt{3}$ is [2010]

(a) $\frac{(\sqrt{2}+\sqrt{3})}{2}$ (b) $\sqrt{6}$ (c) 1.6 (d) 1.9

23. Which of the following is a rational number? [2010]

(a) $\sqrt{5}$ (b) 0.101001000100001 …

(c) π (d) 0.853853853 …

24. The product of a nonzero rational number with an irrational number is always a/an

(a) irrational number (b) rational number

(c) whole number (d) natural number

25. The value of $0.\overline{2}$ in the form $\frac{p}{q}$, where p and q are integers and $q \neq 0$, is [2011]

(a) $\frac{1}{5}$ (b) $\frac{2}{9}$ (c) $\frac{2}{5}$ (d) $\frac{1}{8}$

26. The simplest form of $1.\overline{6}$ is

(a) $\frac{833}{500}$ (b) $\frac{8}{5}$ (c) $\frac{5}{3}$ (d) none of these

27. The simplest form of $0.\overline{54}$ is

(a) $\frac{27}{50}$ (b) $\frac{6}{11}$ (c) $\frac{4}{7}$ (d) none of these

28. The simplest form of $0.3\overline{2}$ is

(a) $\frac{16}{45}$ (b) $\frac{32}{99}$ (c) $\frac{29}{90}$ (d) none of these

29. The simplest form of $0.12\overline{3}$ is

(a) $\frac{41}{330}$ (b) $\frac{37}{330}$ (c) $\frac{41}{333}$ (d) none of these

30. An irrational number between 5 and 6 is

(a) $\frac{1}{2}(5+6)$ (b) $\sqrt{5+6}$ (c) $\sqrt{5\times6}$ (d) none of these

31. An irrational number between $\sqrt{2}$ and $\sqrt{3}$ is

(a) $(\sqrt{2}+\sqrt{3})$ (b) $\sqrt{2}\times\sqrt{3}$ (c) $5^{1/4}$ (d) $6^{1/4}$

32. An irrational number between $\frac{1}{7}$ and $\frac{2}{7}$ is

(a) $\frac{1}{2}\left(\frac{1}{7}+\frac{2}{7}\right)$ (b) $\left(\frac{1}{7}\times\frac{2}{7}\right)$ (c) $\sqrt{\frac{1}{7}\times\frac{2}{7}}$ (d) none of these

33. The sum of $0.\overline{3}$ and $0.\overline{4}$ is [2010]

(a) $\frac{7}{10}$ (b) $\frac{7}{9}$ (c) $\frac{7}{11}$ (d) $\frac{7}{99}$

34. The value of $2.\overline{45}+0.\overline{36}$ is

(a) $\frac{67}{33}$ (b) $\frac{24}{11}$ (c) $\frac{31}{11}$ (d) $\frac{167}{110}$

35. Which of the following is the value of $(\sqrt{11}-\sqrt{7})(\sqrt{11}+\sqrt{7})$? [2010]

(a) -4 (b) 4 (c) $\sqrt{11}$ (d) $\sqrt{7}$

36. $(-2-\sqrt{3})(-2+\sqrt{3})$ when simplified is

(a) positive and irrational (b) positive and rational

(c) negative and irrational (d) negative and rational

37. $(6+\sqrt{27})-(3+\sqrt{3})+(1-2\sqrt{3})$ when simplified is

(a) positive and irrational (b) positive and rational

(c) negative and irrational (d) negative and rational

38. When $15\sqrt{15}$ is divided by $3\sqrt{3}$, the quotient is

(a) $5\sqrt{3}$ (b) $3\sqrt{5}$ (c) $5\sqrt{5}$ (d) $3\sqrt{3}$

39. The value of $\sqrt{20}\times\sqrt{5}$ is [2010]

(a) 10 (b) $2\sqrt{5}$ (c) $20\sqrt{5}$ (d) $4\sqrt{5}$

40. The value of $\frac{4\sqrt{12}}{12\sqrt{27}}$ is

(a) $\frac{1}{9}$ (b) $\frac{2}{9}$ (c) $\frac{4}{9}$ (d) $\frac{8}{9}$

41. $\sqrt{10} \times \sqrt{15} = ?$

(a) $\sqrt{25}$ (b) $5\sqrt{6}$ (c) $6\sqrt{5}$ (d) None of these

42. $\dfrac{\sqrt{32}+\sqrt{48}}{\sqrt{8}+\sqrt{12}} = ?$

(a) $\sqrt{2}$ (b) 2 (c) 4 (d) 8

43. $(125)^{-1/3} = ?$

(a) 5 (b) –5 (c) $\frac{1}{5}$ (d) $-\frac{1}{5}$

44. The value of $7^{\frac{1}{2}} \cdot 8^{\frac{1}{2}}$ is [2010]

(a) $(28)^{1/2}$ (b) $(56)^{1/2}$ (c) $(14)^{1/2}$ (d) $(42)^{1/2}$

45. After simplification, $\dfrac{13^{1/5}}{13^{1/3}}$ is

(a) $13^{2/15}$ (b) $13^{8/15}$ (c) $13^{1/3}$ (d) $13^{-2/15}$

46. The value of $\sqrt[4]{(64)^{-2}}$ is

(a) $\frac{1}{8}$ (b) $\frac{1}{2}$ (c) 8 (d) $\frac{1}{64}$

47. The value of $\dfrac{2^0+7^0}{5^0}$ is [2011]

(a) 0 (b) 2 (c) $\frac{9}{5}$ (d) $\frac{1}{5}$

48. The value of $(243)^{1/5}$ is [2014]

(a) 3 (b) –3 (c) 5 (d) $\frac{1}{3}$

49. $9^3 + (-3)^3 - 6^3 = ?$

(a) 432 (b) 270 (c) 486 (d) 540

50. Simplified value of $(16)^{-\frac{1}{4}} \times \sqrt[4]{16}$ is [2010]

(a) 0 (b) 1 (c) 4 (d) 16

51. The value of $\sqrt[4]{\sqrt[3]{2^2}}$ is

(a) $2^{-1/6}$ (b) 2^{-6} (c) $2^{1/6}$ (d) 2^6

52. Simplified value of $(25)^{1/3} \times 5^{1/3}$ is

(a) 25 (b) 3 (c) 1 (d) 5

53. The value of $\left[(81)^{\frac{1}{2}}\right]^{\frac{1}{2}}$ is

(a) 3 (b) –3 (c) 9 (d) $\frac{1}{3}$

54. There is a number x such that x^2 is irrational but x^4 is rational. Then, x can be

(a) $\sqrt{5}$ (b) $\sqrt{2}$ (c) $\sqrt[3]{2}$ (d) $\sqrt[4]{2}$

55. If $x = \frac{\sqrt{7}}{5}$ and $\frac{5}{x} = p\sqrt{7}$ then the value of p is

(a) $\frac{7}{25}$ (b) $\frac{25}{7}$ (c) $\frac{7}{15}$ (d) $\frac{15}{7}$

56. The value of $\left(\frac{256x^{16}}{81y^4}\right)^{-\frac{1}{4}}$ is

(a) $\frac{3y}{8x^4}$ (b) $\frac{3y}{4x^4}$ (c) $\frac{4y}{5x^4}$ (d) $\frac{4x^4}{3y}$

57. The value of $x^{p-q} \cdot x^{q-r} \cdot x^{r-p}$ is equal to

(a) 0 (b) 1 (c) x (d) x^{pqr}

58. The value of $\sqrt{p^{-1}q} \cdot \sqrt{q^{-1}r} \cdot \sqrt{r^{-1}p}$ is

(a) –1 (b) 0 (c) 1 (d) 2

59. $\sqrt[3]{2} \times \sqrt[4]{2} \times \sqrt[12]{32} = ?$

(a) 2 (b) $\sqrt{2}$ (c) $2\sqrt{2}$ (d) $4\sqrt{2}$

60. If $\left(\frac{2}{3}\right)^x \left(\frac{3}{2}\right)^{2x} = \frac{81}{16}$ then $x = ?$

(a) 1 (b) 2 (c) 3 (d) 4

61. If $(3^3)^2 = 9^x$ then $5^x = ?$

(a) 1 (b) 5 (c) 25 (d) 125

62. On simplification, the expression $\frac{5^{n+2} - 6 \times 5^{n+1}}{13 \times 5^n - 2 \times 5^{n+1}}$ equals

(a) $\frac{5}{3}$ (b) $-\frac{5}{3}$ (c) $\frac{3}{5}$ (d) $-\frac{3}{5}$

63. The simplest rationalisation factor of $\sqrt[3]{500}$ is

(a) $\sqrt{5}$ (b) $\sqrt{3}$ (c) $\sqrt[3]{5}$ (d) $\sqrt[3]{2}$

64. The simplest rationalisation factor of $(2\sqrt{2} - \sqrt{3})$ is

(a) $2\sqrt{2} + 3$ (b) $2\sqrt{2} + \sqrt{3}$ (c) $\sqrt{2} + \sqrt{3}$ (d) $\sqrt{2} - \sqrt{3}$

65. The rationalisation factor of $\frac{1}{2\sqrt{3} - \sqrt{5}}$ is [2011]

(a) $\sqrt{5} - 2\sqrt{3}$ (b) $\sqrt{3} + 2\sqrt{5}$ (c) $(\sqrt{3} + \sqrt{5})$ (d) $\sqrt{12} + \sqrt{5}$

66. Rationalisation of the denominator of $\frac{1}{\sqrt{5} + \sqrt{2}}$ gives

(a) $\frac{1}{\sqrt{10}}$ (b) $\sqrt{5} + \sqrt{2}$ (c) $\sqrt{5} - \sqrt{2}$ (d) $\frac{\sqrt{5} - \sqrt{2}}{3}$

67. If $x = 2 + \sqrt{3}$ then $\left(x + \frac{1}{x}\right)$ equals

(a) $-2\sqrt{3}$ (b) 2 (c) 4 (d) $4 - 2\sqrt{3}$

68. $\dfrac{1}{(3+2\sqrt{2})} = ?$

(a) $\dfrac{3-2\sqrt{2}}{17}$ (b) $\dfrac{(3-2\sqrt{2})}{13}$ (c) $(3-2\sqrt{2})$ (d) None of these

69. If $x = (7+4\sqrt{3})$ then $\left(x+\dfrac{1}{x}\right) = ?$

(a) $8\sqrt{3}$ (b) 14 (c) 49 (d) 48

70. If $\sqrt{2} = 1.41$ then $\dfrac{1}{\sqrt{2}} = ?$

(a) 0.075 (b) 0.75 (c) 0.705 (d) 7.05

71. If $\sqrt{7} = 2.646$ then $\dfrac{1}{\sqrt{7}} = ?$

(a) 0.375 (b) 0.378 (c) 0.441 (d) None of these

72. The value of $\sqrt{3-2\sqrt{2}}$ is

(a) $\sqrt{3}+\sqrt{2}$ (b) $\sqrt{3}-\sqrt{2}$ (c) $\sqrt{2}+1$ (d) $\sqrt{2}-1$

73. The value of $\sqrt{5+2\sqrt{6}}$ is

(a) $\sqrt{5}+\sqrt{6}$ (b) $\sqrt{5}-\sqrt{6}$ (c) $\sqrt{3}+\sqrt{2}$ (d) $\sqrt{3}-\sqrt{2}$

74. If $\sqrt{2} = 1.414$ then $\sqrt{\dfrac{(\sqrt{2}-1)}{(\sqrt{2}+1)}} = ?$

(a) 0.207 (b) 2.414 (c) 0.414 (d) 0.621

75. If $x = 3+\sqrt{8}$ then $\left(x^2+\dfrac{1}{x^2}\right) = ?$

(a) 34 (b) 56 (c) 28 (d) 63

Assertion-and-Reason Type

Each question consists of two statements, namely, Assertion (A) and Reason (R). For selecting the correct answer, use the following code:

(a) Both Assertion (A) and Reason (R) are true and Reason (R) is a correct explanation of Assertion (A).

(b) Both Assertion (A) and Reason (R) are true but Reason (R) is not a correct explanation of Assertion (A).

(c) Assertion (A) is true and Reason (R) is false.

(d) Assertion (A) is false and Reason (R) is true.

76.

Assertion (A)	Reason (R)
Three rational numbers between $\frac{2}{5}$ and $\frac{3}{5}$ are $\frac{9}{20}, \frac{10}{20}$ and $\frac{11}{20}$.	A rational number between two rational numbers p and q is $\frac{1}{2}(p+q)$.

The correct answer is: (a)/(b)/(c)/(d).

77.

Assertion (A)	Reason (R)
$\sqrt{3}$ is an irrational number.	Square root of a positive integer which is not a perfect square is an irrational number.

The correct answer is: (a)/(b)/(c)/(d).

78.

Assertion (A)	Reason (R)
e is an irrational number.	π is an irrational number.

The correct answer is: (a)/(b)/(c)/(d).

79.

Assertion (A)	Reason (R)
$\sqrt{3}$ is an irrational number.	The sum of a rational number and an irrational number is an irrational number.

The correct answer is: (a)/(b)/(c)/(d).

Matching of Columns

80. Match the following columns:

Column I	Column II
(a) $6.\overline{54}$ is	(p) 14
(b) π is	(q) 6
(c) The length of period of $\frac{1}{7}$ = ...	(r) a rational number
(d) If $x = (2-\sqrt{3})$ then $\left(x^2+\frac{1}{x^2}\right)$ = ...	(s) an irrational number

The correct answer is:

(a)—...... , (b)—...... , (c)—...... , (d)—...... .

81. Match the following columns:

Column I	Column II
(a) $\sqrt[4]{(81)^{-2}}$ = ...	(p) 4
(b) If $\left(\frac{a}{b}\right)^{x-2} = \left(\frac{b}{a}\right)^{x-4}$ then x = ...	(q) $\frac{2}{9}$
(c) If $x = (9+4\sqrt{5})$ then $\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)$ = ...	(r) $\frac{1}{9}$
(d) $\left(\frac{81}{16}\right)^{-3/4} \times \left(\frac{64}{27}\right)^{-1/3}$ = ?	(s) 3

The correct answer is:

(a)—...... , (b)—...... , (c)—...... , (d)—...... .

ANSWERS (MCQ)

1. (d)	**2.** (a)	**3.** (c)	**4.** (d)	**5.** (c)	**6.** (d)	**7.** (c)	**8.** (b)
9. (d)	**10.** (d)	**11.** (d)	**12.** (d)	**13.** (b)	**14.** (c)	**15.** (d)	**16.** (a)
17. (b)	**18.** (c)	**19.** (d)	**20.** (d)	**21.** (d)	**22.** (c)	**23.** (d)	**24.** (a)
25. (b)	**26.** (c)	**27.** (b)	**28.** (c)	**29.** (d)	**30.** (c)	**31.** (d)	**32.** (c)
33. (b)	**34.** (c)	**35.** (b)	**36.** (b)	**37.** (b)	**38.** (c)	**39.** (a)	**40.** (b)
41. (b)	**42.** (b)	**43.** (c)	**44.** (b)	**45.** (d)	**46.** (a)	**47.** (b)	**48.** (a)
49. (c)	**50.** (b)	**51.** (c)	**52.** (d)	**53.** (a)	**54.** (d)	**55.** (b)	**56.** (b)
57. (b)	**58.** (c)	**59.** (a)	**60.** (d)	**61.** (d)	**62.** (b)	**63.** (d)	**64.** (b)
65. (d)	**66.** (d)	**67.** (c)	**68.** (c)	**69.** (b)	**70.** (c)	**71.** (b)	**72.** (d)
73. (c)	**74.** (c)	**75.** (a)	**76.** (a)	**77.** (a)	**78.** (b)	**79.** (b)	

80. (a)–(r), (b)–(s), (c)–(q), (d)–(p) **81.** (a)–(r), (b)–(s), (c)–(p), (d)–(q)

HINTS TO SOME SELECTED QUESTIONS

62. $$\frac{5^{n+2}-6\times5^{n+1}}{13\times5^n-2\times5^{n+1}}=\frac{5^{n+1}(5-6)}{5^n(13-2\times5)}=\frac{5^{n+1}\cdot(-1)}{5^n\cdot(13-10)}$$
$$=-\frac{1}{3}\cdot\frac{5^{n+1}}{5^n}=-\frac{1}{3}\cdot5^{(n+1)-n}=-\frac{1}{3}\times5=-\frac{5}{3}.$$

72. $$\sqrt{3-2\sqrt{2}}=\sqrt{(\sqrt{2})^2+(1)^2-2\times\sqrt{2}\times1}=\sqrt{(\sqrt{2}-1)^2}=(\sqrt{2}-1).$$

73. $$\sqrt{5+2\sqrt{6}}=\sqrt{(\sqrt{2})^2+(\sqrt{3})^2+2\times\sqrt{2}\times\sqrt{3}}=\sqrt{(\sqrt{2}+\sqrt{3})^2}=(\sqrt{2}+\sqrt{3}).$$

74. $$\sqrt{\frac{(\sqrt{2}-1)}{(\sqrt{2}+1)}}=\sqrt{\frac{(\sqrt{2}-1)}{(\sqrt{2}+1)}\times\frac{(\sqrt{2}-1)}{(\sqrt{2}-1)}}=\sqrt{\frac{(\sqrt{2}-1)^2}{(\sqrt{2})^2-(1)^2}}=\sqrt{\frac{(\sqrt{2}-1)^2}{2-1}}$$
$$=\sqrt{(\sqrt{2}-1)^2}=\sqrt{2}-1=1.414-1=0.414.$$

VERY-SHORT-ANSWER QUESTIONS

1. What can you say about the sum of a rational number and an irrational number? [2014]

2. Solve $(3-\sqrt{11})(3+\sqrt{11})$. [2014]

3. The number $\frac{665}{625}$ will terminate after how many decimal places? [2015]

4. Find the value of $(1296)^{0.17} \times (1296)^{0.08}$.
5. Simplify $6\sqrt{36} + 5\sqrt{12}$.
6. Find an irrational number between 5 and 6.
7. Find the value of $\frac{21\sqrt{12}}{10\sqrt{27}}$. [2015]
8. Rationalise $\frac{1}{\sqrt{3}+\sqrt{2}}$. [2013]
9. Solve for x $\left(\frac{2}{5}\right)^{2x-2} = \frac{32}{3125}$.
10. Simplify $(32)^{1/5} + (-7)^0 + (64)^{1/2}$. [2010]
11. Evaluate $\left(\frac{81}{49}\right)^{-3/2}$. [2011]
12. Simplify $\sqrt[4]{81x^8y^4z^{16}}$. [2014]
13. If $a = 1$, $b = 2$ then find the value of $(a^b + b^a)^{-1}$.
14. Simplify $\left(\frac{3125}{243}\right)^{4/5}$.
15. Give an example of two irrational numbers whose sum as well as product is rational.
16. Is the product of a rational and an irrational number always irrational? Give an example.
17. Give an example of a number x such that x^2 is an irrational number and x^3 is a rational number.
18. Write the reciprocal of $(2+\sqrt{3})$.
19. If $\sqrt{10} = 3.162$, find the value of $\frac{1}{\sqrt{10}}$.
20. Simplify $(2\sqrt{5} + 3\sqrt{2})^2$.
21. If $10^x = 64$, find the value of $10^{\left(\frac{x}{2}+1\right)}$.
22. Evaluate $\frac{2^n + 2^{n-1}}{2^{n+1} - 2^n}$.
23. Simplify $\left[\left\{(256)^{-\frac{1}{2}}\right\}^{-\frac{1}{4}}\right]^2$.

ANSWERS

(VERY-SHORT-ANSWER QUESTIONS)

1. Irrational **2.** -2 **3.** 3 **4.** 6 **5.** $16\sqrt{3}$ **6.** $\sqrt{30}$ **7.** $\frac{7}{5}$

8. $\sqrt{3}-\sqrt{2}$ **9.** $x=\dfrac{7}{2}$ **10.** 11 **11.** $\dfrac{343}{729}$ **12.** $3x^2yz^4$ **13.** $\dfrac{1}{3}$ **14.** $\dfrac{625}{81}$

15. $(2+\sqrt{3})$ and $(2-\sqrt{3})$ **16.** Yes; $2\times\sqrt{3}=2\sqrt{3}$, which is irrational

17. $\sqrt[3]{3}$ **18.** $(2-\sqrt{3})$ **19.** 0.3162 **20.** $(38+12\sqrt{10})$ **21.** 80

22. $\dfrac{3}{2}$ **23.** 4

HINTS TO SOME SELECTED QUESTIONS

21. $10^{\left(\frac{x}{2}+1\right)}=10^{\frac{x}{2}}\cdot 10=(10^x)^{\frac{1}{2}}\times 10=(64)^{\frac{1}{2}}\times 10=8\times 10=80.$

22. $\dfrac{2^n+2^{n-1}}{2^{n+1}-2^n}=\dfrac{2^{n-1}\cdot(2+1)}{2^n\cdot(2-1)}=\dfrac{3\cdot 2^{n-1}}{2^n}=\dfrac{3}{2^{n-(n-1)}}=\dfrac{3}{2}\cdot$

23. $\left[\left\{(256)^{-\frac{1}{2}}\right\}^{-\frac{1}{4}}\right]^2=\left[\left\{\left(\dfrac{1}{256}\right)^{\frac{1}{2}}\right\}^{-\frac{1}{4}}\right]^2=\left[\left\{\dfrac{1}{16}\right\}^{-\frac{1}{4}}\right]^2$

$=\left[(16)^{\frac{1}{4}}\right]^2=\left[(2^4)^{\frac{1}{4}}\right]^2=2^2=4.$

2 Polynomials

In our previous classes, we have learnt about expressions and various operations on them. Here, we shall review these concepts and extend them to particular types of expressions, known as polynomials.

We shall come across two types of symbols, namely, constants and variables, defined below.

CONSTANTS *A symbol having a fixed numerical value is called a constant.*

Examples $9, -6, \frac{4}{7}, \sqrt{2}, \pi$ are all constants.

VARIABLES *A symbol which may be assigned different numerical values is known as a variable.*

Example We know that the circumference of a circle is given by the formula $C = 2\pi r$, where r is the radius of the circle.

Here, 2 and π are constants, while C and r are variables.

ALGEBRAIC EXPRESSIONS

A combination of constants and variables, connected by some or all of the operations $+, -, \times$ and $\div$, is known as an algebraic expression.

TERMS OF AN ALGEBRAIC EXPRESSION

The several parts of an algebraic expression separated by + or – operations are called the terms of the expression.

Examples (i) $5x^2 - 9x + 4$ is an algebraic expression containing three terms, namely $5x^2$, $-9x$ and 4.

(ii) $4x^3 + \sqrt{2}x^2 - \frac{5}{9}x - 8$ is an algebraic expression containing four terms, namely $4x^3$, $\sqrt{2}x^2$, $\frac{-5}{9}x$ and -8.

(iii) $y^6 - 64$ is an expression containing two terms, namely y^6 and -64.

POLYNOMIALS IN ONE VARIABLE

An expression of the form $a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n$, where $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ are real numbers, $a_n \neq 0$ and n is a non-negative integer, is called a polynomial in x of degree n.

Here, a_0 is called the *constant term* of the given polynomial and $a_1, a_2, a_3, \ldots, a_{n-1}, a_n$ are called the *coefficients* of $x, x^2, x^3, \ldots, x^{n-1}$ and x^n respectively.

Examples (i) $f(x) = 7x^2 - 4x + 3$ is a polynomial in x of degree 2.

(ii) $g(x) = \sqrt{3}x^3 + 5x^2 - \frac{3}{4}x - 9$ is a polynomial in x of degree 3.

(iii) $h(x) = x^4 - 2x^3 + 5x + \sqrt{2}$ is a polynomial in x of degree 4.

(iv) $p(y) = 2y^5 - 6y^4 + 8y^2 - \frac{5}{7}$ is a polynomial in y of degree 5.

(v) $q(z) = z^9 - 1$ is a polynomial in z of degree 9.

Standard Form of a Polynomial *A polynomial in x expressed either in ascending powers of x or in descending powers of x is said to be in standard form.*

An illustrative example

Give reasons to show that none of the following expressions is a polynomial.

(i) $f(x) = x + \frac{1}{x}$ *(ii)* $g(x) = \sqrt{x} - 3$ *(iii)* $h(y) = \sqrt[3]{y} - 6$

(iv) $p(x) = \frac{(x-1)(x-3)}{x}$ *(v)* $q(x) = \frac{1}{x+2}$ *(vi)* $r(x) = \frac{x+3}{x+4}$

SOLUTION (i) $f(x) = x + \frac{1}{x}$ may be written as $f(x) = x + x^{-1}$.

Here, in one term, the exponent of x is –1, which is a negative integer.

$\therefore$ $f(x)$ is not a polynomial.

(ii) $g(x) = \sqrt{x} - 3$ may be written as $g(x) = x^{1/2} - 3$.

Here, in one term, the exponent of x is $\frac{1}{2}$, which is not a non-negative integer.

$\therefore$ $g(x)$ is not a polynomial.

(iii) $h(y) = \sqrt[3]{y} - 6$ may be written as $h(y) = y^{1/3} - 6$.

Here, in one term, the exponent of y is $\frac{1}{3}$, which is not a non-negative integer.

$\therefore$ $h(y)$ is not a polynomial.

(iv) $p(x) = \frac{(x-1)(x-3)}{x}$

$\Rightarrow$ $p(x) = \frac{x^2 - 4x + 3}{x} = \left(x - 4 + \frac{3}{x}\right) = (x - 4 + 3x^{-1})$.

Here, in one term, the exponent of x is –1, which is a negative integer.

$\therefore$ $p(x)$ is not a polynomial.

(v) $q(x) = \dfrac{1}{(x+2)} = (x+2)^{-1}$, which is not a polynomial in $(x+2)$ and therefore, it is not a polynomial in x.

(vi) On dividing $(x+3)$ by $(x+4)$, we get 1 as quotient and –1 as remainder.

$$\begin{array}{r|l} x+4 & x+3\ (1 \\ & x+4 \\ & -\ \ - \\ \hline & \quad -1 \end{array}$$

$\therefore$ we may write,

$$\frac{x+3}{x+4} = 1 - \frac{1}{(x+4)} = 1 - (x+4)^{-1},$$

which is not a polynomial.

Hence, $r(x) = \dfrac{x+3}{x+4}$ is not a polynomial.

POLYNOMIALS OF VARIOUS DEGREES

(i) **LINEAR POLYNOMIAL** *A polynomial of degree* 1 *is called a linear polynomial.*

A linear polynomial in x is of the form, $f(x) = ax + b$, *where a and b are real numbers and* $a \neq 0$.

Examples (i) $5x + 7$ is a linear polynomial in x.

(ii) $\sqrt{2}y - 6$ is a linear polynomial in y.

(iii) $-8z$ is a linear polynomial in z.

(ii) **QUADRATIC POLYNOMIAL** *A polynomial of degree* 2 *is called a quadratic polynomial.*

A quadratic polynomial in x is of the form $f(x) = ax^2 + bx + c$, *where a, b, c are real numbers and* $a \neq 0$.

Examples (i) $3x^2 - 8x + 9$ is a quadratic polynomial in x.

(ii) $4y^2 - 9y + 5$ is a quadratic polynomial in y.

(iii) $z^2 - 11z + 18$ is a quadratic polynomial in z.

(iii) **CUBIC POLYNOMIAL** *A polynomial of degree* 3 *is called a cubic polynomial.*

A cubic polynomial in x is of the form $ax^3 + bx^2 + cx + d$, *where a, b, c, d are real numbers and* $a \neq 0$.

Examples (i) $4x^3 - x^2 + 6x + 3$ is a cubic polynomial in x.

(ii) $7y^3 + 5y^2 - 8y + 9$ is a cubic polynomial in y.

(iii) $8 - z^3$ is a cubic polynomial in z.

(iv) **BIQUADRATIC POLYNOMIAL** *A polynomial of degree 4 is called a biquadratic or quartic polynomial.*

A biquadratic polynomial is of the form $ax^4 + bx^3 + cx^2 + dx + e$, *where a, b, c, d, e are real numbers and* $a \neq 0$.

Examples (i) $x^4 - 2x^3 + 3x^2 - 4x + 5$ is a biquadratic polynomial in x.

(ii) $3y^4 + 8y^3 - 5y^2 + 7$ is a biquadratic polynomial in y.

(iii) $6z^4 - 9z^2 + 1$ is a biquadratic polynomial in z.

NUMBER OF TERMS IN A POLYNOMIAL

(i) **MONOMIAL** *A polynomial containing one term is called a monomial.* (*'Mono'* means *'one'*.)

Examples $9, -14, 6x, -8x^2, 5x^3, 2x^4$, etc., are all monomials.

(ii) **BINOMIAL** *A polynomial containing two nonzero terms is called a binomial.* (*'Bi'* means *'two'*.)

Examples $(9 + 4x), (x - 3x^2), (8 + x^3), (-x^4 + 7)$ are all binomials.

(iii) **TRINOMIAL** *A polynomial containing three nonzero terms is called a trinomial.* (*'Tri'* means *'three'*.)

Examples $(x^2 + 2x - 3), (2x^3 + 5x^2 - 4), (-7x^4 + 5x^2 + 6), (5x^6 - 3x^4 + x)$ are all trinomials.

CONSTANT POLYNOMIAL *A polynomial containing one term, consisting of a nonzero constant, is called a constant polynomial.*

In general, *every nonzero real number is a constant polynomial. The degree of a nonzero constant polynomial is zero.*

Examples $6, -8, \frac{9}{14}, \frac{-7}{12}, \pi$, etc., are all constant polynomials.

ZERO POLYNOMIAL *A polynomial consisting of one term, namely zero, is called a zero polynomial.*

The degree of a zero polynomial is not defined.

SOLVED EXAMPLES

EXAMPLE 1 *Which of the following expressions are polynomials? In case of a polynomial write its degree.*

(i) $x^3 - 5x + 2$ (ii) $y^2 + \sqrt{2}y - \sqrt{5}$ (iii) $2\sqrt{x} + 7$ (iv) -6

(v) $4t^2 + \frac{1}{6}t + 2\sqrt{3}$ (vi) $z^2 + \frac{5}{z^2} + 1$ (vii) $\frac{1}{3x}$

(viii) $1 - \sqrt{5x}$ (ix) $\frac{1}{4x^{-2}} + 3x + 5$ (x) $\frac{6\sqrt{x} + x^{3/2}}{\sqrt{x}}$

SOLUTION (i) $x^3 - 5x + 2$ is an expression having only non-negative integral powers of x. So, it is a polynomial.

The highest power of x is 3.

So, it is a polynomial of degree 3.

(ii) $y^2 + \sqrt{2}y - \sqrt{5}$ is an expression having only non-negative integral powers of y. So, it is a polynomial.

The highest power of y is 2.

So, it is a polynomial of degree 2.

(iii) $2\sqrt{x} + 7$ is an expression in which one term, namely $2\sqrt{x}$ (written as $2x^{1/2}$), has rational power of x. So, it is not a polynomial.

(iv) –6 is clearly a constant polynomial of degree 0.

(v) $4t^2 + \frac{1}{6}t + 2\sqrt{3}$ is an expression having only non-negative integral powers of t. So, it is a polynomial in t.

The highest power of t is 2. So, it is a polynomial of degree 2.

(vi) $z^2 + \frac{5}{z^2} + 1$ may be written as $z^2 + 5z^{-2} + 1$.

Here, in one term, the exponent of z is –2, which is a negative integer.

So, the given expression is not a polynomial.

(vii) $\frac{1}{3x}$ may be written as $\frac{1}{3}x^{-1}$, which is not a polynomial.

(viii) $1 - \sqrt{5x}$ may be written as $1 - \sqrt{5}\,x^{1/2}$.

Thus, in one term, the exponent of x is $\frac{1}{2}$, which is rational.

So, the given expression is not a polynomial.

(ix) $\frac{1}{4x^{-2}} + 3x + 5$ may be written as $\frac{1}{4}x^2 + 3x + 5$.

Thus, it is an expression having only non-negative integral powers of x.

So, the given expression is a polynomial of degree 2.

(x) $\frac{6\sqrt{x} + x^{3/2}}{\sqrt{x}}$ may be written as $\frac{6\sqrt{x}}{\sqrt{x}} + \frac{x^{3/2}}{x^{1/2}} = 6 + x^{\left(\frac{3}{2} - \frac{1}{2}\right)} = 6 + x$, which is clearly a polynomial of degree 1.

EXAMPLE 2 *Classify the following as constant, linear, quadratic, cubic and quartic polynomials.*

(i) $x - x^3$ *(ii)* $y^4 - y$ *(iii)* $y + y^2 + 4$

(iv) $\sqrt{2}x - 1$ *(v)* $5x^3$ *(vi)* 3

(vii) t^2 *(viii)* $2 + x$ *(ix)* $5t - \sqrt{7}$

SOLUTION (i) $x - x^3$ is a polynomial of degree 3. So, it is a cubic polynomial.

(ii) $y^4 - y$ is a polynomial of degree 4. So, it is a quartic polynomial.

(iii) $y + y^2 + 4$ is a polynomial of degree 2. So, it is a quadratic polynomial.

(iv) $\sqrt{2}x - 1$ is a polynomial of degree 1. So, it is a linear polynomial.

(v) $5x^3$ is a polynomial of degree 3. So, it is a cubic polynomial.

(vi) Clearly, 3 is a constant polynomial and its degree is 0.

(vii) t^2 is a polynomial of degree 2. So, it is a quadratic polynomial.

(viii) $2 + x$ is a polynomial of degree 1. So, it is a linear polynomial.

(ix) $5t - \sqrt{7}$ is a polynomial of degree 1. So, it is a linear polynomial.

EXAMPLE 3 *Write the coefficient of* x^2 *in each of the following.*

(i) $(x-1)(3x-4)$ *(ii)* $(2x-5)(2x^2-3x+1)$ *(iii)* $5x - 3$

(iv) $\frac{\pi}{2}x^2 + x$

SOLUTION We have:

(i) $(x-1)(3x-4) = 3x^2 - 7x + 4$.

$\therefore$ coefficient of x^2 in $(x-1)(3x-4)$ is 3.

(ii) $(2x-5)(2x^2-3x+1) = 4x^3 - 16x^2 + 17x - 5$.

$\therefore$ coefficient of x^2 in $(2x-5)(2x^2-3x+1)$ is –16.

(iii) Coefficient of x^2 in $5x - 3$ is 0.

(iv) Coefficient of x^2 in $\frac{\pi}{2}x^2 + x$ is $\frac{\pi}{2}$.

EXAMPLE 4 *Give an example of a polynomial, which is*

(i) a monomial of degree 1 *(ii) a monomial of degree* 5

(iii) a binomial of degree 20 *(iv) a trinomial of degree* 3

SOLUTION (i) x is a monomial of degree 1.

(ii) x^5 is a monomial of degree 5.

(iii) $(5+x^{20})$ is a binomial of degree 20.

(iv) $(2+3x+5x^3)$ is a trinomial of degree 3.

EXAMPLE 5 *For the polynomial* $\frac{x^3+2x+3}{5}-\frac{7}{2}x^2-x^6$, *write*

(i) the degree of the polynomial,

(ii) the coefficient of x^3,

(iii) the coefficient of x^6,

(iv) the constant term.

SOLUTION The given polynomial may be written as

$$-x^6+\frac{1}{5}x^3-\frac{7}{2}x^2+\frac{2}{5}x+\frac{3}{5}\cdot$$

(i) The degree of the given polynomial is 6.

(ii) The coefficient of x^3 is $\frac{1}{5}\cdot$

(iii) The coefficient of x^6 is –1.

(iv) The constant term is $\frac{3}{5}\cdot$

EXAMPLE 6 *Determine the degree of each of the following polynomials.*

(i) $x^3-9x+3x^5$ *(ii)* $y^3(1-y^4)$ *(iii)* $-2x+1$ *(iv)* -10

SOLUTION (i) The given polynomial is $3x^5+x^3-9x$.

Clearly, it is a polynomial of degree 5.

(ii) $y^3(1-y^4)=y^3-y^7=-y^7+y^3$, which is a polynomial of degree 7.

(iii) $-2x+1$ is a polynomial of degree 1.

(iv) -10 is a constant polynomial. So, its degree is 0.

EXERCISE 2A

1. Which of the following expressions are polynomials? In case of a polynomial, write its degree.

(i) $x^5-2x^3+x+\sqrt{3}$ (ii) $y^3+\sqrt{3}y$ (iii) $t^2-\frac{2}{5}t+\sqrt{5}$

(iv) $x^{100}-1$ (v) $\frac{1}{\sqrt{2}}x^2-\sqrt{2}x+2$ (vi) $x^{-2}+2x^{-1}+3$

(vii) 1 (viii) $\frac{-3}{5}$ (ix) $\frac{x^2}{2} - \frac{2}{x^2}$

(x) $\sqrt[3]{2}x^2 - 8$ (xi) $\frac{1}{2x^2}$ (xii) $\frac{1}{\sqrt{5}}x^{1/2} + 1$

(xiii) $\frac{3}{5}x^2 - \frac{7}{3}x + 9$ (xiv) $x^4 - x^{3/2} + x - 3$ (xv) $2x^3 + 3x^2 + \sqrt{x} - 1$

2. Identify constant, linear, quadratic, cubic and quartic polynomials from the following.

(i) $-7 + x$ (ii) $6y$ (iii) $-z^3$

(iv) $1 - y - y^3$ (v) $x - x^3 + x^4$ (vi) $1 + x + x^2$

(vii) $-6x^2$ (viii) -13 (ix) $-p$

3. Write

(i) the coefficient of x^3 in $x + 3x^2 - 5x^3 + x^4$.

(ii) the coefficient of x in $\sqrt{3} - 2\sqrt{2}x + 6x^2$.

(iii) the coefficient of x^2 in $2x - 3 + x^3$.

(iv) the coefficient of x in $\frac{3}{8}x^2 - \frac{2}{7}x + \frac{1}{6}$.

(v) the constant term in $\frac{\pi}{2}x^2 + 7x - \frac{2}{5}\pi$.

4. Determine the degree of each of the following polynomials.

(i) $\frac{4x - 5x^2 + 6x^3}{2x}$ (ii) $y^2(y - y^3)$ (iii) $(3x - 2)(2x^3 + 3x^2)$

(iv) $-\frac{1}{2}x + 3$ (v) -8 (vi) $x^{-2}(x^4 + x^2)$

5. (i) Give an example of a monomial of degree 5.

(ii) Give an example of a binomial of degree 8.

(iii) Given an example of a trinomial of degree 4.

(iv) Give an example of a monomial of degree 0.

6. Rewrite each of the following polynomials in standard form.

(i) $x - 2x^2 + 8 + 5x^3$ (ii) $\frac{2}{3} + 4y^2 - 3y + 2y^3$

(iii) $6x^3 + 2x - x^5 - 3x^2$ (iv) $2 + t - 3t^3 + t^4 - t^2$

ANSWERS (EXERCISE 2A)

1. (i) Polynomial, degree 5 (ii) Polynomial, degree 3

(iii) Polynomial, degree 2 (iv) Polynomial, degree 100

(v) Polynomial, degree 2 (vii) Polynomial, degree 0
(viii) Polynomial, degree 0 (x) Polynomial, degree 2
(xiii) Polynomial, degree 2

2. (i) Linear (ii) Linear (iii) Cubic (iv) Cubic
(v) Quartic (vi) Quadratic (vii) Quadratic (viii) Constant
(ix) Linear

3. (i) -5 (ii) $-2\sqrt{2}$ (iii) 0 (iv) $-\frac{2}{7}$ (v) $-\frac{2}{5}\pi$

4. (i) 2 (ii) 5 (iii) 4 (iv) 1 (v) 0
(vi) 2

5. (i) $2x^5$ (ii) $x+x^8$ (iii) x^4+x^3+x (iv) 5

6. (i) $5x^3-2x^2+x+8$ (ii) $2y^3+4y^2-3y+\frac{2}{3}$
(iii) $-x^5+6x^3-3x^2+2x$ (iv) $t^4-3t^3-t^2+t+2$

VALUE OF A POLYNOMIAL

The value of a polynomial $p(x)$ at $x=\alpha$ is obtained by putting $x=\alpha$ in $p(x)$ and it is denoted by $p(\alpha)$.

EXAMPLE 1 *(i) If $p(x)=3x^2-5x+6$, find $p(2)$.*
(ii) If $q(x)=x^2-2\sqrt{2}x+1$, find $q(2\sqrt{2})$.
(iii) If $r(x)=5x-4x^2+3$, find $r(-1)$.

SOLUTION We have:

(i) $p(x)=3x^2-5x+6$

$\Rightarrow\quad p(2)=3\times 2^2-5\times 2+6=(12-10+6)=8.$

$\therefore\quad p(2)=8.$

(ii) $q(x)=x^2-2\sqrt{2}x+1$

$\Rightarrow\quad q(2\sqrt{2})=(2\sqrt{2})^2-(2\sqrt{2})(2\sqrt{2})+1=(8-8+1)=1.$

$\therefore\quad q(2\sqrt{2})=1.$

(iii) $r(x)=5x-4x^2+3 \Rightarrow r(x)=-4x^2+5x+3.$

$\therefore\quad r(-1)=(-4)\times(-1)^2+5\times(-1)+3=(-4)\times 1-5+3$

$=(-4-5+3)=-6.$

$\therefore\quad r(-1)=-6.$

ZEROS OF A POLYNOMIAL

Let $p(x)$ be a polynomial. If $p(\alpha) = 0$ then we say that α is a zero of the polynomial $p(x)$.

REMARK Finding the zeros of a polynomial $p(x)$ means solving the equation $p(x) = 0$.

EXAMPLE 2 *If $p(x) = x^3 + 2x^2 - 5x - 6$, find $p(2), p(-1), p(-3)$ and $p(0)$. What do you conclude about the zeros of $p(x)$? Is 0 a zero of $p(x)$?*

SOLUTION Given, $p(x) = x^3 + 2x^2 - 5x - 6$.

$\therefore$ $p(2) = 2^3 + 2 \times 2^2 - 5 \times 2 - 6 = (8 + 8 - 10 - 6) = 0.$

$p(-1) = (-1)^3 + 2 \times (-1)^2 - 5 \times (-1) - 6 = (-1 + 2 + 5 - 6) = 0.$

$p(-3) = (-3)^3 + 2 \times (-3)^2 - 5 \times (-3) - 6$

$= (-27 + 18 + 15 - 6) = 0.$

$p(0) = 0^3 + 2 \times 0^2 - 5 \times 0 - 6 = (0 + 0 - 0 - 6) = -6 \neq 0.$

Since $p(2) = 0$, $p(-1) = 0$ and $p(-3) = 0$, it follows that 2, −1 and –3 are the zeros of $p(x)$.

But, $p(0) = -6 \neq 0$.

Therefore, 0 is not a zero of $p(x)$.

EXAMPLE 3 *Check whether −2 and 2 are the zeros of the polynomial $x + 2$.*

SOLUTION Let $p(x) = x + 2$. Then,

$p(-2) = (-2 + 2) = 0$ and $p(2) = (2 + 2) = 4 \neq 0$.

$\therefore$ –2 is a zero of $x + 2$ and 2 is not a zero of $x + 2$.

EXAMPLE 4 *Find a zero of the polynomial*

(i) $p(x) = x - 3$ *(ii)* $q(x) = 3x + 2$

SOLUTION We have:

(i) $p(x) = 0 \Rightarrow x - 3 = 0 \Rightarrow x = 3.$

$\therefore$ 3 is a zero of the polynomial $p(x) = x - 3$.

(ii) $q(x) = 0 \Rightarrow 3x + 2 = 0$

$\Rightarrow 3x = -2 \Rightarrow x = \frac{-2}{3}.$

$\therefore$ $\frac{-2}{3}$ is a zero of the polynomial $q(x) = 3x + 2$.

EXAMPLE 5 *Find a zero of the polynomial $p(x) = ax + b$, $a \neq 0$ and a, b are real numbers.*

SOLUTION Let $p(x) = 0$. Then,

$$p(x) = 0 \Rightarrow ax + b = 0$$

$$\Rightarrow ax = -b \Rightarrow x = \frac{-b}{a}.$$

$\therefore$ $\frac{-b}{a}$ is a zero of the polynomial, $p(x) = ax + b$.

EXAMPLE 6 *If $p(x) = x + 3$ then find $p(x) + p(-x)$.*

SOLUTION We have, $p(x) = x + 3$. ... (i)

Replacing x by $-x$, we get

$$p(-x) = -x + 3. \quad \text{... (ii)}$$

Adding the corresponding sides of (i) and (ii), we get

$$p(x) + p(-x) = 6.$$

SOME IMPORTANT OBSERVATIONS

(i) A constant polynomial does not have any zero.

(ii) Every linear polynomial has one and only one zero.

(iii) 0 may or may not be the zero of a given polynomial.

(iv) A polynomial can have repeated zeros.
For example, $p(x) = x^2 - 2x + 1$ has 1 as repeated zeros.

(v) Number of zeros of a polynomial cannot exceed its degree.

EXERCISE 2B

1. If $p(x) = 5 - 4x + 2x^2$, find (i) $p(0)$, (ii) $p(3)$, (iii) $p(-2)$.

2. If $p(y) = 4 + 3y - y^2 + 5y^3$, find (i) $p(0)$, (ii) $p(2)$, (iii) $p(-1)$.

3. If $f(t) = 4t^2 - 3t + 6$, find (i) $f(0)$, (ii) $f(4)$, (iii) $f(-5)$.

4. If $p(x) = x^3 - 3x^2 + 2x$, find $p(0)$, $p(1)$, $p(2)$. What do you conclude?

5. If $p(x) = x^3 + x^2 - 9x - 9$, find $p(0)$, $p(3)$, $p(-3)$ and $p(-1)$. What do you conclude about the zeros of $p(x)$? Is 0 a zero of $p(x)$?

6. Verify that

(i) 4 is a zero of the polynomial, $p(x) = x - 4$.

(ii) –3 is a zero of the polynomial, $q(x) = x + 3$.

(iii) $\frac{2}{5}$ is a zero of the polynomial, $f(x) = 2 - 5x$.

(iv) $\frac{-1}{2}$ is a zero of the polynomial, $g(y) = 2y + 1$.

7. Verify that

(i) 1 and 2 are the zeros of the polynomial, $p(x) = x^2 - 3x + 2$.

(ii) 2 and –3 are the zeros of the polynomial, $q(x) = x^2 + x - 6$.

(iii) 0 and 3 are the zeros of the polynomial, $r(x) = x^2 - 3x$.

8. Find the zero of the polynomial:

(i) $p(x) = x - 5$ (ii) $q(x) = x + 4$ (iii) $r(x) = 2x + 5$

(iv) $f(x) = 3x + 1$ (v) $g(x) = 5 - 4x$ (vi) $h(x) = 6x - 2$

(vii) $p(x) = ax, a \neq 0$ (viii) $q(x) = 4x$

9. If 2 and 0 are the zeros of the polynomial $f(x) = 2x^3 - 5x^2 + ax + b$ then find the values of a and b.

HINT $f(2) = 0$ and $f(0) = 0$.

ANSWERS (EXERCISE 2B)

1. (i) 5 (ii) 11 (iii) 21 **2.** (i) 4 (ii) 46 (iii) –5

3. (i) 6 (ii) 58 (iii) 121

4. $p(0) = 0, p(1) = 0, p(2) = 0$.

Conclusion: 0, 1 and 2 are the zeros of $p(x) = x^3 - 3x^2 + 2x$.

5. $p(0) = -9, p(3) = 0, p(-3) = 0$ and $p(-1) = 0$.

$3, -3$ and -1 are the zeros of $p(x)$.

0 is not a zero of $p(x)$.

8. (i) 5 (ii) –4 (iii) $\frac{-5}{2}$ (iv) $\frac{-1}{3}$ (v) $\frac{5}{4}$ (vi) $\frac{1}{3}$ (vii) 0 (viii) 0

9. $a = 2, b = 0$

DIVISION ALGORITHM IN POLYNOMIALS

Let $p(x)$ and $g(x)$ be two given polynomials such that degree $p(x) \geq$ degree $g(x)$. On dividing $p(x)$ by $g(x)$, let $q(x)$ be the quotient and $r(x)$ be the remainder. Then, in general, we have

dividend = (*divisor* × *quotient*) + *remainder*,

i.e., $p(x) = g(x) \cdot q(x) + r(x)$, where $r(x) = 0$ or degree $r(x) <$ degree $g(x)$.

EXAMPLE 1 *Verify division algorithm for the polynomials*

$p(x) = 3x^4 - 4x^3 - 3x - 1$ *and* $g(x) = x - 2$.

Find $p(2)$. *What do you observe?*

SOLUTION By long division, we have

$$\begin{array}{l} x-2\overline{)\,3x^4-4x^3+0x^2-3x-1\,}(3x^3+2x^2+4x+5 \\ \quad\;\; 3x^4-6x^3 \\ \quad\;\; -\quad + \\ \hline \qquad\quad 2x^3+0x^2-3x-1 \\ \qquad\quad 2x^3-4x^2 \\ \qquad\quad -\quad + \\ \hline \qquad\qquad 4x^2-3x-1 \\ \qquad\qquad 4x^2-8x \\ \qquad\qquad -\quad + \\ \hline \qquad\qquad\quad 5x-1 \\ \qquad\qquad\quad 5x-10 \\ \qquad\qquad\quad -\quad + \\ \hline \qquad\qquad\qquad 9 \end{array}$$

$\therefore$ quotient, $q(x) = 3x^3 + 2x^2 + 4x + 5$ and remainder, $r(x) = 9$.

Now, $g(x) \times q(x) + r(x) = (x-2)(3x^3 + 2x^2 + 4x + 5) + 9$

$$= 3x^4 - 4x^3 - 3x - 10 + 9$$

$$= 3x^4 - 4x^3 - 3x - 1 = p(x).$$

$\therefore$ $p(x) = g(x) \times q(x) + r(x)$,

where degree $r(x) = 0 < 1 =$ degree $g(x)$.

Thus, division algorithm is verified.

Now, $p(2) = 3 \times 2^4 - 4 \times 2^3 - 3 \times 2 - 1 = 48 - 32 - 6 - 1 = 9$.

Observation When $p(x)$ is divided by $(x-2)$, then the remainder is $p(2)$.

EXAMPLE 2 *Verify division algorithm for the polynomials*

$p(x) = x^3 + x^2 + 2x + 3$ *and* $g(x) = x + 2$.

Find $p(-2)$. *What do you observe?*

SOLUTION By long division, we have

$$\begin{array}{l} x+2\overline{)\,x^3+\;x^2+2x+3\,}(x^2-x+4 \\ \quad\;\; x^3+2x^2 \\ \quad\;\; -\quad - \\ \hline \qquad\; -x^2+2x+3 \\ \qquad\; -x^2-2x \\ \qquad\; +\quad + \\ \hline \qquad\qquad 4x+3 \\ \qquad\qquad 4x+8 \\ \qquad\qquad -\quad - \\ \hline \qquad\qquad\; -5 \end{array}$$

$\therefore$ quotient, $q(x) = x^2 - x + 4$ and remainder, $r(x) = -5$.

Now, $g(x) \times q(x) + r(x) = (x+2)(x^2 - x + 4) - 5$

$= x^3 + x^2 + 2x + 8 - 5$

$= x^3 + x^2 + 2x + 3 = p(x)$.

$\therefore$ $p(x) = g(x) \times q(x) + r(x)$,

where $r(x) = -5$ and degree $r(x) = 0 < 1 =$ degree $g(x)$.

Thus, division algorithm is verified.

Now, $p(-2) = (-2)^3 + (-2)^2 + 2 \times (-2) + 3 = (-8 + 4 - 4 + 3) = -5$.

Observation When $p(x)$ is divided by $(x+2)$, then the remainder is $p(-2)$.

REMAINDER THEOREM *Let $p(x)$ be a polynomial of degree 1 or more and let α be any real number. If $p(x)$ is divided by $(x - \alpha)$ then the remainder is $p(\alpha)$.*

PROOF Let $p(x)$ be a polynomial of degree 1 or more.

Suppose that when $p(x)$ is divided by $(x - \alpha)$ then the quotient is $q(x)$ and remainder is $r(x)$.

By division algorithm, we have

$p(x) = (x - \alpha) \cdot q(x) + r(x)$,

where degree $r(x) <$ degree $(x - \alpha) = 1$ [$\because$ degree $(x - \alpha) = 1$].

But, degree $r(x) < 1 \Rightarrow$ degree $r(x) = 0$

$\Rightarrow r(x)$ is a constant, equal to r (say).

$\therefore$ $p(x) = (x - \alpha) \cdot q(x) + r$. ... (i)

Putting $x = \alpha$ on both sides of (i), we get

$p(\alpha) = 0 \times q(x) + r \Rightarrow r = p(\alpha)$.

Thus, when $p(x)$ is divided by $(x - \alpha)$, then the remainder is $p(\alpha)$.

SOLVED EXAMPLES

EXAMPLE 1 *Find the remainder when the polynomial $p(x) = x^4 + 2x^3 - 3x^2 + x - 1$ is divided by $g(x) = x - 2$.*

SOLUTION $g(x) = 0 \Rightarrow x - 2 = 0 \Rightarrow x = 2$.

By the remainder theorem, we know that when $p(x)$ is divided by $(x - 2)$ then the remainder is $p(2)$.

Now, $p(2) = (2^4 + 2 \times 2^3 - 3 \times 2^2 + 2 - 1)$

$= (16 + 16 - 12 + 2 - 1) = 21$.

Hence, the required remainder is 21.

EXAMPLE 2 *Find the remainder when the polynomial* $p(x) = x^3 - 3x^2 + 4x + 50$ *is divided by* $g(x) = x + 3$.

SOLUTION $g(x) = 0 \Rightarrow x + 3 = 0 \Rightarrow x = -3.$

By the remainder theorem, we know that when $p(x)$ is divided by $(x + 3)$ then the remainder is $p(-3)$.

Now, $p(-3) = [(-3)^3 - 3 \times (-3)^2 + 4 \times (-3) + 50]$

$$= (-27 - 27 - 12 + 50) = -16.$$

Hence, the required remainder is -16.

EXAMPLE 3 *Find the remainder when the polynomial* $p(x) = 4x^3 - 12x^2 + 14x - 3$ *is divided by* $g(x) = (2x - 1)$.

SOLUTION $g(x) = 0 \Rightarrow 2x - 1 = 0 \Rightarrow x = \frac{1}{2}\cdot$

By the remainder theorem, we know that when $p(x)$ is divided by $(2x - 1)$ then the remainder is $p\left(\frac{1}{2}\right)\cdot$

Now, $p\left(\frac{1}{2}\right) = \left[4 \times \left(\frac{1}{2}\right)^3 - 12 \times \left(\frac{1}{2}\right)^2 + 14 \times \frac{1}{2} - 3\right]$

$$= \left(\frac{1}{2} - 3 + 7 - 3\right) = \frac{3}{2}\cdot$$

Hence, the required remainder is $\frac{3}{2}\cdot$

EXAMPLE 4 *Find the remainder when the polynomial* $p(x) = 12x^3 - 13x^2 - 5x + 7$ *is divided by* $g(x) = (2 + 3x)$.

SOLUTION $g(x) = 0 \Rightarrow 2 + 3x = 0 \Rightarrow 3x = -2 \Rightarrow x = \frac{-2}{3}\cdot$

By the remainder theorem, we know that when $p(x)$ is divided by $(2 + 3x)$ then the remainder is $p\left(\frac{-2}{3}\right)\cdot$

Now, $p\left(\frac{-2}{3}\right) = \left[12 \times \left(\frac{-2}{3}\right)^3 - 13 \times \left(\frac{-2}{3}\right)^2 - 5 \times \left(\frac{-2}{3}\right) + 7\right]$

$$= \left\{12 \times \frac{(-8)}{27} - 13 \times \frac{4}{9} + \frac{10}{3} + 7\right\} = 1.$$

Hence, the required remainder is 1.

EXAMPLE 5 *Find the remainder when* $x^3 + 3x^2 + 3x + 1$ *is divided by* $(x + \pi)$.

SOLUTION Let $p(x) = x^3 + 3x^2 + 3x + 1$ and $g(x) = x + \pi$.

Now, $g(x) = 0 \Rightarrow x + \pi = 0 \Rightarrow x = -\pi.$

By the remainder theorem, we know that when $p(x)$ is divided by $(x+\pi)$ then the remainder is $p(-\pi)$.

And, $p(-\pi) = (-\pi)^3 + 3\times(-\pi)^2 + 3\times(-\pi) + 1$

$$= -\pi^3 + 3\pi^2 - 3\pi + 1.$$

Hence, the required remainder is $(-\pi^3 + 3\pi^2 - 3\pi + 1)$.

EXAMPLE 6 *Let* $p(x) = x^3 - x + 1$ *and* $g(x) = 2 - 3x$. *Check whether* $p(x)$ *is a multiple of* $g(x)$ *or not.*

SOLUTION $g(x) = 0 \Rightarrow 2 - 3x = 0 \Rightarrow 3x = 2 \Rightarrow x = \dfrac{2}{3}$.

By the remainder theorem, we know that when $p(x)$ is divided by $(2-3x)$ then the remainder is $p\left(\dfrac{2}{3}\right)$.

And, $p\left(\dfrac{2}{3}\right) = \left\{\left(\dfrac{2}{3}\right)^3 - \left(\dfrac{2}{3}\right) + 1\right\} = \left(\dfrac{8}{27} - \dfrac{2}{3} + 1\right)$

$$= \left(\frac{8 - 18 + 27}{27}\right) = \frac{17}{27} \neq 0.$$

Thus, when $p(x)$ is divided by $g(x)$, the remainder is nonzero.

Hence, $p(x)$ is not a multiple of $g(x)$.

EXAMPLE 7 *Check whether* $(7+3x)$ *is a factor of* $(3x^3 + 7x)$.

SOLUTION Let $p(x) = 3x^3 + 7x$ and $g(x) = 7 + 3x$. Then,

$$g(x) = 0 \Rightarrow 7 + 3x = 0 \Rightarrow 3x = -7 \Rightarrow x = \frac{-7}{3}.$$

By the remainder theorem, we know that when $p(x)$ is divided by $(7+3x)$ then the remainder is $p\left(\dfrac{-7}{3}\right)$.

Now, $p\left(\dfrac{-7}{3}\right) = \left\{3\times\left(\dfrac{-7}{3}\right)^3 + 7\times\left(\dfrac{-7}{3}\right)\right\} = \left\{3\times\dfrac{(-343)}{27} - \dfrac{49}{3}\right\}$

$$= \left(\frac{-343}{9} - \frac{49}{3}\right) = \left(\frac{-343 - 147}{9}\right) = \frac{-490}{9} \neq 0.$$

Thus, when $p(x)$ is divided by $g(x)$, the remainder is nonzero.

$\therefore$ $(7+3x)$ is not a factor of $(3x^3 + 7x)$.

EXAMPLE 8 *If the polynomials* $(2x^3 + ax^2 + 3x - 5)$ *and* $(x^3 + x^2 - 2x + a)$ *leave the same remainder when divided by* $(x-2)$, *find the value of a. Also, find the remainder in each case.*

SOLUTION Let $f(x) = 2x^3 + ax^2 + 3x - 5$ and $g(x) = x^3 + x^2 - 2x + a$.

When $f(x)$ is divided by $(x-2)$, remainder $= f(2)$.

When $g(x)$ is divided by $(x-2)$, remainder $= g(2)$.

Now, $f(2) = (2 \times 2^3 + a \times 2^2 + 3 \times 2 - 5) = (17 + 4a)$.

And, $g(2) = (2^3 + 2^2 - 2 \times 2 + a) = (8 + a)$.

$\therefore \quad 17 + 4a = 8 + a \Rightarrow 3a = -9 \Rightarrow a = -3.$

Hence, $a = -3$.

Remainder in each case $= (8 - 3) = 5$.

EXERCISE 2C

1. By actual division, find the quotient and the remainder when $(x^4 + 1)$ is divided by $(x - 1)$.
   Verify that remainder $= f(1)$.
2. Verify the division algorithm for the polynomials
   $$p(x) = 2x^4 - 6x^3 + 2x^2 - x + 2 \text{ and } g(x) = x + 2.$$

Using the remainder theorem, find the remainder, when p(x) is divided by g(x), where

3. $p(x) = x^3 - 6x^2 + 9x + 3$, $g(x) = x - 1$.
4. $p(x) = 2x^3 - 7x^2 + 9x - 13$, $g(x) = x - 3$.
5. $p(x) = 3x^4 - 6x^2 - 8x - 2$, $g(x) = x - 2$.
6. $p(x) = 2x^3 - 9x^2 + x + 15$, $g(x) = 2x - 3$.
7. $p(x) = x^3 - 2x^2 - 8x - 1$, $g(x) = x + 1$.
8. $p(x) = 2x^3 + x^2 - 15x - 12$, $g(x) = x + 2$.
9. $p(x) = 6x^3 + 13x^2 + 3$, $g(x) = 3x + 2$.
10. $p(x) = x^3 - 6x^2 + 2x - 4$, $g(x) = 1 - \frac{3}{2}x$.
11. $p(x) = 2x^3 + 3x^2 - 11x - 3$, $g(x) = \left(x + \frac{1}{2}\right)$.
12. $p(x) = x^3 - ax^2 + 6x - a$, $g(x) = x - a$.
13. The polynomials $(2x^3 + x^2 - ax + 2)$ and $(2x^3 - 3x^2 - 3x + a)$ when divided by $(x - 2)$ leave the same remainder. Find the value of a.
14. The polynomial $p(x) = x^4 - 2x^3 + 3x^2 - ax + b$ when divided by $(x - 1)$ and $(x + 1)$ leaves the remainders 5 and 19 respectively. Find the values of a and b. Hence, find the remainder when $p(x)$ is divided by $(x - 2)$.

15. If $p(x) = x^3 - 5x^2 + 4x - 3$ and $g(x) = x - 2$, show that $p(x)$ is not a multiple of $g(x)$.

16. If $p(x) = 2x^3 - 11x^2 - 4x + 5$ and $g(x) = 2x + 1$, show that $g(x)$ is not a factor of $p(x)$.

ANSWERS (EXERCISE 2C)

1. Quotient $= (x^3 + x^2 + x + 1)$, remainder $= 2$ **3.** 7 **4.** 5 **5.** 6

6. 3 **7.** 4 **8.** 6 **9.** 7 **10.** $\frac{-136}{27}$ **11.** 3 **12.** $5a$ **13.** $a = 8$

14. $(a = 5, b = 8)$; 10

HINTS TO SOME SELECTED QUESTIONS

1. By actual division, we get

$$
\begin{array}{r|l}
x-1 & x^4 + 0x^3 + 0x^2 + 0x + 1 \;(x^3 + x^2 + x + 1 \\
 & x^4 - x^3 \\
 & -\quad + \\
\hline
 & x^3 + 0x^2 + 0x + 1 \\
 & x^3 - x^2 \\
 & -\quad + \\
\hline
 & x^2 + 0x + 1 \\
 & x^2 - x \\
 & -\quad + \\
\hline
 & x + 1 \\
 & x - 1 \\
 & -\quad + \\
\hline
 & 2
\end{array}
$$

$\therefore$ quotient $= (x^3 + x^2 + x + 1)$ and remainder $= 2$.

6. $g(x) = 0 \Rightarrow 2x - 3 = 0 \Rightarrow 2x = 3 \Rightarrow x = \frac{3}{2}$.

Remainder is $p\left(\frac{3}{2}\right) = \left\{2 \times \left(\frac{3}{2}\right)^3 - 9 \times \left(\frac{3}{2}\right)^2 + \frac{3}{2} + 15\right\}$

$$= \left(\frac{27}{4} - \frac{81}{4} + \frac{3}{2} + 15\right) = \left(\frac{27 - 81 + 6 + 60}{4}\right) = \frac{12}{4} = 3.$$

9. $g(x) = 0 \Rightarrow 3x + 2 = 0 \Rightarrow 3x = -2 \Rightarrow x = \frac{-2}{3}$.

Remainder is $p\left(\frac{-2}{3}\right) = \left\{6 \times \left(\frac{-2}{3}\right)^3 + 13 \times \left(\frac{-2}{3}\right)^2 + 3\right\} = \left(\frac{-16}{9} + \frac{52}{9} + 3\right)$

$$= \left(\frac{-16 + 52 + 27}{9}\right) = \frac{63}{9} = 7.$$

10. $g(x) = 0 \Rightarrow 1 - \frac{3}{2}x = 0 \Rightarrow \frac{3}{2}x = 1 \Rightarrow x = \frac{2}{3} \cdot$ Find $p\left(\frac{2}{3}\right) \cdot$

11. $g(x) = 0 \Rightarrow x + \frac{1}{2} = 0 \Rightarrow x = \frac{-1}{2} \cdot$ Find $p\left(\frac{-1}{2}\right) \cdot$

15. $g(x) = 0 \Rightarrow x - 2 = 0 \Rightarrow x = 2.$ Show that $p(2) \neq 0.$

16. $g(x) = 0 \Rightarrow 2x + 1 = 0 \Rightarrow x = \frac{-1}{2} \cdot$ Show that $p\left(\frac{-1}{2}\right) \neq 0.$

FACTOR THEOREM *Let $p(x)$ be a polynomial of degree 1 or more and let α be any real number.*

(i) If $p(\alpha) = 0$ then $(x - \alpha)$ is a factor of $p(x)$.

(ii) If $(x - \alpha)$ is a factor of $p(x)$ then $p(\alpha) = 0$.

PROOF When $p(x)$ is divided by $(x - \alpha)$, let $q(x)$ be the quotient and by remainder theorem, the remainder is $p(\alpha)$.

$\therefore \quad p(x) = (x - \alpha) \cdot q(x) + p(\alpha)$ [by division algorithm].

(i) If $p(\alpha) = 0$ then $p(x) = (x - \alpha) \cdot q(x)$.

This shows that $(x - \alpha)$ is a factor of $p(x)$.

(ii) If $(x - \alpha)$ is a factor of $p(x)$ then we have

$p(x) = (x - \alpha) \cdot q(x)$ for some polynomial $q(x)$.

Putting $x = \alpha$ on both sides, we get $p(\alpha) = 0$.

Thus, when $(x - \alpha)$ is a factor of $p(x)$, then $p(\alpha) = 0$.

SOLVED EXAMPLES

EXAMPLE 1 *If $p(x) = 8x^3 - 6x^2 - 4x + 3$ and $g(x) = \frac{x}{3} - \frac{1}{4}$ then check whether $g(x)$ is a factor of $p(x)$ or not.*

SOLUTION $g(x) = 0 \Rightarrow \frac{x}{3} - \frac{1}{4} = 0 \Rightarrow \frac{x}{3} = \frac{1}{4} \Rightarrow x = \frac{3}{4} \cdot$

By factor theorem, $g(x)$ will be a factor of $p(x)$, if $p\left(\frac{3}{4}\right) = 0$.

$$\text{Now, } p\left(\frac{3}{4}\right) = \left\{8 \times \left(\frac{3}{4}\right)^3 - 6 \times \left(\frac{3}{4}\right)^2 - 4 \times \frac{3}{4} + 3\right\}$$

$$= \left\{\left(8 \times \frac{27}{64}\right) - \left(6 \times \frac{9}{16}\right) - 3 + 3\right\} = \left(\frac{27}{8} - \frac{27}{8}\right) = 0.$$

Since $p\left(\frac{3}{4}\right) = 0$, it follows that $g(x)$ is a factor of $p(x)$.

EXAMPLE 2 *Show that* $(2x-3)$ *is a factor of* $(x+2x^3-9x^2+12)$.

SOLUTION Let $p(x)=2x^3-9x^2+x+12$ and $g(x)=2x-3$.

Now, $g(x)=0 \Rightarrow 2x-3=0 \Rightarrow 2x=3 \Rightarrow x=\frac{3}{2}\cdot$

By factor theorem, $g(x)$ will be a factor of $p(x)$, if $p\left(\frac{3}{2}\right)=0$.

$$\text{Now, } p\left(\frac{3}{2}\right)=\left\{2\times\left(\frac{3}{2}\right)^3-9\times\left(\frac{3}{2}\right)^2+\frac{3}{2}+12\right\}$$

$$=\left\{\left(2\times\frac{27}{8}\right)-\left(9\times\frac{9}{4}\right)+\frac{3}{2}+12\right\}$$

$$=\left(\frac{27}{4}-\frac{81}{4}+\frac{3}{2}+12\right)=\left(\frac{27-81+6+48}{4}\right)=\frac{0}{4}=0.$$

Since $p\left(\frac{3}{2}\right)=0$, so $g(x)$ is a factor of $p(x)$.

EXAMPLE 3 *Use factor theorem to show that* $x^4+2x^3-2x^2+2x-3$ *is exactly divisible by* $(x+3)$.

SOLUTION Let $p(x)=x^4+2x^3-2x^2+2x-3$ and $g(x)=x+3$. Then,

$$g(x)=0 \Rightarrow x+3=0 \Rightarrow x=-3.$$

By factor theorem, $p(x)$ will be exactly divisible by $x+3$, if $p(-3)=0$.

$$\text{Now, } p(-3)=\{(-3)^4+2\times(-3)^3-2\times(-3)^2+2\times(-3)-3\}$$

$$=(81-54-18-6-3)=0.$$

Since $p(-3)=0$, it follows that $p(x)$ is exactly divisible by $(x+3)$.

EXAMPLE 4 *If* $(x-a)$ *is a factor of* $(x^3-ax^2+2x+a-1)$, *find the value of* a.

SOLUTION Let $p(x)=x^3-ax^2+2x+a-1$ and $g(x)=x-a$. Then,

$$g(x)=0 \Rightarrow x-a=0 \Rightarrow x=a.$$

By factor theorem, $(x-a)$ will be a factor of $p(x)$, if $p(a)=0$.

$$\text{Now, } p(a)=0 \Rightarrow a^3-a\times a^2+2a+a-1=0$$

$$\Rightarrow a^3-a^3+3a-1=0$$

$$\Rightarrow 3a-1=0 \Rightarrow 3a=1 \Rightarrow a=\frac{1}{3}\cdot$$

Hence, the required value of a is $\frac{1}{3}\cdot$

EXAMPLE 5 *For what value of* m *is* (x^3-2mx^2+16) *divisible by* $(x+2)$?

SOLUTION Let $p(x)=x^3-2mx^2+16$ and $g(x)=x+2$. Then,

$$g(x) = 0 \Rightarrow x + 2 = 0 \Rightarrow x = -2.$$

By factor theorem, $(x+2)$ will be a factor of $p(x)$, if $p(-2) = 0$.

Now, $p(-2) = 0 \Rightarrow (-2)^3 - 2m \times (-2)^2 + 16 = 0$

$$\Rightarrow -8 - 8m + 16 = 0 \Rightarrow 8m = 8 \Rightarrow m = 1.$$

Hence, the required value of m is 1.

EXAMPLE 6 *Without actual division, prove that* $(2x^4 + 3x^3 - 12x^2 - 7x + 6)$ *is exactly divisible by* $(x^2 + x - 6)$.

SOLUTION Let $p(x) = 2x^4 + 3x^3 - 12x^2 - 7x + 6$ and $g(x) = x^2 + x - 6$. Then,

$$g(x) = x^2 + x - 6 = x^2 + 3x - 2x - 6 = x(x+3) - 2(x+3)$$
$$= (x+3)(x-2).$$

Clearly, $p(x)$ will be exactly divisible by $g(x)$ only when it is exactly divisible by $(x+3)$ as well as $(x-2)$.

Now, $(x + 3 = 0 \Rightarrow x = -3)$ and $(x - 2 = 0 \Rightarrow x = 2)$.

By factor theorem, $g(x)$ will be a factor of $p(x)$, if $p(-3) = 0$ and $p(2) = 0$.

Now, $p(-3) = \{2 \times (-3)^4 + 3 \times (-3)^3 - 12 \times (-3)^2 - 7 \times (-3) + 6\}$

$$= \{(2 \times 81) + 3 \times (-27) - (12 \times 9) + 21 + 6\}$$
$$= (162 - 81 - 108 + 21 + 6) = 0.$$

And, $p(2) = \{(2 \times 2^4) + (3 \times 2^3) - (12 \times 2^2) - (7 \times 2) + 6\}$

$$= (32 + 24 - 48 - 14 + 6) = 0.$$

Thus, $p(x)$ is exactly divisible by each one of $(x+3)$ and $(x-2)$.

Hence, $p(x)$ is exactly divisible by $(x+3)(x-2)$, i.e., by $(x^2 + x - 6)$.

EXAMPLE 7 *Find the values of a and b so that* $(2x^3 + ax^2 + x + b)$ *has* $(x+2)$ *and* $(2x-1)$ *as factors.*

SOLUTION Let $p(x) = 2x^3 + ax^2 + x + b$. Then,

$$(x + 2 = 0 \Rightarrow x = -2) \text{ and } \left(2x - 1 = 0 \Rightarrow 2x = 1 \Rightarrow x = \frac{1}{2}\right).$$

Now, $(x+2)$ and $(2x-1)$ will be factors of $p(x)$, if $p(-2) = 0$ and $p\left(\frac{1}{2}\right) = 0$.

$$p(-2) = 0 \Rightarrow 2 \times (-2)^3 + a \times (-2)^2 + (-2) + b = 0$$
$$\Rightarrow -16 + 4a - 2 + b = 0 \Rightarrow 4a + b = 18. \qquad \ldots \text{(i)}$$

$$p\left(\frac{1}{2}\right) = 0 \Rightarrow 2 \times \left(\frac{1}{2}\right)^3 + a \times \left(\frac{1}{2}\right)^2 + \frac{1}{2} + b = 0$$

$$\Rightarrow \left(2 \times \frac{1}{8}\right) + \left(a \times \frac{1}{4}\right) + \frac{1}{2} + b = 0$$

$$\Rightarrow \frac{1}{4} + \frac{a}{4} + \frac{1}{2} + b = 0 \Rightarrow a + 4b = -3. \qquad \ldots \text{(ii)}$$

On solving (i) and (ii), we get $a = 5$ and $b = -2$.

Hence, $a = 5$ and $b = -2$.

EXAMPLE 8 *If* $(ax^3 + bx^2 - 5x + 2)$ *has* $(x + 2)$ *as a factor and leaves a remainder* 12 *when divided by* $(x - 2)$*, find the values of a and b.*

SOLUTION Let $p(x) = ax^3 + bx^2 - 5x + 2$, $g(x) = x + 2$ and $h(x) = x - 2$. Then,

$$g(x) = 0 \Rightarrow x + 2 = 0 \Rightarrow x = -2.$$

$$h(x) = 0 \Rightarrow x - 2 = 0 \Rightarrow x = 2.$$

$(x + 2)$ is a factor of $p(x) \Rightarrow p(-2) = 0$.

Now, $p(-2) = 0 \Rightarrow \{a \times (-2)^3 + b \times (-2)^2 - 5 \times (-2) + 2\} = 0$

$$\Rightarrow -8a + 4b + 12 = 0$$

$$\Rightarrow 8a - 4b = 12 \Rightarrow 2a - b = 3. \qquad \ldots \text{(i)}$$

When $p(x)$ is divided by $(x - 2)$, then the remainder is $p(2)$.

$\therefore \quad p(2) = 12 \Rightarrow \{(a \times 2^3) + (b \times 2^2) - (5 \times 2) + 2\} = 12$

$$\Rightarrow 8a + 4b = 20 \Rightarrow 2a + b = 5. \qquad \ldots \text{(ii)}$$

On solving (i) and (ii), we get $a = 2$ and $b = 1$.

Hence, $a = 2$ and $b = 1$.

EXAMPLE 9 *What must be added to* $(x^3 - 3x^2 + 4x - 15)$ *to obtain a polynomial which is exactly divisible by* $(x - 3)$?

SOLUTION When the given polynomial is divided by a linear polynomial then the remainder is constant.

Let the required number to be added be k.

Let $p(x) = x^3 - 3x^2 + 4x - 15 + k$ and $g(x) = x - 3$.

Then, $g(x) = 0 \Rightarrow x - 3 = 0 \Rightarrow x = 3$.

By factor theorem, $p(x)$ will be divisible by $(x - 3)$, if $p(3) = 0$.

Now, $p(3) = 0 \Rightarrow \{3^3 - 3 \times 3^2 + 4 \times 3 - 15 + k\} = 0$

$$\Rightarrow (27 - 27 + 12 - 15 + k) = 0 \Rightarrow k = 3.$$

Hence, the required number to be added is 3.

EXAMPLE 10 *What must be subtracted from* $(4x^4 - 2x^3 - 6x^2 + 2x + 6)$ *so that the result is exactly divisible by* $(2x^2 + x - 1)$?

SOLUTION When the given polynomial is divided by a quadratic polynomial, then the remainder is a linear expression, say $(ax + b)$.

Let $p(x) = (4x^4 - 2x^3 - 6x^2 + 2x + 6) - (ax + b)$ and $g(x) = 2x^2 + x - 1$. Then,

$$p(x) = 4x^4 - 2x^3 - 6x^2 + (2 - a)x + (6 - b)$$

and $g(x) = (2x^2 + x - 1) = (2x^2 + 2x - x - 1) = 2x(x + 1) - (x + 1)$

$= (x + 1)(2x - 1)$.

Now, $p(x)$ will be divisible by $g(x)$ only when it is divisible by $(x + 1)$ as well as by $(2x - 1)$.

Now, $(x + 1 = 0 \Rightarrow x = -1)$ and $\left(2x - 1 = 0 \Rightarrow 2x = 1 \Rightarrow x = \frac{1}{2}\right)$.

By factor theorem, $p(x)$ will be divisible by $g(x)$, if $p(-1) = 0$ and $p\left(\frac{1}{2}\right) = 0$.

$$p(-1) = 0 \Rightarrow 4 \times (-1)^4 - 2 \times (-1)^3 - 6 \times (-1)^2 + (2 - a) \times (-1) + 6 - b = 0$$

$$\Rightarrow 4 + 2 - 6 - 2 + a + 6 - b = 0 \Rightarrow a - b = -4. \qquad \ldots \text{(i)}$$

$$p\left(\frac{1}{2}\right) = 0 \Rightarrow 4 \times \left(\frac{1}{2}\right)^4 - 2 \times \left(\frac{1}{2}\right)^3 - 6 \times \left(\frac{1}{2}\right)^2 + (2 - a) \times \frac{1}{2} + (6 - b) = 0$$

$$\Rightarrow \left(4 \times \frac{1}{16}\right) - \left(2 \times \frac{1}{8}\right) - \left(6 \times \frac{1}{4}\right) + 1 - \frac{a}{2} + 6 - b = 0$$

$$\Rightarrow \frac{1}{4} - \frac{1}{4} - \frac{3}{2} + 1 - \frac{a}{2} + 6 - b = 0 \Rightarrow \frac{a}{2} + b = \frac{11}{2}$$

$$\Rightarrow a + 2b = 11. \qquad \ldots \text{(ii)}$$

On solving (i) and (ii), we get $a = 1$ and $b = 5$.

Hence, the required expression to be subtracted is $(x + 5)$.

EXERCISE 2D

Using factor theorem, show that g(x) is a factor of p(x), when

1. $p(x) = x^3 - 8$, $g(x) = x - 2$
2. $p(x) = 2x^3 + 7x^2 - 24x - 45$, $g(x) = x - 3$
3. $p(x) = 2x^4 + 9x^3 + 6x^2 - 11x - 6$, $g(x) = x - 1$

4. $p(x) = x^4 - x^2 - 12$, $g(x) = x + 2$

5. $p(x) = 69 + 11x - x^2 + x^3$, $g(x) = x + 3$

6. $p(x) = 2x^3 + 9x^2 - 11x - 30$, $g(x) = x + 5$

7. $p(x) = 2x^4 + x^3 - 8x^2 - x + 6$, $g(x) = 2x - 3$

8. $p(x) = 3x^3 + x^2 - 20x + 12$, $g(x) = 3x - 2$

9. $p(x) = 7x^2 - 4\sqrt{2}\,x - 6$, $g(x) = x - \sqrt{2}$

10. $p(x) = 2\sqrt{2}\,x^2 + 5x + \sqrt{2}$, $g(x) = x + \sqrt{2}$

11. Show that $(p-1)$ is a factor of $(p^{10} - 1)$ and also of $(p^{11} - 1)$.

12. Find the value of k for which $(x-1)$ is a factor of $(2x^3 + 9x^2 + x + k)$.

13. Find the value of a for which $(x-4)$ is a factor of $(2x^3 - 3x^2 - 18x + a)$.

14. Find the value of a for which $(x+1)$ is a factor of $(ax^3 + x^2 - 2x + 4a - 9)$.

15. Find the value of a for which $(x+2a)$ is a factor of $(x^5 - 4a^2x^3 + 2x + 2a + 3)$.

16. Find the value of m for which $(2x-1)$ is a factor of $(8x^4 + 4x^3 - 16x^2 + 10x + m)$.

17. Find the value of a for which the polynomial $(x^4 - x^3 - 11x^2 - x + a)$ is divisible by $(x+3)$.

18. Without actual division, show that $(x^3 - 3x^2 - 13x + 15)$ is exactly divisible by $(x^2 + 2x - 3)$.

19. If $(x^3 + ax^2 + bx + 6)$ has $(x-2)$ as a factor and leaves a remainder 3 when divided by $(x-3)$, find the values of a and b.

20. Find the values of a and b so that the polynomial $(x^3 - 10x^2 + ax + b)$ is exactly divisible by $(x-1)$ as well as $(x-2)$.

21. Find the values of a and b so that the polynomial $(x^4 + ax^3 - 7x^2 - 8x + b)$ is exactly divisible by $(x+2)$ as well as $(x+3)$.

22. If both $(x-2)$ and $\left(x - \frac{1}{2}\right)$ are factors of $px^2 + 5x + r$, prove that $p = r$.

23. Without actual division, prove that $2x^4 - 5x^3 + 2x^2 - x + 2$ is divisible by $x^2 - 3x + 2$.

24. What must be added to $2x^4 - 5x^3 + 2x^2 - x - 3$ so that the result is exactly divisible by $(x-2)$?

25. What must be subtracted from $(x^4 + 2x^3 - 2x^2 + 4x + 6)$ so that the result is exactly divisible by $(x^2 + 2x - 3)$?

26. Use factor theorem to prove that $(x+a)$ is a factor of $(x^n + a^n)$ for any odd positive integer n.

ANSWERS (EXERCISE 2D)

12. $k = -12$ **13.** $a = -8$ **14.** $a = 2$ **15.** $a = \frac{3}{2}$

16. $m = -2$ **17.** $a = -12$ **19.** $a = -3, b = -1$ **20.** $a = 23, b = -14$

21. $a = 2, b = 12$ **24.** 5 **25.** $2x + 9$

HINTS TO SOME SELECTED QUESTIONS

11. Let $f(p) = (p^{10} - 1)$, $g(p) = (p^{11} - 1)$ and $h(p) = p - 1$. Then,

$h(p) = 0 \Rightarrow p - 1 = 0 \Rightarrow p = 1.$

Now, $f(1) = \{f(1)^{10} - 1\} = (1 - 1) = 0 \Rightarrow (p - 1)$ is a factor of $f(p)$.

And, $g(1) = \{(1)^{11} - 1\} = (1 - 1) = 0 \Rightarrow (p - 1)$ is a factor of $g(p)$.

24. Let the required number to be added be k. Then,

$p(x) = \{2x^4 - 5x^3 + 2x^2 - x - 3 + k\}$ and $g(x) = (x - 2)$.

Now, $g(x) = 0 \Rightarrow x - 2 = 0 \Rightarrow x = 2.$

$\therefore$ $p(2) = 0 \Rightarrow (32 - 40 + 8 - 5 + k) = 0 \Rightarrow k = 5.$

25. Let $g(x) = (x^2 + 2x - 3) = (x^2 + 3x - x - 3) = x(x + 3) - (x + 3) = (x + 3)(x - 1)$.

When a polynomial is divided by a quadratic polynomial, then the remainder is a linear expression, say $ax + b$.

Let $p(x) = (x^4 + 2x^3 - 2x^2 + 4x + 6) - (ax + b)$. Then,

$p(x) = x^4 + 2x^3 - 2x^2 + (4 - a)x + (6 - b)$.

$\therefore$ $p(-3) = 0$ and $p(1) = 0 \Rightarrow 3a - b = -3$ and $a + b = 11$.

On solving these equations, we get $a = 2$ and $b = 9$.

Hence, the required expression is $(2x + 9)$.

26. Let $p(x) = (x^n + a^n)$, where n is any odd positive integer.

Let $g(x) = x + a$. Then, $g(x) = 0 \Rightarrow x + a = 0 \Rightarrow x = -a$.

Now, $p(-a) = \{(-a)^n + a^n\} = \{(-1)^n a^n + a^n\} = \{(-1)^n + 1\}a^n$

$= (-1 + 1)a^n = 0$ [$\because$ n being odd, $(-1)^n = -1$].

By factor theorem, $(x + a)$ is a factor of $(x^n + a^n)$, when n is an odd positive integer.

MULTIPLE-CHOICE QUESTIONS (MCQ)

Choose the correct answer in each of the following questions.

1. Which of the following expressions is a polynomial in one variable?

(a) $x + \frac{2}{x} + 3$ (b) $3\sqrt{x} + \frac{2}{\sqrt{x}} + 5$

(c) $\sqrt{2}x^2 - \sqrt{3}x + 6$ (d) $x^{10} + y^5 + 8$

2. Which of the following expressions is a polynomial?

(a) $\sqrt{x} - 1$ (b) $\frac{x - 1}{x + 1}$ (c) $x^2 - \frac{2}{x^2} + 5$ (d) $x^2 + \frac{2x^{3/2}}{\sqrt{x}} + 6$

3. Which of the following is a polynomial?

(a) $\sqrt[3]{y}+4$ (b) $\sqrt{y}-3$ (c) y (d) $\frac{1}{\sqrt{y}}+7$

4. Which of the following is a polynomial?

(a) $x-\frac{1}{x}+2$ (b) $\frac{1}{x}+5$ (c) $\sqrt{x}+3$ (d) -4

5. Which of the following is a polynomial?

(a) $x^{-2}+x^{-1}+3$ (b) $x+x^{-1}+2$ (c) x^{-1} (d) 0

6. Which of the following is a quadratic polynomial?

(a) $x+4$ (b) x^3+x (c) x^3+2x+6 (d) x^2+5x+4

7. Which of the following is a linear polynomial?

(a) $x+x^2$ (b) $x+1$ (c) $5x^2-x+3$ (d) $x+\frac{1}{x}$

8. Which of the following is a binomial?

(a) x^2+x+3 (b) x^2+4 (c) $2x^2$ (d) $x+3+\frac{1}{x}$

9. $\sqrt{3}$ is a polynomial of degree

(a) $\frac{1}{2}$ (b) 2 (c) 1 (d) 0

10. Degree of the zero polynomial is

(a) 1 (b) 0 (c) not defined (d) none of these

11. Zero of the zero polynomial is

(a) 0 (b) 1

(c) every real number (d) not defined

12. If $p(x)=x+4$ then $p(x)+p(-x)=$?

(a) 0 (b) 4 (c) $2x$ (d) 8

13. If $p(x)=x^2-2\sqrt{2}x+1$ then $p(2\sqrt{2})=$?

(a) 0 (b) 1 (c) $4\sqrt{2}$ (d) –1

14. If $p(x)=5x-4x^2+3$ then $p(-1)=$?

(a) 2 (b) –2 (c) 6 (d) –6

15. If $(x^{51}+51)$ is divided by $(x+1)$ then the remainder is

(a) 0 (b) 1 (c) 49 (d) 50

16. If $(x+1)$ is a factor of the polynomial $(2x^2+kx)$ then $k=$?

(a) 4 (b) –3 (c) 2 (d) –2

17. When $p(x)=x^4+2x^3-3x^2+x-1$ is divided by $(x-2)$, the remainder is

(a) 0 (b) –1 (c) –15 (d) 21

18. When $p(x) = x^3 - 3x^2 + 4x + 32$ is divided by $(x + 2)$, the remainder is

(a) 0 (b) 32 (c) 36 (d) 4

19. When $p(x) = 4x^3 - 12x^2 + 11x - 5$ is divided by $(2x - 1)$, the remainder is

(a) 0 (b) –5 (c) –2 (d) 2

20. When $p(x) = x^3 - ax^2 + x$ is divided by $(x - a)$, the remainder is

(a) 0 (b) a (c) $2a$ (d) $3a$

21. When $p(x) = x^3 + ax^2 + 2x + a$ is divided by $(x + a)$, the remainder is

(a) 0 (b) a (c) $-a$ (d) $2a$

22. $(x + 1)$ is a factor of the polynomial

(a) $x^3 + x^2 - x + 1$ (b) $x^3 + 2x^2 - x - 2$

(c) $x^3 + 2x^2 - x + 2$ (d) $x^4 + x^3 + x^2 + 1$

23. Zero of the polynomial $p(x) = 2x + 5$ is

(a) $\frac{-2}{5}$ (b) $\frac{-5}{2}$ (c) $\frac{2}{5}$ (d) $\frac{5}{2}$

24. The zeros of the polynomial $p(x) = x^2 + x - 6$ are

(a) 2, 3 (b) –2, 3 (c) 2, –3 (d) –2, –3

25. The zeros of the polynomial $p(x) = 2x^2 + 5x - 3$ are

(a) $\frac{1}{2}, 3$ (b) $\frac{1}{2}, -3$ (c) $\frac{-1}{2}, 3$ (d) $1, \frac{-1}{2}$

26. The zeros of the polynomial $p(x) = 2x^2 + 7x - 4$ are

(a) $4, \frac{-1}{2}$ (b) $4, \frac{1}{2}$ (c) $-4, \frac{1}{2}$ (d) $-4, \frac{-1}{2}$

27. If $(x + 5)$ is a factor of $p(x) = x^3 - 20x + 5k$ then $k = ?$

(a) –5 (b) 5 (c) 3 (d) –3

28. If $(x + 2)$ and $(x - 1)$ are factors of the polynomial $p(x) = x^3 + 10x^2 + mx + n$ then

(a) $m = 5, n = -3$ (b) $m = 7, n = -18$

(c) $m = 17, n = -8$ (d) $m = 23, n = -19$

29. If $(x^{100} + 2x^{99} + k)$ is divisible by $(x + 1)$ then the value of k is

(a) 1 (b) 2 (c) –2 (d) –3

30. For what value of k is the polynomial $p(x) = 2x^3 - kx^2 + 3x + 10$ exactly divisible by $(x + 2)$?

(a) $-\frac{1}{3}$ (b) $\frac{1}{3}$ (c) 3 (d) –3

31. The zeros of the polynomial $p(x) = x^2 - 3x$ are

(a) 0, 0 (b) 0, 3 (c) 0, –3 (d) 3, –3

32. The zeros of the polynomial $p(x) = 3x^2 - 1$ are

(a) $\frac{1}{3}$ and 3 (b) $\frac{1}{\sqrt{3}}$ and $\sqrt{3}$ (c) $\frac{-1}{\sqrt{3}}$ and $\sqrt{3}$ (d) $\frac{1}{\sqrt{3}}$ and $\frac{-1}{\sqrt{3}}$

ANSWERS (MCQ)

1. (c)	**2.** (d)	**3.** (c)	**4.** (d)	**5.** (d)	**6.** (d)	**7.** (b)	**8.** (b)
9. (d)	**10.** (c)	**11.** (d)	**12.** (d)	**13.** (b)	**14.** (d)	**15.** (d)	**16.** (c)
17. (d)	**18.** (d)	**19.** (c)	**20.** (b)	**21.** (c)	**22.** (b)	**23.** (b)	**24.** (c)
25. (b)	**26.** (c)	**27.** (b)	**28.** (b)	**29.** (a)	**30.** (d)	**31.** (b)	**32.** (d)

REVIEW OF FACTS AND FORMULAE

1. *An expression of the form* $a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n$, *where* $a_0, a_1, a_2, \ldots, a_{n-1}, a_n$ *are real numbers and* $a_n \neq 0$ *and* n *is a non-negative integer, is called a polynomial in* x *of degree* n.

2. (i) *A polynomial of degree* 1 *is called a linear polynomial.*

It is of the form $(ax + b)$, where a and b are real numbers and $a \neq 0$.

(ii) *A polynomial of degree* 2 *is called a quadratic polynomial.*

It is of the form $(ax^2 + bx + c)$, where a, b, c are real numbers and $a \neq 0$.

(iii) *A polynomial of degree* 3 *is called a cubic polynomial.*

It is of the form $(ax^3 + bx^2 + cx + d)$, where a, b, c, d are real numbers and $a \neq 0$.

(iv) *A polynomial of degree* 4 *is called a biquadratic polynomial.*

It is of the form $(ax^4 + bx^3 + cx^2 + dx + e)$, where a, b, c, d, e are real numbers and $a \neq 0$.

3. (i) *A polynomial containing one term is called a monomial.*

(ii) *A polynomial containing two terms is called a binomial.*

(iii) *A polynomial containing three terms is called a trinomial.*

(iv) *A polynomial containing more than three terms is called a multinomial.*

4. (i) **Constant polynomial** *A polynomial containing one term consisting of a nonzero constant, is called a constant polynomial.*

(ii) The degree of a constant polynomial is zero.

5. (i) **Zero polynomial** *A polynomial consisting of one term, namely* 0, *is called a zero polynomial.*

(ii) The degree of a zero polynomial is not defined.

6. (i) **Zeros of a polynomial** Let $p(x)$ be a polynomial. If $p(\alpha) = 0$ then α is called the zero of the polynomial $p(x)$.

(ii) Finding the zeros of a polynomial $p(x)$ means solving the equation $p(x) = 0$.

7. Division algorithm in polynomials

Let $p(x)$ and $g(x)$ be two polynomials such that degree $p(x) \geq$ degree $g(x)$. On dividing $p(x)$ by $g(x)$, let $q(x)$ be the quotient and $r(x)$ be the remainder. Then,

dividend = (*divisor* $\times$ *quotient*) + *remainder*,

i.e., $p(x) = g(x) \cdot q(x) + r(x)$, where $r(x) = 0$ or degree $r(x) <$ degree $g(x)$.

8. Remainder theorem Let $p(x)$ be a polynomial of degree 1 or more and let α be any real number. If $p(x)$ is divided by $(x - \alpha)$ then the remainder is $p(\alpha)$.

9. Factor theorem Let $p(x)$ be a polynomial of degree 1 or more and let α be any real number.

(i) If $p(\alpha) = 0$ then $(x - \alpha)$ is a factor of $p(x)$.

(ii) If $(x - \alpha)$ is a factor of $p(x)$ then $p(\alpha) = 0$.

✿

3 Factorisation of Polynomials

FACTOR *Let $p(x)$ and $q(x)$ be two polynomials. We say that $q(x)$ is a factor of $p(x)$, if $q(x)$ divides $p(x)$ exactly.*

Examples (i) $(x-2)$ is a factor of (x^2-3x+2).

(ii) $(x+1)$ is a factor of (x^2-1).

FACTORISATION *To express a given polynomial as the product of polynomials, each of degree less than that of the given polynomial such that no such a factor has a factor of lower degree, is called factorisation.*

Examples It is easy to verify that

(i) $(x^2-16)=(x-4)(x+4)$.

(ii) $(x^2-3x+2)=(x-1)(x-2)$.

SOME EXPANSIONS We know that

(i) $(a+b)^2=(a^2+2ab+b^2)$.

(ii) $(a-b)^2=(a^2-2ab+b^2)$.

(iii) $(a+b+c)^2=(a^2+b^2+c^2)+2(ab+bc+ca)$.

(iv) $(a+b)^3=a^3+b^3+3ab(a+b)$.

(v) $(a-b)^3=a^3-b^3-3ab(a-b)$.

SOME FORMULAE FOR FACTORISATION

(i) $(a^2-b^2)=(a-b)(a+b)$.

(ii) $(a^3+b^3)=(a+b)(a^2-ab+b^2)$.

(iii) $(a^3-b^3)=(a-b)(a^2+ab+b^2)$.

METHODS OF FACTORISATION

FACTORISATION BY TAKING OUT THE COMMON FACTOR

METHOD When each term of an expression has a common factor, we divide each term by this factor and take it out as a multiple, as shown below.

SOLVED EXAMPLES

EXAMPLE 1 *Factorise*

(i) $5a^2 - 20ab$ (ii) $36a^3b - 60a^2bc$

(iii) $5a(b+c) - 7b(b+c)$ (iv) $6(2a+3b)^2 - 8(2a+3b)$

SOLUTION We have

(i) $5a^2 - 20ab = 5a(a-4b)$.

(ii) $36a^3b - 60a^2bc = 12a^2b(3a-5c)$.

(iii) $5a(b+c) - 7b(b+c) = (b+c)(5a-7b)$.

(iv) $6(2a+3b)^2 - 8(2a+3b) = 2(2a+3b)[3(2a+3b)-4]$
$= 2(2a+3b)(6a+9b-4)$.

EXAMPLE 2 *Factorise* $x(x-y)^3 + 3x^2y(x-y)$.

SOLUTION $x(x-y)^3 + 3x^2y(x-y) = x(x-y)[(x-y)^2 + 3xy]$
$= x(x-y)(x^2+y^2-2xy+3xy)$
$= x(x-y)(x^2+y^2+xy)$.

FACTORISATION BY GROUPING

METHOD Sometimes in a given expression it is not possible to take out a common factor directly. However, the terms of the given expression are grouped in such a manner that we may have a common factor. This can now be factorised as discussed above.

EXAMPLE 3 *Factorise*

(i) $ab + bc + ax + cx$ (ii) $x^2 + 3x + x + 3$

(iii) $6ab - b^2 + 12ac - 2bc$ (iv) $a^2 + b - ab - a$

SOLUTION We have

(i) $ab + bc + ax + cx = (ab+bc) + (ax+cx)$
$= b(a+c) + x(a+c) = (a+c)(b+x)$.

(ii) $x^2 + 3x + x + 3 = (x^2+3x) + (x+3)$
$= x(x+3) + (x+3) = (x+3)(x+1)$.

(iii) $6ab - b^2 + 12ac - 2bc = (6ab+12ac) - (b^2+2bc)$
$= 6a(b+2c) - b(b+2c)$
$= (b+2c)(6a-b)$.

(iv) $a^2 + b - ab - a = a^2 - a - ab + b = (a^2-a) - (ab-b)$
$= a(a-1) - b(a-1) = (a-1)(a-b)$.

EXAMPLE 4 *Factorise* $x^2+\frac{1}{x^2}+2-2x-\frac{2}{x}\cdot$

SOLUTION $x^2+\frac{1}{x^2}+2-2x-\frac{2}{x}=\left(x^2+\frac{1}{x^2}+2\right)-2\left(x+\frac{1}{x}\right)$

$$=\left(x+\frac{1}{x}\right)^2-2\left(x+\frac{1}{x}\right)$$

$$=\left(x+\frac{1}{x}\right)\left(x+\frac{1}{x}-2\right)\cdot$$

EXAMPLE 5 *Factorise* $1+x+y-z+xy-yz-zx-xyz.$

SOLUTION We have

$$\begin{aligned}&1+x+y-z+xy-yz-zx-xyz\\&=(1+x)+(y+xy)-z-zx-yz-xyz\\&=(1+x)+y(1+x)-z(1+x)-yz(1+x)\\&=(1+x)(1+y-z-yz)\\&=(1+x)\times[(1+y)-z(1+y)]\\&=(1+x)(1+y)(1-z).\end{aligned}$$

EXAMPLE 6 *Factorise* $(2x-3)^2-8x+12.$

SOLUTION We have

$$\begin{aligned}(2x-3)^2-8x+12&=(2x-3)^2-4(2x-3)\\&=(2x-3)\times[(2x-3)-4]\\&=(2x-3)(2x-7).\end{aligned}$$

EXERCISE 3A

Factorise:

1. $9x^2+12xy$

2. $18x^2y-24xyz$

3. $27a^3b^3-45a^4b^2$

4. $2a(x+y)-3b(x+y)$

5. $2x(p^2+q^2)+4y(p^2+q^2)$

6. $x(a-5)+y(5-a)$

7. $4(a+b)-6(a+b)^2$

8. $8(3a-2b)^2-10(3a-2b)$

9. $x(x+y)^3-3x^2y(x+y)$

10. $x^3+2x^2+5x+10$

11. $x^2+xy-2xz-2yz$

12. $a^3b-a^2b+5ab-5b$

13. $8-4a-2a^3+a^4$

14. $x^3-2x^2y+3xy^2-6y^3$

15. $px-5q+pq-5x$

16. $x^2+y-xy-x$

17. $(3a-1)^2-6a+2$

18. $(2x-3)^2-8x+12$

19. a^3+a-3a^2-3

20. $3ax-6ay-8by+4bx$

21. $abx^2+a^2x+b^2x+ab$

22. $x^3-x^2+ax+x-a-1$

23. $2x+4y-8xy-1$

24. $ab(x^2+y^2)-xy(a^2+b^2)$

25. $a^2+ab(b+1)+b^3$

26. $a^3+ab(1-2a)-2b^2$

27. $2a^2+bc-2ab-ac$

28. $(ax+by)^2+(bx-ay)^2$

29. $a(a+b-c)-bc$

30. $a(a-2b-c)+2bc$

31. $a^2x^2+(ax^2+1)x+a$

32. $ab(x^2+1)+x(a^2+b^2)$

33. $x^2-(a+b)x+ab$

34. $x^2+\dfrac{1}{x^2}-2-3x+\dfrac{3}{x}$

ANSWERS (EXERCISE 3A)

1. $3x(3x+4y)$

2. $6xy(3x-4z)$

3. $9a^3b^2(3b-5a)$

4. $(x+y)(2a-3b)$

5. $2(p^2+q^2)(x+2y)$

6. $(a-5)(x-y)$

7. $2(a+b)(2-3a-3b)$

8. $2(3a-2b)(12a-8b-5)$

9. $x(x+y)(x^2+y^2-xy)$

10. $(x+2)(x^2+5)$

11. $(x+y)(x-2z)$

12. $b(a-1)(a^2+5)$

13. $(2-a)(4-a^3)$

14. $(x-2y)(x^2+3y^2)$

15. $(x+q)(p-5)$

16. $(x-y)(x-1)$

17. $3(3a-1)(a-1)$

18. $(2x-3)(2x-7)$

19. $(a-3)(a^2+1)$

20. $(3a+4b)(x-2y)$

21. $(bx+a)(ax+b)$

22. $(x-1)(x^2+a+1)$

23. $(1-4y)(2x-1)$

24. $(bx-ay)(ax-by)$

25. $(a+b)(a+b^2)$

26. $(a-2b)(a^2+b)$

27. $(2a-c)(a-b)$

28. $(a^2+b^2)(x^2+y^2)$

29. $(a-c)(a+b)$

30. $(a-c)(a-2b)$

31. $(a+x)(ax^2+1)$

32. $(ax+b)(bx+a)$

33. $(x-a)(x-b)$

34. $\left(x-\dfrac{1}{x}\right)\left(x-\dfrac{1}{x}-3\right)$

HINTS TO SOME SELECTED QUESTIONS

16. $x^2+y-xy-x=x^2-xy-x+y$.

19. Given expression $=(a^3-3a^2)+(a-3)$.

20. Given expression $=(3ax+4bx)-(6ay+8by)$.

22. Given expression $=(x^3-x^2)+(ax-a)+(x-1)$.

23. Given expression $=(2x-8xy)-1(1-4y)$.

28. Given expression $=a^2x^2+b^2y^2+b^2x^2+a^2y^2$

$$=(a^2x^2+b^2x^2)+(a^2y^2+b^2y^2).$$

30. Given expression $= (a^2 - ac) - (2ab - 2bc)$.

34. Given expression $= \left(x - \frac{1}{x}\right)^2 - 3\left(x - \frac{1}{x}\right)$.

FACTORISING THE DIFFERENCE OF TWO SQUARES

FORMULA $(a^2 - b^2) = (a - b)(a + b)$.

SOLVED EXAMPLES

EXAMPLE 1 *Factorise*

(i) $\left(x^2 - \frac{y^2}{100}\right)$ *(ii)* $(100 - 9x^2)$ *(iii)* $\left(49x^2 - \frac{1}{4}\right)$

SOLUTION We have

(i) $\left(x^2 - \frac{y^2}{100}\right) = x^2 - \left(\frac{y}{10}\right)^2$

$= \left(x - \frac{y}{10}\right)\left(x + \frac{y}{10}\right)$ $[\because (a^2 - b^2) = (a - b)(a + b)]$.

$\therefore \left(x^2 - \frac{y^2}{100}\right) = \left(x - \frac{y}{10}\right)\left(x + \frac{y}{10}\right)$.

(ii) $(100 - 9x^2) = (10)^2 - (3x)^2$

$= (10 - 3x)(10 + 3x)$ $[\because (a^2 - b^2) = (a - b)(a + b)]$.

$\therefore (100 - 9x^2) = (10 - 3x)(10 + 3x)$.

(iii) $\left(49x^2 - \frac{1}{4}\right) = (7x)^2 - \left(\frac{1}{2}\right)^2$

$= \left(7x - \frac{1}{2}\right)\left(7x + \frac{1}{2}\right)$ $[\because (a^2 - b^2) = (a - b)(a + b)]$.

$\therefore \left(49x^2 - \frac{1}{4}\right) = \left(7x - \frac{1}{2}\right)\left(7x + \frac{1}{2}\right)$.

EXAMPLE 2 *Factorise*

(i) $(x^3 - x)$ *(ii)* $(18a^2x^2 - 32)$ *(iii)* $(2a^5 - 32a)$

SOLUTION We have

(i) $(x^3 - x) = x(x^2 - 1)$

$= x(x - 1)(x + 1)$ $[\because (a^2 - b^2) = (a - b)(a + b)]$.

$\therefore (x^3 - x) = x(x - 1)(x + 1)$.

(ii) $(18a^2x^2 - 32) = 2(9a^2x^2 - 16)$

$= 2 \times [(3ax)^2 - 4^2]$

$= 2(3ax - 4)(3ax + 4)$

$[\because (a^2 - b^2) = (a - b)(a + b)]$.

$\therefore \quad (18a^2x^2 - 32) = 2(3ax - 4)(3ax + 4)$.

(iii) $(2a^5 - 32a) = 2a(a^4 - 16)$

$= 2a \times [(a^2)^2 - 4^2]$

$= 2a(a^2 - 4)(a^2 + 4) \qquad [\because (a^2 - b^2) = (a - b)(a + b)]$

$= 2a(a - 2)(a + 2)(a^2 + 4)$.

$\therefore \quad (2a^5 - 32a) = 2a(a - 2)(a + 2)(a^2 + 4)$.

EXAMPLE 3 *Factorise*

(i) $x^2 - 1 - 2a - a^2$ *(ii)* $1 + 2ab - (a^2 + b^2)$

SOLUTION We have

(i) $x^2 - 1 - 2a - a^2$

$= x^2 - (1 + 2a + a^2)$

$= x^2 - (1 + a)^2$

$= \{x - (1 + a)\}\{x + (1 + a)\} \qquad [\because (a^2 - b^2) = (a - b)(a + b)]$

$= (x - 1 - a)(x + 1 + a)$.

$\therefore \quad (x^2 - 1 - 2a - a^2) = (x - 1 - a)(x + 1 + a)$.

(ii) $1 + 2ab - (a^2 + b^2)$

$= 1 - (a^2 + b^2 - 2ab)$

$= 1^2 - (a - b)^2$

$= \{1 - (a - b)\}\{1 + (a - b)\} \qquad [\because (a^2 - b^2) = (a - b)(a + b)]$

$= (1 - a + b)(1 + a - b)$.

$\therefore \quad 1 + 2ab - (a^2 + b^2) = (1 - a + b)(1 + a - b)$.

EXAMPLE 4 *Factorise*

(i) $(x^4 + 4)$ *(ii)* $\left(x^2 + \dfrac{4}{x^2}\right)$ *(iii)* $\left(x^4 + \dfrac{1}{x^4} + 1\right)$

SOLUTION We have

(i) $(x^4 + 4)$

$= (x^2)^2 + 2^2 + (2 \times x^2 \times 2) - (2 \times x^2 \times 2)$

[adding and subtracting $(2 \times x^2 \times 2)$]

$= (x^2 + 2)^2 - 4x^2$

$= (x^2 + 2)^2 - (2x)^2$

$$= (x^2 + 2 - 2x)(x^2 + 2 + 2x) = (x^2 - 2x + 2)(x^2 + 2x + 2).$$

$$\therefore \quad (x^4 + 4) = (x^2 - 2x + 2)(x^2 + 2x + 2).$$

(ii) $\left(x^2 + \frac{4}{x^2}\right)$

$$= x^2 + \left(\frac{2}{x}\right)^2 + \left(2 \times x \times \frac{2}{x}\right) - \left(2 \times x \times \frac{2}{x}\right)$$

[adding and subtracting $\left(2 \times x \times \frac{2}{x}\right)$]

$$= \left(x + \frac{2}{x}\right)^2 - 4 = \left\{\left(x + \frac{2}{x}\right)^2 - 2^2\right\}$$

$$= \left(x + \frac{2}{x} - 2\right)\left(x + \frac{2}{x} + 2\right).$$

$$\therefore \quad \left(x^2 + \frac{4}{x^2}\right) = \left(x + \frac{2}{x} - 2\right)\left(x + \frac{2}{x} + 2\right).$$

(iii) $\left(x^4 + \frac{1}{x^4} + 1\right)$

$$= \left(x^4 + \frac{1}{x^4} + 2\right) - 1 = \left(x^2 + \frac{1}{x^2}\right)^2 - 1^2$$

$$= \left(x^2 + \frac{1}{x^2} - 1\right)\left(x^2 + \frac{1}{x^2} + 1\right)$$

$$= \left(x^2 + \frac{1}{x^2} - 1\right)\left\{\left(x^2 + \frac{1}{x^2} + 2\right) - 1\right\}$$

$$= \left(x^2 + \frac{1}{x^2} - 1\right)\left\{\left(x + \frac{1}{x}\right)^2 - 1^2\right\}$$

$$= \left(x^2 + \frac{1}{x^2} - 1\right)\left(x + \frac{1}{x} - 1\right)\left(x + \frac{1}{x} + 1\right).$$

$$\therefore \quad \left(x^4 + \frac{1}{x^4} + 1\right) = \left(x^2 + \frac{1}{x^2} - 1\right)\left(x + \frac{1}{x} - 1\right)\left(x + \frac{1}{x} + 1\right).$$

EXAMPLE 5 *Factorise* $(x^4 + x^2y^2 + y^4)$.

SOLUTION We have

$$(x^4 + x^2y^2 + y^4) = (x^4 + 2x^2y^2 + y^4) - x^2y^2$$
$$= (x^2 + y^2)^2 - (xy)^2$$
$$= (x^2 + y^2 - xy)(x^2 + y^2 + xy).$$

$$\therefore \quad (x^4 + x^2y^2 + y^4) = (x^2 + y^2 - xy)(x^2 + y^2 + xy).$$

EXAMPLE 6 *Factorise* $a(a-1) - b(b-1)$.

SOLUTION We have

$$a(a-1) - b(b-1) = a^2 - a - b^2 + b$$

$$= (a^2 - b^2) - (a - b)$$
$$= (a - b)(a + b) - (a - b)$$
$$= (a - b)\{(a + b) - 1\}$$
$$= (a - b)(a + b - 1).$$

Hence, $a(a - 1) - b(b - 1) = (a - b)(a + b - 1)$.

EXAMPLE 7 *Factorise (i)* $(x^4 + 3x^2 + 4)$ *(ii)* $(x^4 + 5x^2 + 9)$

SOLUTION We have

(i) $(x^4 + 3x^2 + 4) = (x^4 + 4x^2 + 4) - x^2$
$$= (x^2 + 2)^2 - x^2$$
$$= (x^2 + 2 - x)(x^2 + 2 + x)$$
$$= (x^2 - x + 2)(x^2 + x + 2).$$

$\therefore \quad (x^4 + 3x^2 + 4) = (x^2 - x + 2)(x^2 + x + 2).$

(ii) $(x^4 + 5x^2 + 9) = (x^4 + 6x^2 + 9) - x^2$
$$= (x^2 + 3)^2 - x^2$$
$$= (x^2 + 3 - x)(x^2 + 3 + x)$$
$$= (x^2 - x + 3)(x^2 + x + 3).$$

$\therefore \quad (x^4 + 5x^2 + 9) = (x^2 - x + 3)(x^2 + x + 3).$

EXAMPLE 8 *Factorise* $(x^8 - y^8)$.

SOLUTION We have

$(x^8 - y^8) = (x^4)^2 - (y^4)^2$
$$= (x^4 - y^4)(x^4 + y^4)$$
$$= \{(x^2)^2 - (y^2)^2\}(x^4 + y^4 + 2x^2y^2 - 2x^2y^2)$$
$$= (x^2 - y^2)(x^2 + y^2)[(x^2 + y^2)^2 - (\sqrt{2}\,xy)^2]$$
$$= (x - y)(x + y)(x^2 + y^2)(x^2 + y^2 - \sqrt{2}\,xy)(x^2 + y^2 + \sqrt{2}\,xy).$$

$\therefore \quad (x^8 - y^8) = (x - y)(x + y)(x^2 + y^2)(x^2 + y^2 - \sqrt{2}\,xy)$
$(x^2 + y^2 + \sqrt{2}\,xy)$.

EXAMPLE 9 *Factorise* $\left(2x + \frac{1}{3}\right)^2 - \left(x - \frac{1}{2}\right)^2$.

SOLUTION We have

$$\left(2x + \frac{1}{3}\right)^2 - \left(x - \frac{1}{2}\right)^2$$
$$= \left\{\left(2x + \frac{1}{3}\right) + \left(x - \frac{1}{2}\right)\right\}\left\{\left(2x + \frac{1}{3}\right) - \left(x - \frac{1}{2}\right)\right\}$$
$$= \left(2x + \frac{1}{3} + x - \frac{1}{2}\right)\left(2x + \frac{1}{3} - x + \frac{1}{2}\right)$$

$$= \left\{3x + \left(\frac{1}{3} - \frac{1}{2}\right)\right\}\left\{x + \left(\frac{1}{3} + \frac{1}{2}\right)\right\}$$

$$= \left(3x - \frac{1}{6}\right)\left(x + \frac{5}{6}\right).$$

$$\therefore \quad \left(2x + \frac{1}{3}\right)^2 - \left(x - \frac{1}{2}\right)^2 = \left(3x - \frac{1}{6}\right)\left(x + \frac{5}{6}\right).$$

EXERCISE 3B

Factorise:

1. $9x^2 - 16y^2$

2. $\left(\frac{25}{4}x^2 - \frac{1}{9}y^2\right)$

3. $81 - 16x^2$

4. $5 - 20x^2$

5. $2x^4 - 32$

6. $3a^3b - 243ab^3$

7. $3x^3 - 48x$

8. $27a^2 - 48b^2$

9. $x - 64x^3$

10. $8ab^2 - 18a^3$

11. $150 - 6x^2$

12. $2 - 50x^2$

13. $20x^2 - 45$

14. $(3a + 5b)^2 - 4c^2$

15. $a^2 - b^2 - a - b$

16. $4a^2 - 9b^2 - 2a - 3b$

17. $a^2 - b^2 + 2bc - c^2$

18. $4a^2 - 4b^2 + 4a + 1$

19. $a^2 + 2ab + b^2 - 9c^2$

20. $108a^2 - 3(b - c)^2$

21. $(a + b)^3 - a - b$

22. $x^2 + y^2 - z^2 - 2xy$

23. $x^2 + 2xy + y^2 - a^2 + 2ab - b^2$

24. $25x^2 - 10x + 1 - 36y^2$

25. $a - b - a^2 + b^2$

26. $a^2 - b^2 - 4ac + 4c^2$

27. $9 - a^2 + 2ab - b^2$

28. $x^3 - 5x^2 - x + 5$

29. $1 + 2ab - (a^2 + b^2)$

30. $9a^2 + 6a + 1 - 36b^2$

31. $x^2 - y^2 + 6y - 9$

32. $4x^2 - 9y^2 - 2x - 3y$

33. $9a^2 + 3a - 8b - 64b^2$

34. $x^2 + \frac{1}{x^2} - 3$

35. $x^2 - 2 + \frac{1}{x^2} - y^2$

36. $x^4 + \frac{4}{x^4}$

37. $x^8 - 1$

38. $16x^4 - 1$

39. $81x^4 - y^4$

40. $x^4 - 625$

ANSWERS (EXERCISE 3B)

1. $(3x - 4y)(3x + 4y)$

2. $\left(\frac{5}{2}x - \frac{1}{3}y\right)\left(\frac{5}{2}x + \frac{1}{3}y\right)$

3. $(9-4x)(9+4x)$

4. $5(1-2x)(1+2x)$

5. $2(x-2)(x+2)(x^2+4)$

6. $3ab(a-9b)(a+9b)$

7. $3x(x-4)(x+4)$

8. $3(3a-4b)(3a+4b)$

9. $x(1-8x)(1+8x)$

10. $2a(2b-3a)(2b+3a)$

11. $6(5-x)(5+x)$

12. $2(1-5x)(1+5x)$

13. $5(2x-3)(2x+3)$

14. $(3a+5b-2c)(3a+5b+2c)$

15. $(a+b)(a-b-1)$

16. $(2a+3b)(2a-3b-1)$

17. $(a-b+c)(a+b-c)$

18. $(2a+2b+1)(2a-2b+1)$

19. $(a+b+3c)(a+b-3c)$

20. $3(6a+b-c)(6a-b+c)$

21. $(a+b)(a+b+1)(a+b-1)$

22. $(x-y+z)(x-y-z)$

23. $(x+y+a-b)(x+y-a+b)$

24. $(5x-1+6y)(5x-1-6y)$

25. $(a-b)(1-a-b)$

26. $(a-2c+b)(a-2c-b)$

27. $(3+a-b)(3-a+b)$

28. $(x-5)(x-1)(x+1)$

29. $(1+a-b)(1-a+b)$

30. $(3a+1-6b)(3a+1+6b)$

31. $(x+y-3)(x-y+3)$

32. $(2x+3y)(2x-3y-1)$

33. $(3a-8b)(3a+8b+1)$

34. $\left(x-\frac{1}{x}+1\right)\left(x-\frac{1}{x}-1\right)$

35. $\left(x-\frac{1}{x}-y\right)\left(x-\frac{1}{x}+y\right)$

36. $\left(x^2+\frac{2}{x^2}-2\right)\left(x^2+\frac{2}{x^2}+2\right)$

37. $(x-1)(x+1)(x^2+1)(x^2+1-\sqrt{2}\,x)(x^2+1+\sqrt{2}\,x)$

38. $(2x-1)(2x+1)(4x^2+1)$

39. $(3x-y)(3x+y)(9x^2+y^2)$

40. $(x-5)(x+5)(x^2+25)$

HINTS TO SOME SELECTED QUESTIONS

17. $a^2-b^2+2bc-c^2=a^2-(b^2-2bc+c^2)=a^2-(b-c)^2$.

18. $4a^2-4b^2+4a+1=(4a^2+4a+1)-4b^2=(2a+1)^2-(2b)^2$.

21. $(a+b)^3-a-b=(a+b)^3-(a+b)=(a+b)[(a+b)^2-1]=(a+b)[(a+b)^2-1^2]$.

22. $x^2+y^2-z^2-2xy=(x^2+y^2-2xy)-z^2=(x-y)^2-z^2$.

26. $a^2-b^2-4ac+4c^2=(a^2-4ac+4c^2)-b^2=(a-2c)^2-b^2$.

28. $x^3-5x^2-x+5=x^2(x-5)-(x-5)=(x-5)(x^2-1)$.

29. $1+2ab-(a^2+b^2)=1-(a^2+b^2-2ab)=1^2-(a-b)^2$.

31. $x^2-y^2+6y-9=x^2-(y^2-6y+9)=x^2-(y-3)^2$.

32. $4x^2-9y^2-2x-3y=(4x^2-9y^2)-(2x+3y)$.

33. $9a^2+3a-8b-64b^2=(9a^2-64b^2)+(3a-8b)$.

34. $x^2+\frac{1}{x^2}-3=\left(x^2+\frac{1}{x^2}-2\right)-1=\left(x-\frac{1}{x}\right)^2-1^2$.

35. $x^2 - 2 + \frac{1}{x^2} - y^2 = \left(x - \frac{1}{x}\right)^2 - y^2.$

36. $x^4 + \frac{4}{x^4} = \left(x^4 + \frac{4}{x^4} + 4\right) - 4 = \left(x^2 + \frac{2}{x^2}\right)^2 - 2^2.$

FACTORISATION OF QUADRATIC TRINOMIALS

Polynomials of The Form $x^2 + bx + c$.

We find integers p and q such that $p + q = b$ and $pq = c$. Then,

$$\begin{aligned} x^2 + bx + c &= x^2 + (p+q)x + pq \\ &= x^2 + px + qx + pq \\ &= x(x+p) + q(x+p) \\ &= (x+p)(x+q). \end{aligned}$$

SOLVED EXAMPLES

EXAMPLE 1 *Factorise* $x^2 + 9x + 18.$

SOLUTION The given expression is $x^2 + 9x + 18.$

We try to split 9 into two parts whose sum is 9 and product 18.

Clearly, $6 + 3 = 9$ and $6 \times 3 = 18.$

$$\therefore \quad \begin{aligned} x^2 + 9x + 18 &= x^2 + 6x + 3x + 18 \\ &= x(x+6) + 3(x+6) \\ &= (x+6)(x+3). \end{aligned}$$

Hence, $x^2 + 9x + 18 = (x+6)(x+3).$

EXAMPLE 2 *Factorise* $x^2 + 5x - 24.$

SOLUTION The given expression is $x^2 + 5x - 24.$

We try to split 5 into two parts whose sum is 5 and product –24.

Clearly, $8 + (-3) = 5$ and $8 \times (-3) = -24.$

$$\therefore \quad \begin{aligned} x^2 + 5x - 24 &= x^2 + 8x - 3x - 24 \\ &= x(x+8) - 3(x+8) \\ &= (x+8)(x-3). \end{aligned}$$

Hence, $x^2 + 5x - 24 = (x+8)(x-3).$

EXAMPLE 3 *Factorise* $x^2 - 4x - 21.$

SOLUTION The given expression is $x^2 - 4x - 21.$

We split -4 into two parts whose sum is -4 and product -21.

Clearly, $(-7)+3=-4$ and $(-7)\times 3=-21$.

$$\therefore\quad x^2-4x-21 = x^2-7x+3x-21$$
$$= x(x-7)+3(x-7)$$
$$= (x-7)(x+3).$$

Hence, $x^2-4x-21=(x-7)(x+3)$.

EXAMPLE 4 *Factorise*

(i) $x^2+5\sqrt{3}x+12$ (ii) $x^2+3\sqrt{3}x-30$

SOLUTION (i) The given expression is $x^2+5\sqrt{3}x+12$.

We split $5\sqrt{3}$ into two parts whose sum is $5\sqrt{3}$ and product 12.

Clearly, $(4\sqrt{3}+\sqrt{3})=5\sqrt{3}$ and $(4\sqrt{3}\times\sqrt{3})=12$.

$$\therefore\quad x^2+5\sqrt{3}x+12 = x^2+4\sqrt{3}x+\sqrt{3}x+12$$
$$= x(x+4\sqrt{3})+\sqrt{3}(x+4\sqrt{3})$$
$$= (x+4\sqrt{3})(x+\sqrt{3}).$$

Hence, $x^2+5\sqrt{3}x+12=(x+4\sqrt{3})(x+\sqrt{3})$.

(ii) The given expression is $x^2+3\sqrt{3}x-30$.

We split $3\sqrt{3}$ into two parts whose sum is $3\sqrt{3}$ and product -30.

Clearly, $(5\sqrt{3}-2\sqrt{3})=3\sqrt{3}$ and $5\sqrt{3}\times\{-2\sqrt{3}\}=-30$.

$$\therefore\quad x^2+3\sqrt{3}x-30 = x^2+5\sqrt{3}x-2\sqrt{3}x-30$$
$$= x(x+5\sqrt{3})-2\sqrt{3}(x+5\sqrt{3})$$
$$= (x+5\sqrt{3})(x-2\sqrt{3}).$$

Hence, $x^2+3\sqrt{3}x-30=(x+5\sqrt{3})(x-2\sqrt{3})$.

EXAMPLE 5 *Factorise* $(p+q)^2-20(p+q)-125$.

SOLUTION The given expression is $(p+q)^2-20(p+q)-125$.

Putting $(p+q)=x$, it becomes $x^2-20x-125$.

We split -20 into two parts whose sum is -20 and product -125.

Clearly, $(-25+5)=-20$ and $(-25)\times 5=-125$.

$$\therefore\quad x^2-20x-125 = x^2-25x+5x-125$$
$$= x(x-25)+5(x-25)$$
$$= (x-25)(x+5).$$

$\therefore\quad (p+q)^2-20(p+q)-125=(p+q-25)(p+q+5)$ [$\because\ x=p+q$].

EXAMPLE 6 *Factorise* $a^4 + 4a^2 + 3$.

SOLUTION The given expression is $a^4 + 4a^2 + 3$.

Putting $a^2 = x$, it becomes $x^2 + 4x + 3$.

We split 4 into two parts whose sum is 4 and product 3.

Clearly, $(3 + 1) = 4$ and $(3 \times 1) = 3$.

$$\therefore \quad x^2 + 4x + 3 = x^2 + 3x + x + 3$$
$$= x(x+3) + (x+3) = (x+3)(x+1)$$
$$= (a^4 + 4a^2 + 3) = (a^2+3)(a^2+1) \qquad [\because \; x = a^2].$$

Factorisation of Polynomials of The Form $ax^2 + bx + c$

We find integers p and q such that $p + q = b$ and $pq = ac$. Then,

$$ax^2 + bx + c = ax^2 + (p+q)x + \frac{pq}{a}$$
$$= a^2x^2 + apx + aqx + pq$$
$$= ax(ax+p) + q(ax+p)$$
$$= (ax+p)(ax+q).$$

Hence, $(ax^2 + bx + c) = (ax+p)(ax+q)$.

EXAMPLE 7 *Factorise* $6x^2 + 17x + 5$.

SOLUTION The given expression is $6x^2 + 17x + 5$.

Here, $6 \times 5 = 30$.

We split 17 into two parts whose sum is 17 and product 30.

Clearly, $(15 + 2) = 17$ and $(15 \times 2) = 30$.

$$\therefore \quad 6x^2 + 17x + 5 = 6x^2 + 15x + 2x + 5$$
$$= 3x(2x+5) + (2x+5)$$
$$= (2x+5)(3x+1).$$

Hence, $6x^2 + 17x + 5 = (2x+5)(3x+1)$.

EXAMPLE 8 *Factorise* $\sqrt{2}x^2 + 9x + 4\sqrt{2}$.

SOLUTION The given expression is $\sqrt{2}x^2 + 9x + 4\sqrt{2}$.

Here, $\sqrt{2} \times 4\sqrt{2} = 8$.

We split 9 into two parts whose sum is 9 and product 8.

Clearly, $(8 + 1) = 9$ and $(8 \times 1) = 8$.

$$\therefore \quad \sqrt{2}x^2 + 9x + 4\sqrt{2} = \sqrt{2}x^2 + 8x + x + 4\sqrt{2}$$
$$= \sqrt{2}x(x + 4\sqrt{2}) + (x + 4\sqrt{2})$$

$$= (x + 4\sqrt{2})(\sqrt{2}x + 1).$$

Hence, $(\sqrt{2}x^2 + 9x + 4\sqrt{2}) = (x + 4\sqrt{2})(\sqrt{2}x + 1)$.

EXAMPLE 9 *Factorise* $2x^2 + 11x - 21$.

SOLUTION The given expression is $2x^2 + 11x - 21$.

Here, $2 \times (-21) = -42$.

We split 11 into two parts whose sum is 11 and product -42.

Clearly, $14 + (-3) = 11$ and $14 \times (-3) = -42$.

$$\begin{aligned} \therefore \quad 2x^2 + 11x - 21 &= 2x^2 + 14x - 3x - 21 \\ &= 2x(x + 7) - 3(x + 7) \\ &= (x + 7)(2x - 3). \end{aligned}$$

Hence, $(2x^2 + 11x - 21) = (x + 7)(2x - 3)$.

EXAMPLE 10 *Factorise* $6x^2 + 7x - 3$.

SOLUTION The given expression is $6x^2 + 7x - 3$.

Here, $6 \times (-3) = -18$.

We split 7 into two parts whose sum is 7 and product –18.

Clearly, $9 + (-2) = 7$ and $9 \times (-2) = -18$.

$$\begin{aligned} \therefore \quad 6x^2 + 7x - 3 &= 6x^2 + 9x - 2x - 3 \\ &= 3x(2x + 3) - (2x + 3) \\ &= (2x + 3)(3x - 1). \end{aligned}$$

Hence, $(6x^2 + 7x - 3) = (2x + 3)(3x - 1)$.

EXAMPLE 11 *Factorise* $9x^2 - 22x + 8$.

SOLUTION The given expression is $9x^2 - 22x + 8$.

Here, $9 \times 8 = 72$.

We split –22 into two parts whose sum is –22 and product 72.

Clearly, $(-18) + (-4) = -22$ and $(-18) \times (-4) = 72$.

$$\begin{aligned} \therefore \quad 9x^2 - 22x + 8 &= 9x^2 - 18x - 4x + 8 \\ &= 9x(x - 2) - 4(x - 2) \\ &= (x - 2)(9x - 4). \end{aligned}$$

Hence, $(9x^2 - 22x + 8) = (x - 2)(9x - 4)$.

EXAMPLE 12 *Factorise* $35y^2 + 13y - 12$.

SOLUTION The given expression is $35y^2 + 13y - 12$.

Here, $35 \times (-12) = -420$.

We split 13 into two parts whose sum is 13 and product -420.

Clearly, $28 + (-15) = 13$ and $28 \times (-15) = -420$.

$$\therefore \quad 35y^2 + 13y - 12 = 35y^2 + 28y - 15y - 12$$
$$= 7y(5y+4) - 3(5y+4)$$
$$= (5y+4)(7y-3).$$

Hence, $(35y^2 + 13y - 12) = (5y+4)(7y-3)$.

EXAMPLE 13 *Factorise* $4\sqrt{3}x^2 + 5x - 2\sqrt{3}$.

SOLUTION The given expression is $4\sqrt{3}x^2 + 5x - 2\sqrt{3}$.

Here, $4\sqrt{3} \times (-2\sqrt{3}) = -24$.

So, we split 5 into two parts whose sum is 5 and product –24.

Clearly, $8 + (-3) = 5$ and $8 \times (-3) = -24$.

$$\therefore \quad 4\sqrt{3}x^2 + 5x - 2\sqrt{3} = 4\sqrt{3}x^2 + 8x - 3x - 2\sqrt{3}$$
$$= 4x(\sqrt{3}x + 2) - \sqrt{3}(\sqrt{3}x + 2)$$
$$= (\sqrt{3}x + 2)(4x - \sqrt{3}).$$

Hence, $(4\sqrt{3}x^2 + 5x - 2\sqrt{3}) = (\sqrt{3}x + 2)(4x - \sqrt{3})$.

EXAMPLE 14 *Factorise* $2x^2 - 7x - 39$.

SOLUTION The given expression is $2x^2 - 7x - 39$.

Here, $2 \times (-39) = -78$.

So, we split –7 into two parts whose sum is –7 and product –78.

Clearly, $(-13) + 6 = -7$ and $(-13) \times 6 = -78$.

$$\therefore \quad 2x^2 - 7x - 39 = 2x^2 - 13x + 6x - 39$$
$$= x(2x - 13) + 3(2x - 13)$$
$$= (2x - 13)(x + 3).$$

Hence, $(2x^2 - 7x - 39) = (2x - 13)(x + 3)$.

EXAMPLE 15 *Factorise* $42 - r - r^2$.

SOLUTION The given expression may be written as $-r^2 - r + 42$.

Here, $(-1) \times 42 = -42$.

So, we split –1 into two parts whose sum is –1 and product -42.

Clearly, $(-7) + 6 = -1$ and $(-7) \times 6 = -42$.

$$\therefore \quad 42 - r - r^2 = -r^2 - r + 42$$
$$= -r^2 - 7r + 6r + 42$$

$$= -r(r+7) + 6(r+7)$$
$$= (r+7)(-r+6) = (7+r)(6-r).$$

Hence, $(42 - r - r^2) = (7+r)(6-r)$.

EXAMPLE 16 *Factorise* $2x^2 - \frac{5}{6}x + \frac{1}{12}$.

SOLUTION We have

$$2x^2 - \frac{5}{6}x + \frac{1}{12} = \frac{(24x^2 - 10x + 1)}{12}$$
$$= \frac{1}{12}(24x^2 - 10x + 1)$$
$$= \frac{1}{12}(24x^2 - 6x - 4x + 1)$$
$$= \frac{1}{12}[6x(4x-1) - (4x-1)]$$
$$= \frac{1}{12}(4x-1)(6x-1)$$
$$= \left(\frac{4x}{12} - \frac{1}{12}\right)(6x-1) = \left(\frac{1}{3}x - \frac{1}{12}\right)(6x-1).$$

Hence, $\left(2x^2 - \frac{5}{6}x + \frac{1}{12}\right) = \left(\frac{1}{3}x - \frac{1}{12}\right)(6x-1)$.

EXAMPLE 17 *Factorise* $5(3x+y)^2 + 6(3x+y) - 8$.

SOLUTION Putting $(3x+y) = z$ in the given expression, we get

$$5(3x+y)^2 + 6(3x+y) - 8$$
$$= 5z^2 + 6z - 8, \text{ where } 3x+y = z$$
$$= 5z^2 + 10z - 4z - 8 \qquad [\because\ 10 + (-4) = 6 \text{ and } 10 \times (-4) = -40]$$
$$= 5z(z+2) - 4(z+2)$$
$$= (z+2)(5z-4)$$
$$= (3x+y+2)[5(3x+y) - 4]$$
$$= (3x+y+2)(15x+5y-4).$$

Hence, $5(3x+y)^2 + 6(3x+y) - 8 = (3x+y+2)(15x+5y-4)$.

EXAMPLE 18 *Factorise* $x^4 - 3x^2 + 2$.

SOLUTION Putting $x^2 = y$, we get

$$x^4 - 3x^2 + 2$$
$$= y^2 - 3y + 2, \text{ where } x^2 = y$$
$$= y^2 - 2y - y + 2$$
$$= y(y-2) - (y-2)$$

$= (y-2)(y-1)$

$= (x^2-2)(x^2-1)$ $[\because\ y = x^2]$

$= (x+\sqrt{2})(x-\sqrt{2})(x+1)(x-1)$.

Hence, $(x^4-3x^2+2) = (x+\sqrt{2})(x-\sqrt{2})(x+1)(x-1)$.

EXAMPLE 19 *Factorise* $7\sqrt{2}x^2-10x-4\sqrt{2}$.

SOLUTION The given expression is $7\sqrt{2}x^2-10x-4\sqrt{2}$.

Here, $7\sqrt{2}\times(-4\sqrt{2}) = -56$.

So, we split -10 into two parts whose sum is -10 and product -56.

Clearly, $(-14+4) = -10$ and $(-14)\times 4 = -56$.

$$\begin{aligned}\therefore\quad 7\sqrt{2}x^2-10x-4\sqrt{2} &= 7\sqrt{2}x^2-14x+4x-4\sqrt{2}\\ &= 7\sqrt{2}x(x-\sqrt{2})+4(x-\sqrt{2})\\ &= (x-\sqrt{2})(7\sqrt{2}x+4).\end{aligned}$$

Hence, $(7\sqrt{2}x^2-10x-4\sqrt{2}) = (x-\sqrt{2})(7\sqrt{2}x+4)$.

EXAMPLE 20 *Factorise* $5\sqrt{5}x^2+30x+8\sqrt{5}$.

SOLUTION The given expression is $5\sqrt{5}x^2+30x+8\sqrt{5}$.

Here, $(5\sqrt{5}\times 8\sqrt{5}) = 200$.

So, we split 30 into two parts whose sum is 30 and product 200.

Clearly, $(20+10) = 30$ and $(20\times 10) = 200$.

$$\begin{aligned}\therefore\quad 5\sqrt{5}x^2+30x+8\sqrt{5} &= 5\sqrt{5}x^2+20x+10x+8\sqrt{5}\\ &= 5x(\sqrt{5}x+4)+2\sqrt{5}(\sqrt{5}x+4)\\ &= (\sqrt{5}x+4)(5x+2\sqrt{5}).\end{aligned}$$

Hence, $(5\sqrt{5}x^2+30x+8\sqrt{5}) = (\sqrt{5}x+4)(5x+2\sqrt{5})$.

EXAMPLE 21 *Factorise* $4x^2+20x+25$.

SOLUTION The given expression is $4x^2+20x+25$.

Here, $(4\times 25) = 100$.

So, we split 20 into two parts whose sum is 20 and product 100.

Clearly, $(10+10) = 20$ and $(10\times 10) = 100$.

$$\begin{aligned}\therefore\quad 4x^2+20x+25 &= 4x^2+10x+10x+25\\ &= 2x(2x+5)+5(2x+5)\\ &= (2x+5)(2x+5).\end{aligned}$$

Hence, $(4x^2+20x+25) = (2x+5)(2x+5)$.

EXERCISE 3C

Factorise:

1. $x^2+11x+30$

2. $x^2+18x+32$

3. $x^2+20x-69$

4. $x^2+19x-150$

5. $x^2+7x-98$

6. $x^2+2\sqrt{3}\,x-24$

7. $x^2-21x+90$

8. $x^2-22x+120$

9. x^2-4x+3

10. $x^2+7\sqrt{6}\,x+60$

11. $x^2+3\sqrt{3}\,x+6$

12. $x^2+6\sqrt{6}\,x+48$

13. $x^2+5\sqrt{5}\,x+30$

14. $x^2-24x-180$

15. $x^2-32x-105$

16. $x^2-11x-80$

17. $6-x-x^2$

18. $x^2-\sqrt{3}\,x-6$

19. $40+3x-x^2$

20. $x^2-26x+133$

21. $x^2-2\sqrt{3}\,x-24$

22. $x^2-3\sqrt{5}\,x-20$

23. $x^2+\sqrt{2}\,x-24$

24. $x^2-2\sqrt{2}\,x-30$

25. $x^2-x-156$

26. $x^2-32x-105$

27. $9x^2+18x+8$

28. $6x^2+17x+12$

29. $18x^2+3x-10$

30. $2x^2+11x-21$

31. $15x^2+2x-8$

32. $21x^2+5x-6$

33. $24x^2-41x+12$

34. $3x^2-14x+8$

35. $2x^2+3x-90$

36. $\sqrt{5}\,x^2+2x-3\sqrt{5}$

37. $2\sqrt{3}\,x^2+x-5\sqrt{3}$

38. $7x^2+2\sqrt{14}\,x+2$

39. $6\sqrt{3}\,x^2-47x+5\sqrt{3}$

40. $5\sqrt{5}\,x^2+20x+3\sqrt{5}$

41. $\sqrt{3}\,x^2+10x+8\sqrt{3}$

42. $\sqrt{2}\,x^2+3x+\sqrt{2}$

43. $2x^2+3\sqrt{3}\,x+3$

44. $15x^2-x-28$

45. $6x^2-5x-21$

46. $2x^2-7x-15$

47. $5x^2-16x-21$

48. $6x^2-11x-35$

49. $9x^2-3x-20$

50. $10x^2-9x-7$

51. $x^2-2x+\dfrac{7}{16}$

52. $\dfrac{1}{3}x^2-2x-9$

53. $x^2+\dfrac{12}{35}x+\dfrac{1}{35}$

54. $21x^2-2x+\dfrac{1}{21}$

55. $\dfrac{3}{2}x^2+16x+10$

56. $\dfrac{2}{3}x^2-\dfrac{17}{3}x-28$

57. $\dfrac{3}{5}x^2-\dfrac{19}{5}x+4$

58. $2x^2-x+\dfrac{1}{8}$

59. $2(x+y)^2-9(x+y)-5$

60. $9(2a-b)^2-4(2a-b)-13$

61. $7(x-2y)^2-25(x-2y)+12$

62. $10\left(3x+\frac{1}{x}\right)^2-\left(3x+\frac{1}{x}\right)-3$

63. $6\left(2x-\frac{3}{x}\right)^2+7\left(2x-\frac{3}{x}\right)-20$

64. $(a+2b)^2+101(a+2b)+100$

65. $4x^4+7x^2-2$

66. Evaluate $\{(999)^2-1\}$.

ANSWERS (EXERCISE 3C)

1. $(x+6)(x+5)$

2. $(x+16)(x+2)$

3. $(x+23)(x-3)$

4. $(x+25)(x-6)$

5. $(x+14)(x-7)$

6. $(x+4\sqrt{3})(x-2\sqrt{3})$

7. $(x-15)(x-6)$

8. $(x-12)(x-10)$

9. $(x-3)(x-1)$

10. $(x+5\sqrt{6})(x+2\sqrt{6})$

11. $(x+2\sqrt{3})(x+\sqrt{3})$

12. $(x+4\sqrt{6})(x+2\sqrt{6})$

13. $(x+3\sqrt{5})(x+2\sqrt{5})$

14. $(x-30)(x+6)$

15. $(x-35)(x+3)$

16. $(x-16)(x+5)$

17. $(2-x)(3+x)$

18. $(x-2\sqrt{3})(x+\sqrt{3})$

19. $(8-x)(5+x)$

20. $(x-19)(x-7)$

21. $(x-4\sqrt{3})(x+2\sqrt{3})$

22. $(x-4\sqrt{5})(x+\sqrt{5})$

23. $(x+4\sqrt{2})(x-3\sqrt{2})$

24. $(x-5\sqrt{2})(x+3\sqrt{2})$

25. $(x-13)(x+12)$

26. $(x-35)(x+3)$

27. $(3x+4)(3x+2)$

28. $(2x+3)(3x+4)$

29. $(6x+5)(3x-2)$

30. $(x+7)(2x-3)$

31. $(5x+4)(3x-2)$

32. $(3x+2)(7x-3)$

33. $(3x-4)(8x-3)$

34. $(x-4)(3x-2)$

35. $(2x+15)(x-6)$

36. $(x+\sqrt{5})(\sqrt{5}\,x-3)$

37. $(x+\sqrt{3})(2\sqrt{3}\,x-5)$

38. $(\sqrt{7}\,x+\sqrt{2})(\sqrt{7}\,x+\sqrt{2})$

39. $(2x-5\sqrt{3})(3\sqrt{3}\,x-1)$

40. $(\sqrt{5}\,x+3)(5x+\sqrt{5})$

41. $(x+2\sqrt{3})(\sqrt{3}\,x+4)$

42. $(x+\sqrt{2})(\sqrt{2}\,x+1)$

43. $(x+\sqrt{3})(2x+\sqrt{3})$

44. $(5x-7)(3x+4)$

45. $(3x-7)(2x+3)$

46. $(x-5)(2x+3)$

47. $(5x-21)(x+1)$

48. $(2x-7)(3x+5)$

49. $(3x-5)(3x+4)$

50. $(5x-7)(2x+1)$

51. $\left(\frac{x}{4}-\frac{7}{16}\right)(4x-1)$

52. $\left(1+\frac{x}{3}\right)(x-9)$

53. $(5x+1)\left(\frac{x}{5}+\frac{1}{35}\right)$

54. $\left(x-\frac{1}{21}\right)(21x-1)$

55. $\left(\frac{x}{2}+5\right)(3x+2)$

56. $\left(\frac{x}{3}-4\right)(2x+7)$

57. $\left(\frac{x}{5}-1\right)(3x-4)$

58. $\left(\frac{x}{2}-\frac{1}{8}\right)(4x-1)$

59. $(x+y-5)(2x+2y+1)$

60. $(18a-9b-13)(2a-b+1)$

61. $(x-2y-3)(7x-14y-4)$

62. $\left(15x+\frac{5}{x}-3\right)\left(6x+\frac{2}{x}+1\right)$

63. $\left(4x-\frac{6}{x}+5\right)\left(6x-\frac{9}{x}-4\right)$

64. $(a+2b+100)(a+2b+1)$

65. $(x^2+2)(2x-1)(2x+1)$

66. 998000

HINTS TO SOME SELECTED QUESTIONS

6. $x^2+2\sqrt{3}x-24=x^2+4\sqrt{3}x-2\sqrt{3}x-24.$

10. $x^2+7\sqrt{6}x+60=x^2+5\sqrt{6}x+2\sqrt{6}x+60.$

11. $x^2+3\sqrt{3}x+6=x^2+2\sqrt{3}x+\sqrt{3}x+6.$

12. $x^2+6\sqrt{6}x+48=x^2+4\sqrt{6}x+2\sqrt{6}x+48.$

13. $x^2+5\sqrt{5}x+30=x^2+3\sqrt{5}x+2\sqrt{5}x+30.$

18. $x^2-\sqrt{3}x-6=x^2-2\sqrt{3}x+\sqrt{3}x-6.$

21. $x^2-2\sqrt{3}x-24=x^2-4\sqrt{3}x+2\sqrt{3}x-24.$

22. $x^2-3\sqrt{5}x-20=x^2-4\sqrt{5}x+\sqrt{5}x-20.$

23. $x^2+\sqrt{2}x-24=x^2+4\sqrt{2}x-3\sqrt{2}x-24.$

24. $x^2-2\sqrt{2}x-30=x^2-5\sqrt{2}x+3\sqrt{2}x-30.$

36. $\sqrt{5}x^2+2x-3\sqrt{5}=\sqrt{5}x^2+5x-3x-3\sqrt{5}.$

37. $2\sqrt{3}x^2+x-5\sqrt{3}=2\sqrt{3}x^2+6x-5x-5\sqrt{3}.$

38. $7x^2+2\sqrt{14}x+2=7x^2+\sqrt{14}x+\sqrt{14}x+2=\sqrt{7}x(\sqrt{7}x+\sqrt{2})+\sqrt{2}(\sqrt{7}x+\sqrt{2}).$

39. $6\sqrt{3}x^2-47x+5\sqrt{3}=6\sqrt{3}x^2-45x-2x+5\sqrt{3}=3\sqrt{3}x(2x-5\sqrt{3})-(2x-5\sqrt{3}).$

40. $5\sqrt{5}x^2+20x+3\sqrt{5}=5\sqrt{5}x^2+15x+5x+3\sqrt{5}=5x(\sqrt{5}x+3)+\sqrt{5}(\sqrt{5}x+3).$

42. $\sqrt{2}x^2+3x+\sqrt{2}=\sqrt{2}x^2+2x+x+\sqrt{2}.$

43. $2x^2+3\sqrt{3}x+3=2x^2+2\sqrt{3}x+\sqrt{3}x+3.$

44. $15x^2-x-28=15x^2-21x+20x-28.$

51. $x^2-2x+\frac{7}{16}=\frac{16x^2-32x+7}{16}=\frac{16x^2-28x-4x+7}{16}=\frac{(4x-7)(4x-1)}{16}$
$$=\left(\frac{4x-7}{16}\right)(4x-1)=\left(\frac{x}{4}-\frac{7}{16}\right)(4x-1).$$

66. $\{(999)^2-1^2\}=(999-1)(999+1).$

SQUARE OF A TRINOMIAL

FORMULA $(x+y+z)^2 = x^2+y^2+z^2+2xy+2yz+2zx.$

SOLVED EXAMPLES

EXAMPLE 1 *Exapand* $(2a+3b+4c)^2$.

SOLUTION Putting $2a = x$, $3b = y$ and $4c = z$, we get

$$\begin{aligned}&(2a+3b+4c)^2\\&=(x+y+z)^2\\&=x^2+y^2+z^2+2xy+2yz+2zx\\&=(2a)^2+(3b)^2+(4c)^2+\{2\times(2a)\times(3b)\}+\{2\times(3b)\times(4c)\}\\&\qquad+\{2\times(4c)\times(2a)\}\\&=4a^2+9b^2+16c^2+12ab+24bc+16ca.\end{aligned}$$

EXAMPLE 2 *Expand each of the following:*

(i) $(4a-b+2c)^2$ *(ii)* $(3a-5b-c)^2$ *(iii)* $(-x+2y-3z)^2$

SOLUTION (i) Putting $4a = x$, $-b = y$ and $2c = z$, we get

$$\begin{aligned}&(4a-b+2c)^2\\&=(x+y+z)^2\\&=x^2+y^2+z^2+2xy+2yz+2zx\\&=(4a)^2+(-b)^2+(2c)^2+\{2\times4a\times(-b)\}+\{2\times(-b)\times2c\}\\&\qquad+\{2\times2c\times4a\}\\&=16a^2+b^2+4c^2-8ab-4bc+16ca.\end{aligned}$$

(ii) Putting $3a = x$, $-5b = y$ and $-c = z$, we get

$$\begin{aligned}&(3a-5b-c)^2\\&=(x+y+z)^2\\&=x^2+y^2+z^2+2xy+2yz+2zx\\&=(3a)^2+(-5b)^2+(-c)^2+\{2\times3a\times(-5b)\}+\{2\times(-5b)\times(-c)\}\\&\qquad+\{2\times(-c)\times3a\}\\&=9a^2+25b^2+c^2-30ab+10bc-6ca.\end{aligned}$$

(iii) Putting $-x = a$, $2y = b$ and $-3z = c$, we get

$$\begin{aligned}&(-x+2y-3z)^2\\&=(a+b+c)^2\\&=a^2+b^2+c^2+2ab+2bc+2ca\end{aligned}$$

$$= (-x)^2 + (2y)^2 + (-3z)^2 + \{2 \times (-x) \times 2y\} + \{2 \times (2y) \times (-3z)\} + \{2 \times (-3z) \times (-x)\}$$

$$= x^2 + 4y^2 + 9z^2 - 4xy - 12yz + 6zx.$$

EXAMPLE 3 *Factorise* $9x^2 + 4y^2 + 16z^2 + 12xy - 16yz - 24xz$.

SOLUTION We observe here that $-16yz$ and $-24xz$ are negative terms in the given expression and z occurs in both the terms. So, we write $16z^2$ as $(-4z)^2$.

Thus, we have

$$9x^2 + 4y^2 + 16z^2 + 12xy - 16yz - 24xz$$

$$= (3x)^2 + (2y)^2 + (-4z)^2 + \{(2 \times (3x) \times (2y)\} + \{2 \times (2y) \times (-4z)\} + \{2 \times (-4z) \times (3x)\}$$

$$= \{3x + 2y + (-4z)^2\} = (3x + 2y - 4z)^2.$$

EXAMPLE 4 *Factorise* $25x^2 + 16y^2 + 4z^2 - 40xy + 16yz - 20xz$.

SOLUTION We observe here that $-40xy$ and $-20xz$ are negative terms in the given expression and x occurs in both the terms.

So, we write, $25x^2$ as $(-5x)^2$.

$\therefore$ $$25x^2 + 16y^2 + 4z^2 - 40xy + 16yz - 20xz$$

$$= (-5x)^2 + (4y)^2 + (2z)^2 + \{2 \times (-5x) \times (4y)\} + \{2 \times (4y) \times (2z)\} + \{2 \times (-5x) \times (2z)\}$$

$$= \{(-5x) + 4y + 2z)\}^2 = (-5x + 4y + 2z)^2.$$

EXAMPLE 5 *Factorise* $2x^2 + y^2 + 8z^2 - 2\sqrt{2}xy + 4\sqrt{2}yz - 8zx$.

SOLUTION We observe here that $-2\sqrt{2}xy$ and $-8zx$ are negative terms in the given expression and x occurs in both the terms.

So, we write $2x^2$ as $(-\sqrt{2}x)^2$.

$\therefore$ $$2x^2 + y^2 + 8z^2 - 2\sqrt{2}xy + 4\sqrt{2}yz - 8zx$$

$$= (-\sqrt{2}x)^2 + y^2 + (2\sqrt{2}z)^2 + \{2 \times (-\sqrt{2}x) \times y\} + \{2 \times y \times (2\sqrt{2}z)\} + \{2 \times (2\sqrt{2}z) \times (-\sqrt{2}x)\}$$

$$= (-\sqrt{2}x + y + 2\sqrt{2}z)^2.$$

EXAMPLE 6 *Evaluate* *(i)* 101×102 *(ii)* $(999)^2$ *(iii)* $(997)^2$

SOLUTION We have

(i) $101 \times 102 = 101 \times (100 + 2)$

$$= (101 \times 100) + (101 \times 2) = 10100 + 202 = 10302.$$

(ii) $(999)^2 = (1000-1)^2$
$= (1000)^2 + 1^2 - (2 \times 1000 \times 1)$
$[\because\ (a-b)^2 = a^2 + b^2 - 2ab]$
$= 1000000 + 1 - 2000$
$= (1000001 - 2000) = 998001.$

(iii) $(997)^2 = (1000-3)^2$
$= (1000)^2 + 3^2 - (2 \times 1000 \times 3)$
$= 1000000 + 9 - 6000 = 994009.$

EXERCISE 3D

1. Expand

(i) $(a+2b+5c)^2$ (ii) $(2a-b+c)^2$ (iii) $(a-2b-3c)^2$

2. Expand

(i) $(2a-5b-7c)^2$ (ii) $(-3a+4b-5c)^2$ (iii) $\left(\frac{1}{2}a-\frac{1}{4}b+2\right)^2$

Factorise

3. $4x^2+9y^2+16z^2+12xy-24yz-16zx$

4. $9x^2+16y^2+4z^2-24xy+16yz-12xz$

5. $25x^2+4y^2+9z^2-20xy-12yz+30xz$

6. $16x^2+4y^2+9z^2-16xy-12yz+24xz$

7. Evaluate

(i) $(99)^2$ (ii) $(995)^2$ (iii) $(107)^2$

ANSWERS (EXERCISE 3D)

1. (i) $a^2+4b^2+25c^2+4ab+20bc+10ac$ (ii) $4a^2+b^2+c^2-4ab-2bc+4ac$
(iii) $a^2+4b^2+9c^2-4ab+12bc-6ac$

2. (i) $4a^2+25b^2+49c^2-20ab+70bc-28ac$
(ii) $9a^2+16b^2+25c^2-24ab-40bc+30ac$
(iii) $\frac{a^2}{4}+\frac{b^2}{16}+4-\frac{ab}{4}-b+2a$

3. $(2x+3y-4z)(2x+3y-4z)$ **4.** $(-3x+4y+2z)(-3x+4y+2z)$

5. $(5x-2y+3z)(5x-2y+3z)$ **6.** $(4x-2y+3z)(4x-2y+3z)$

7. (i) 9801 (ii) 990025 (iii) 11449

CUBE OF A BINOMIAL

FORMULAE (i) $(x+y)^3 = x^3 + y^3 + 3xy(x+y)$.

(ii) $(x-y)^3 = x^3 - y^3 - 3xy(x-y)$.

SOLVED EXAMPLES

EXAMPLE 1 *Expand (i)* $(4a+5b)^3$ *(ii)* $(3a-2b)^3$

SOLUTION (i) Putting $4a = x$ and $5b = y$, we get

$$\begin{aligned}(4a+5b)^3 &= (x+y)^3\\ &= x^3 + y^3 + 3xy(x+y)\\ &= (4a)^3 + (5b)^3 + (3 \times 4a \times 5b)(4a+5b)\\ &= 64a^3 + 125b^3 + 60ab(4a+5b)\\ &= 64a^3 + 125b^3 + 240a^2b + 300ab^2.\end{aligned}$$

(ii) Putting $3a = x$ and $2b = y$, we get

$$\begin{aligned}(3a-2b)^3 &= (x-y)^3\\ &= x^3 - y^3 - 3xy(x-y)\\ &= (3a)^3 - (2b)^3 - (3 \times 3a \times 2b)(3a-2b)\\ &= 27a^3 - 8b^3 - 18ab(3a-2b)\\ &= 27a^3 - 8b^3 - 54a^2b + 36ab^2.\end{aligned}$$

EXAMPLE 2 *Expand (i)* $\left(\frac{1}{x}+\frac{y}{3}\right)^3$ *(ii)* $\left(4-\frac{1}{3x}\right)^3$

SOLUTION (i) Putting $\frac{1}{x} = a$ and $\frac{y}{3} = b$, we get

$$\begin{aligned}\left(\frac{1}{x}+\frac{y}{3}\right)^3 &= (a+b)^3\\ &= a^3 + b^3 + 3ab(a+b)\\ &= \left(\frac{1}{x}\right)^3 + \left(\frac{y}{3}\right)^3 + 3 \times \frac{1}{x} \times \frac{y}{3} \times \left(\frac{1}{x}+\frac{y}{3}\right)\\ &= \frac{1}{x^3} + \frac{y^3}{27} + \frac{y}{x}\left(\frac{1}{x}+\frac{y}{3}\right)\\ &= \frac{1}{x^3} + \frac{y^3}{27} + \frac{y}{x^2} + \frac{y^2}{3x} = \frac{1}{x^3} + \frac{y}{x^2} + \frac{y^2}{3x} + \frac{y^3}{27}.\end{aligned}$$

(ii) Putting $4 = a$ and $\frac{1}{3x} = b$, we get

$$\left(4-\frac{1}{3x}\right)^3 = (a-b)^3$$

$$= a^3 - b^3 - 3ab(a-b)$$
$$= (4)^3 - \left(\frac{1}{3x}\right)^3 - 3 \times 4 \times \frac{1}{3x} \times \left(4 - \frac{1}{3x}\right)$$
$$= 64 - \frac{1}{27x^3} - \frac{16}{x} + \frac{4}{3x^2} = 64 - \frac{16}{x} + \frac{4}{3x^2} - \frac{1}{27x^3}.$$

EXAMPLE 3 *Simplify* $(2x-5y)^3 - (2x+5y)^3$.

SOLUTION Putting $2x = a$ and $5y = b$, we get

$$(2x-5y)^3 - (2x+5y)^3$$
$$= (a-b)^3 - (a+b)^3$$
$$= \{a^3 - b^3 - 3ab(a-b)\} - \{a^3 + b^3 + 3ab(a+b)\}$$
$$= (a^3 - b^3 - 3a^2b + 3ab^2 - a^3 - b^3 - 3a^2b - 3ab^2)$$
$$= -2b^3 - 6a^2b = -2 \times (5y)^3 - 6 \times (2x)^2 \times (5y)$$
$$= (-2 \times 125y^3) - (6 \times 4x^2 \times 5y) = -250y^3 - 120x^2y.$$

EXAMPLE 4 *Factorise*

(i) $8a^3 + b^3 + 12a^2b + 6ab^2$

(ii) $64a^3 - 27b^3 - 144a^2b + 108ab^2$

SOLUTION We have

(i) $8a^3 + b^3 + 12a^2b + 6ab^2$
$$= (2a)^3 + b^3 + 6ab(2a+b)$$
$$= (2a)^3 + b^3 + (3 \times 2a \times b)(2a+b)$$
$$= (2a+b)^3 = (2a+b)(2a+b)(2a+b).$$

(ii) $64a^3 - 27b^3 - 144a^2b + 108ab^2$
$$= (4a)^3 - (3b)^3 - 36ab(4a-3b)$$
$$= (4a)^3 - (3b)^3 - (3 \times 4a \times 3b)(4a-3b)$$
$$= (4a-3b)^3 = (4a-3b)(4a-3b)(4a-3b).$$

EXAMPLE 5 *Factorise*

(i) $27 - 125a^3 - 135a + 225a^2$ *(ii)* $1 - 64a^3 - 12a + 48a^2$

SOLUTION We have

(i) $27 - 125a^3 - 135a + 225a^2$
$$= 3^3 - (5a)^3 - 45a(3-5a)$$
$$= 3^3 - (5a)^3 - (3 \times 3 \times 5a)(3-5a)$$
$$= (3-5a)^3 = (3-5a)(3-5a)(3-5a).$$

(ii) $1 - 64a^3 - 12a + 48a^2$

$= 1^3 - (4a)^3 - 12a(1 - 4a)$

$= 1^3 - (4a)^3 - (3 \times 1 \times 4a)(1 - 4a)$

$= (1 - 4a)^3 = (1 - 4a)(1 - 4a)(1 - 4a)$.

EXAMPLE 6 *Factorise*

(i) $8p^3 + \frac{12}{5}p^2 + \frac{6}{25}p + \frac{1}{125}$ *(ii)* $27p^3 - \frac{9}{2}p^2 + \frac{1}{4}p - \frac{1}{216}$

SOLUTION We have

(i) $8p^3 + \frac{12}{5}p^2 + \frac{6}{25}p + \frac{1}{125}$

$= 8p^3 + \frac{1}{125} + \frac{12}{5}p^2 + \frac{6}{25}p$

$= (2p)^3 + \left(\frac{1}{5}\right)^3 + \frac{6}{5}p\left(2p + \frac{1}{5}\right)$

$= (2p)^3 + \left(\frac{1}{5}\right)^3 + \left(3 \times 2p \times \frac{1}{5}\right) \times \left(2p + \frac{1}{5}\right)$

$= \left(2p + \frac{1}{5}\right)^3 = \left(2p + \frac{1}{5}\right)\left(2p + \frac{1}{5}\right)\left(2p + \frac{1}{5}\right)$.

(ii) $27p^3 - \frac{9}{2}p^2 + \frac{1}{4}p - \frac{1}{216}$

$= 27p^3 - \frac{1}{216} - \frac{9}{2}p^2 + \frac{1}{4}p$

$= (3p)^3 - \left(\frac{1}{6}\right)^3 - \frac{3}{2}p\left(3p - \frac{1}{6}\right)$

$= (3p)^3 - \left(\frac{1}{6}\right)^3 - \left(3 \times 3p \times \frac{1}{6}\right)\left(3p - \frac{1}{6}\right)$

$= \left(3p - \frac{1}{6}\right)^3 = \left(3p - \frac{1}{6}\right)\left(3p - \frac{1}{6}\right)\left(3p - \frac{1}{6}\right)$.

EXAMPLE 7 *Simplify* $(x + y)^3 - (x - y)^3 - 6y(x^2 - y^2)$.

SOLUTION We have

$(x + y)^3 - (x - y)^3 - 6y(x^2 - y^2)$

$= (x + y)^3 - (x - y)^3 - 3(x + y)(x - y) \times \{(x + y) - (x - y)\}$

$[\because 2y = (x + y) - (x - y)]$

$= a^3 - b^3 - 3ab(a - b)$, where $x + y = a$ and $x - y = b$

$= (a - b)^3 = \{(x + y) - (x - y)\}^3 = (2y)^3 = 8y^3$.

EXAMPLE 8 *Evaluate* *(i)* $(102)^3$ *(ii)* $(999)^3$

SOLUTION We have

(i) $(102)^3 = (100+2)^3$
$= (100)^3 + 2^3 + 3 \times 100 \times 2 \times (100+2)$
$= 1000000 + 8 + (600 \times 102)$
$= 1000008 + 61200 = 1061208.$

(ii) $(999)^3 = (1000-1)^3$
$= (1000)^3 - 1^3 - (3 \times 1000 \times 1)(1000-1)$
$= 1000000000 - 1 - (3000 \times 999)$
$= 999999999 - 2997000 = 997002999.$

EXERCISE 3E

1. Expand

(i) $(3x+2)^3$ (ii) $\left(3a+\frac{1}{4b}\right)^3$ (iii) $\left(1+\frac{2}{3}a\right)^3$

2. Expand

(i) $(5a-3b)^3$ (ii) $\left(3x-\frac{5}{x}\right)^3$ (iii) $\left(\frac{4}{5}a-2\right)^3$

Factorise

3. $8a^3 + 27b^3 + 36a^2b + 54ab^2$

4. $64a^3 - 27b^3 - 144a^2b + 108ab^2$

5. $1 + \frac{27}{125}a^3 + \frac{9a}{5} + \frac{27a^2}{25}$

6. $125x^3 - 27y^3 - 225x^2y + 135xy^2$

7. $a^3x^3 - 3a^2bx^2 + 3ab^2x - b^3$

8. $\frac{64}{125}a^3 - \frac{96}{25}a^2 + \frac{48}{5}a - 8$

9. $a^3 - 12a(a-4) - 64$

10. Evaluate

(i) $(103)^3$ (ii) $(99)^3$

ANSWERS (EXERCISE 3E)

1. (i) $27x^3 + 8 + 54x^2 + 36x$ (ii) $27a^3 + \frac{1}{64b^3} + \frac{27a^2}{4b} + \frac{9a}{16b^2}$

(iii) $1 + \frac{8}{27}a^3 + \frac{4}{3}a^2 + 2a$

2. (i) $125a^3 - 27b^3 - 225a^2b + 135ab^2$ (ii) $27x^3 - \frac{125}{x^3} - 135x + \frac{225}{x}$

(iii) $\frac{64}{125}a^3 - 8 - \frac{96}{25}a^2 + \frac{48}{5}a$

3. $(2a+3b)(2a+3b)(2a+3b)$

4. $(4a-3b)(4a-3b)(4a-3b)$

5. $\left(1+\frac{3}{5}a\right)\left(1+\frac{3}{5}a\right)\left(1+\frac{3}{5}a\right)$

6. $(5x-3y)(5x-3y)(5x-3y)$

7. $(ax-b)(ax-b)(ax-b)$

8. $\left(\frac{4}{5}a-2\right)\left(\frac{4}{5}a-2\right)\left(\frac{4}{5}a-2\right)$

9. $(a-4)(a-4)(a-4)$

10. (i) 1092727 (ii) 970299

FACTORISATION OF SUM OR DIFFERENCE OF CUBES

THEOREM *Prove that*

(i) $(x^3+y^3)=(x+y)(x^2-xy+y^2)$

(ii) $(x^3-y^3)=(x-y)(x^2+xy+y^2)$

PROOF We have

(i) $(x+y)^3=(x^3+y^3)+3xy(x+y)$

$$\Rightarrow \quad (x^3+y^3)=(x+y)^3-3xy(x+y)$$

$$=(x+y)\times[(x+y)^2-3xy]$$

$$=(x+y)(x^2-xy+y^2).$$

$$\therefore \quad (x^3+y^3)=(x+y)(x^2-xy+y^2).$$

(ii) $(x-y)^3=(x^3-y^3)-3xy(x-y)$

$$\Rightarrow \quad (x^3-y^3)=(x-y)^3+3xy(x-y)$$

$$=(x-y)\times[(x-y)^2+3xy]$$

$$=(x-y)(x^2+xy+y^2).$$

$$\therefore \quad (x^3-y^3)=(x-y)(x^2+xy+y^2).$$

SUMMARY

(i) $(x^3+y^3)=(x+y)(x^2-xy+y^2)$.

(ii) $(x^3-y^3)=(x-y)(x^2+xy+y^2)$.

SOLVED EXAMPLES

EXAMPLE 1 *Find the following products*

(i) $\left(\frac{x}{2}+2y\right)\left(\frac{x^2}{4}-xy+4y^2\right)$ *(ii)* $(x^2-1)(x^4+x^2+1)$

SOLUTION We have

(i) $\left(\frac{x}{2}+2y\right)\left(\frac{x^2}{4}-xy+4y^2\right)$

$= (a+b)(a^2-ab+b^2)$, where $\frac{x}{2}=a$ and $2y=b$

$= (a^3+b^3)$

$= \left(\frac{x}{2}\right)^3+(2y)^3 = \left(\frac{x^3}{8}+8y^3\right)\cdot$

(ii) $(x^2-1)(x^4+x^2+1)$

$= (a-b)(a^2+ab+b^2)$, where $x^2=a$ and $1=b$

$= (a^3-b^3)$

$= (x^2)^3-(1)^3 = (x^6-1)$.

EXAMPLE 2 *Factorise*

(i) $1+64x^3$ *(ii)* $27x^3+125y^3$ *(iii)* $(x+1)^3+(x-1)^3$

SOLUTION We have

(i) $1+64x^3$

$= (1)^3+(4x)^3$

$= (1+4x)(1-4x+16x^2)$ $[\because\ (a^3+b^3)=(a+b)(a^2-ab+b^2)]$.

(ii) $27x^3+125y^3$

$= (3x)^3+(5y)^3$

$= (3x+5y)(9x^2-15xy+25y^2)$

$[\because\ (a^3+b^3)=(a+b)(a^2-ab+b^2)]$.

(iii) Putting $(x+1)=a$ and $(x-1)=b$, we get

$(x+1)^3+(x-1)^3$

$= (a^3+b^3)$

$= (a+b)(a^2-ab+b^2)$

$= \{(x+1)+(x-1)\}\times\{(x+1)^2-(x+1)(x-1)+(x-1)^2\}$

$= 2x\times\{(x^2+2x+1)-(x^2-1)+(x^2-2x+1)\}$

$= 2x\times(x^2+2x+1-x^2+1+x^2-2x+1) = 2x(x^2+3)$.

EXAMPLE 3 *Factorise*

(i) $a^3-2\sqrt{2}\,b^3$ *(ii)* $1-64a^3$ *(iii)* $8a^3-27b^3$

SOLUTION We have

(i) $(a^3-2\sqrt{2}\,b^3)$

$= (a)^3-(\sqrt{2}\,b)^3$

$= (a - \sqrt{2}\,b) \times \{a^2 + (a \times \sqrt{2}\,b) + (\sqrt{2}\,b)^2\}$

$[\because\ x^3 - y^3 = (x - y)(x^2 + xy + y^2)]$

$= (a - \sqrt{2}\,b)(a^2 + \sqrt{2}\,ab + 2b^2).$

$\therefore\quad (a^3 - 2\sqrt{2}\,b^3) = (a - \sqrt{2}\,b)(a^2 + \sqrt{2}\,ab + 2b^2).$

(ii) $(1 - 64a^3)$

$= (1)^3 - (4a)^3$

$= (1 - 4a) \times \{1^2 + (1 \times 4a) + (4a)^2\}$

$[\because\ x^3 - y^3 = (x - y)(x^2 + xy + y^2)]$

$= (1 - 4a)(1 + 4a + 16a^2).$

$\therefore\quad (1 - 64a^3) = (1 - 4a)(1 + 4a + 16a^2).$

(iii) $(8a^3 - 27b^3)$

$= (2a)^3 - (3b)^3$

$= (2a - 3b) \times \{(2a)^2 + (2a \times 3b) + (3b)^2\}$

$[\because\ x^3 - y^3 = (x - y)(x^2 + xy + y^2)]$

$= (2a - 3b)(4a^2 + 6ab + 9b^2).$

$\therefore\quad (8a^3 - 27b^3) = (2a - 3b)(4a^2 + 6ab + 9b^2).$

EXAMPLE 4 *Factorise*

(i) $2a^7 - 128a$ *(ii)* $a^7 + ab^6$ *(iii)* $32a^3 + 108b^3$

SOLUTION We have

(i) $(2a^7 - 128a)$

$= 2a \times (a^6 - 64) = 2a \times [(a^3)^2 - 8^2]$

$= 2a \times (a^3 - 8) \times (a^3 + 8)$ $[\because\ x^2 - y^2 = (x - y)(x + y)]$

$= 2a \times (a^3 - 2^3) \times (a^3 + 2^3)$

$= 2a \times [(a - 2) \times (a^2 + 2a + 4)] \times [(a + 2) \times (a^2 - 2a + 4)]$

$= 2a(a - 2)(a + 2)(a^2 + 2a + 4)(a^2 - 2a + 4).$

$\therefore\quad (2a^7 - 128a) = 2a(a - 2)(a + 2)(a^2 + 2a + 4)(a^2 - 2a + 4).$

(ii) $(a^7 + ab^6)$

$= a \times (a^6 + b^6) = a \times \{(a^2)^3 + (b^2)^3\}$

$= a \times (a^2 + b^2) \times (a^4 - a^2b^2 + b^4)$

$[\because\ x^3 + y^3 = (x + y)(x^2 - xy + y^2)].$

$\therefore\quad (a^7 + ab^6) = a(a^2 + b^2)(a^4 - a^2b^2 + b^4).$

(iii) $(32a^3 + 108b^3)$

$= 4 \times (8a^3 + 27b^3)$

$= 4 \times \{(2a)^3 + (3b)^3\}$

$= 4 \times (2a + 3b) \times (4a^2 - 6ab + 9b^2)$

$[\because\ x^3 + y^3 = (x + y)(x^2 - xy + y^2)].$

$\therefore\quad (32a^3 + 108b^3) = 4(2a + 3b)(4a^2 - 6ab + 9b^2).$

EXAMPLE 5 *Factorise*

(i) $a^3 + b^3 + a + b$ *(ii)* $a^3 - b^3 - a + b$

SOLUTION We have

(i) $a^3 + b^3 + a + b$

$= (a^3 + b^3) + (a + b)$

$= (a + b)(a^2 - ab + b^2) + (a + b)$

$[\because\ (a^3 + b^3) = (a + b)(a^2 - ab + b^2)]$

$= (a + b) \times \{(a^2 - ab + b^2) + 1\}$

$= (a + b)(a^2 - ab + b^2 + 1).$

$\therefore\quad (a^3 + b^3 + a + b) = (a + b)(a^2 - ab + b^2 + 1).$

(ii) $a^3 - b^3 - a + b$

$= (a^3 - b^3) - (a - b)$

$= (a - b)(a^2 + ab + b^2) - (a - b)$

$[\because\ a^3 - b^3 = (a - b)(a^2 + ab + b^2)]$

$= (a - b)(a^2 + ab + b^2 - 1).$

$\therefore\quad (a^3 - b^3 - a + b) = (a - b)(a^2 + ab + b^2 - 1).$

EXAMPLE 6 *Factorise* $(a^6 - b^6)$.

SOLUTION We have

$(a^6 - b^6)$

$= (a^3)^2 - (b^3)^2$

$= (a^3 - b^3)(a^3 + b^3)$

$= (a - b)(a^2 + ab + b^2)(a + b)(a^2 - ab + b^2)$

$= (a - b)(a + b)(a^2 + ab + b^2)(a^2 - ab + b^2).$

$\therefore\quad (a^6 - b^6) = (a - b)(a + b)(a^2 + ab + b^2)(a^2 - ab + b^2).$

EXAMPLE 7 *Factorise* $(x^6 - 7x^3 - 8)$.

SOLUTION Putting $x^3 = y$, we get

$(x^6 - 7x^3 - 8)$

$= (y^2 - 7y - 8)$

$= (y^2 - 8y + y - 8) = y(y-8) + (y-8)$

$= (y-8)(y+1)$

$= (x^3 - 8)(x^3 + 1)$ $\quad [\because\ y = x^3]$

$= (x^3 - 2^3)(x^3 + 1^3)$

$= (x-2)(x^2 + 2x + 4)(x+1)(x^2 - x + 1)$

$= (x-2)(x+1)(x^2 + 2x + 4)(x^2 - x + 1)$.

$\therefore\quad (x^6 - 7x^3 - 8) = (x-2)(x+1)(x^2 + 2x + 4)(x^2 - x + 1)$.

EXAMPLE 8 *Factorise* $x^3 + 3x^2 + 3x - 7$.

SOLUTION We have

$x^3 + 3x^2 + 3x - 7$

$= (x^3 + 3x^2 + 3x + 1) - 7 - 1$ $\quad$ [adding and subtracting 1]

$= (x+1)^3 - 8 = (x+1)^3 - 2^3$

$= (x + 1 - 2) \times \{(x+1)^2 + 2(x+1) + 2^2\}$

$= (x-1)(x^2 + 4x + 7)$.

$\therefore\quad (x^3 + 3x^2 + 3x - 7) = (x-1)(x^2 + 4x + 7)$.

EXAMPLE 9 *If* $x + y = 12$ *and* $xy = 27$, *find the value of* $(x^3 + y^3)$.

SOLUTION We have

$(x^3 + y^3) = (x+y)(x^2 - xy + y^2)$

$= (x+y)[(x+y)^2 - 3xy]$

$= 12 \times [(12)^2 - 3 \times 27]$

$= 12 \times (144 - 81) = 12 \times 63 = 756$.

$\therefore\quad (x^3 + y^3) = 756$.

EXAMPLE 10 *Factorise* $(2a + 3b)^3 - (2a - 3b)^3$.

SOLUTION We have

$(2a + 3b)^3 - (2a - 3b)^3$

$= (x^3 - y^3)$, where $2a + 3b = x$ and $2a - 3b = y$

$= (x - y)(x^2 + xy + y^2)$

$= (x - y)[(x+y)^2 - xy]$

$= 6b \times [(4a)^2 - (4a^2 - 9b^2)]$

$[\because\ x - y = 6b,\ x + y = 4a,\ xy = 4a^2 - 9b^2]$

$= 6b \times (12a^2 + 9b^2) = 18b(4a^2 + 3b^2)$.

$\therefore\quad (2a + 3b)^3 - (2a - 3b)^3 = 18b(4a^2 + 3b^2)$.

EXERCISE 3F

Factorise

1. $x^3 + 27$

2. $27a^3 + 64b^3$

3. $125a^3 + \frac{1}{8}$

4. $216x^3 + \frac{1}{125}$

5. $16x^4 + 54x$

6. $7a^3 + 56b^3$

7. $x^5 + x^2$

8. $a^3 + 0.008$

9. $1 - 27a^3$

10. $64a^3 - 343$

11. $x^3 - 512$

12. $a^3 - 0.064$

13. $8x^3 - \frac{1}{27y^3}$

14. $\frac{x^3}{216} - 8y^3$

15. $x - 8xy^3$

16. $32x^4 - 500x$

17. $3a^7b - 81a^4b^4$

18. $x^4y^4 - xy$

19. $8x^2y^3 - x^5$

20. $1029 - 3x^3$

21. $x^6 - 729$

22. $x^9 - y^9$

23. $(a + b)^3 - (a - b)^3$

24. $8a^3 - b^3 - 4ax + 2bx$

25. $a^3 + 3a^2b + 3ab^2 + b^3 - 8$

26. $a^3 - \frac{1}{a^3} - 2a + \frac{2}{a}$

27. $2a^3 + 16b^3 - 5a - 10b$

28. $a^6 + b^6$

29. $a^{12} - b^{12}$

30. $x^6 - 7x^3 - 8$

31. $x^3 - 3x^2 + 3x + 7$

32. $(x + 1)^3 + (x - 1)^3$

33. $(2a + 1)^3 + (a - 1)^3$

34. $8(x + y)^3 - 27(x - y)^3$

35. $(x + 2)^3 + (x - 2)^3$

36. $(x + 2)^3 - (x - 2)^3$

37. Prove that $\dfrac{0.85 \times 0.85 \times 0.85 + 0.15 \times 0.15 \times 0.15}{0.85 \times 0.85 - 0.85 \times 0.15 + 0.15 \times 0.15} = 1.$

38. Prove that $\dfrac{59 \times 59 \times 59 - 9 \times 9 \times 9}{59 \times 59 + 59 \times 9 + 9 \times 9} = 50.$

ANSWERS (EXERCISE 3F)

1. $(x + 3)(x^2 - 3x + 9)$

2. $(3a + 4b)(9a^2 - 12ab + 16b^2)$

3. $\left(5a + \frac{1}{2}\right)\left(25a^2 - \frac{5a}{2} + \frac{1}{4}\right)$

4. $\left(6x + \frac{1}{5}\right)\left(36x^2 - \frac{6x}{5} + \frac{1}{25}\right)$

5. $2x(2x + 3)(4x^2 - 6x + 9)$

6. $7(a + 2b)(a^2 - 2ab + 4b^2)$

7. $x^2(x + 1)(x^2 - x + 1)$

8. $(a + 0.2)(a^2 - 0.2a + 0.004)$

9. $(1 - 3a)(1 + 3a + 9a^2)$

10. $(4a - 7)(16a^2 + 28a + 49)$

11. $(x - 8)(x^2 + 8x + 64)$

12. $(a - 0.4)(a^2 + 0.4a + 0.16)$

13. $\left(2x-\frac{1}{3y}\right)\left(4x^2+\frac{2x}{3y}+\frac{1}{9y^2}\right)$ **14.** $\left(\frac{x}{6}-2y\right)\left(\frac{x^2}{36}+\frac{xy}{3}+4y^2\right)$

15. $x(1-2y)(1+2y+4y^2)$ **16.** $4x(2x-5)(4x^2+10x+25)$

17. $3a^4b(a-3b)(a^2+3ab+9b^2)$ **18.** $xy(xy-1)(x^2y^2+xy+1)$

19. $x^2(2y-x)(4y^2+2xy+x^2)$ **20.** $3(7-x)(49+7x+x^2)$

21. $(x-3)(x+3)(x^2+3x+9)(x^2-3x+9)$

22. $(x-y)(x^2+xy+y^2)(x^6+x^3y^3+y^6)$

23. $2b(3a^2+b^2)$ **24.** $(2a-b)(4a^2+2ab+b^2-2x)$

25. $(a+b-2)[(a+b)^2+2(a+b)+4]$ **26.** $\left(a-\frac{1}{a}\right)\left(a^2-1+\frac{1}{a^2}\right)$

27. $(a+2b)(2a^2-4ab+8b^2-5)$ **28.** $(a^2+b^2)(a^4-a^2b^2+b^4)$

29. $(a-b)(a+b)(a^2+b^2)(a^2+ab+b^2)(a^2-ab+b^2)(a^4-a^2b^2+b^4)$

30. $(x-2)(x+1)(x^2+2x+4)(x^2-x+1)$

31. $(x+1)(x^2-4x+7)$ **32.** $2x(x^2+3)$

33. $9a(a^2+a+1)$ **34.** $(-x+5y)(19x^2-10xy+7y^2)$

35. $2x(x^2+12)$ **36.** $4(3x^2+4)$

HINTS TO SOME SELECTED QUESTIONS

5. $16x^4+54x=2x(8x^3+27)$.

6. $7a^3+56b^3=7(a^3+8b^3)$.

7. $x^5+x^2=x^2(x^3+1)$.

15. $x-8xy^3=x(1-8y^3)$.

16. $32x^4-500x=4x(8x^3-125)$.

17. $3a^7b-81a^4b^4=3a^4b(a^3-27b^3)$.

18. $x^4y^4-xy=xy(x^3y^3-1)$.

19. $8x^2y^3-x^5=x^2(8y^3-x^3)$.

20. $1029-3x^3=3(343-x^3)=3(7^3-x^3)$.

23. $(a+b)^3-(a-b)^3=(x^3-y^3)$, where $a+b=x$ and $a-b=y$

$$=(x-y)(x^2+xy+y^2)$$
$$=(x-y)[(x+y)^2-xy]=2b\times[(2a)^2-(a^2-b^2)].$$

24. $8a^3-b^3-4ax+2bx=(8a^3-b^3)-2x(2a-b)$.

25. Given expression $=(a+b)^3-2^3$.

26. Given expression $=\left(a^3-\frac{1}{a^3}\right)-2\left(a-\frac{1}{a}\right)$.

30. $x^6 - 7x^3 - 8 = y^2 - 7y - 8$, where $x^3 = y$

$= y^2 - 8y + y - 8.$

31. $x^3 - 3x^2 + 3x + 7 = (x^3 - 3x^2 + 3x - 1) + 8 = (x-1)^3 + 2^3.$

37. Given expression $= \dfrac{(a^3 + b^3)}{(a^2 - ab + b^2)} = (a + b)$, where $a = 0.85$ and $b = 0.15$.

FACTORISATION OF $(x^3 + y^3 + z^3 - 3xyz)$

THEOREM 1 *Prove that*

$$(x^3 + y^3 + z^3 - 3xyz) = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx).$$

PROOF We have

$$\begin{aligned}(x^3 + y^3 + z^3 - 3xyz) &= (x^3 + y^3) + z^3 - 3xyz \\ &= [(x+y)^3 - 3xy(x+y)] + z^3 - 3xyz \\ &= u^3 - 3xyu + z^3 - 3xyz, \text{ where } (x+y) = u \\ &= (u^3 + z^3) - 3xy(u+z) \\ &= (u+z)(u^2 - uz + z^2) - 3xy(u+z) \\ &= (u+z)(u^2 + z^2 - uz - 3xy) \\ &= (x+y+z)[(x+y)^2 + z^2 - (x+y)z - 3xy] \\ &= (x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx).\end{aligned}$$

$\therefore \quad (x^3 + y^3 + z^3 - 3xyz) = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx).$

THEOREM 2 *If* $(x + y + z) = 0$, *prove that* $(x^3 + y^3 + z^3) = 3xyz.$

PROOF We have

$$\begin{aligned} x + y + z = 0 &\Rightarrow x + y = -z \\ &\Rightarrow (x+y)^3 = (-z)^3 \\ &\Rightarrow x^3 + y^3 + 3xy(x+y) = -z^3 \\ &\Rightarrow x^3 + y^3 + 3xy(-z) = -z^3 \qquad [\because (x+y) = -z] \\ &\Rightarrow x^3 + y^3 - 3xyz = -z^3 \\ &\Rightarrow x^3 + y^3 + z^3 = 3xyz.\end{aligned}$$

Hence, $(x + y + z) = 0 \Rightarrow (x^3 + y^3 + z^3) = 3xyz.$

SUMMARY

I. $(x^3 + y^3 + z^3 - 3xyz) = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

II. $(x + y + z) = 0 \Rightarrow (x^3 + y^3 + z^3) = 3xyz.$

SOLVED EXAMPLES

EXAMPLE 1 *Find the product*

$$(2x - y + 3z)(4x^2 + y^2 + 9z^2 + 2xy + 3yz - 6xz).$$

SOLUTION Putting $2x = a$, $-y = b$ and $3z = c$, we get

$$(2x - y + 3z)(4x^2 + y^2 + 9z^2 + 2xy + 3yz - 6xz)$$
$$= [2x + (-y) + (3z)] \times [(2x)^2 + (-y)^2 + (3z)^2 - (2x)(-y) - (-y)(3z) - (2x)(3z)]$$
$$= (a + b + c) \times (a^2 + b^2 + c^2 - ab - bc - ca)$$
$$= a^3 + b^3 + c^3 - 3abc$$
$$= (2x)^3 + (-y)^3 + (3z)^3 - 3 \times (2x) \times (-y) \times (3z)$$
$$= 8x^3 - y^3 + 27z^3 + 18xyz.$$

EXAMPLE 2 *Find the product*

$$(x - 2y - z)(x^2 + 4y^2 + z^2 + 2xy - 2yz + zx)$$

SOLUTION Putting $x = a$, $-2y = b$ and $-z = c$, we get

$$(x - 2y - z)(x^2 + 4y^2 + z^2 + 2xy - 2yz + zx)$$
$$= [x + (-2y) + (-z)] \times [x^2 + (-2y)^2 + (-z)^2 - x \times (-2y) - (-2y) \times (-z) - (-z) \times x]$$
$$= (a + b + c) \times (a^2 + b^2 + c^2 - ab - bc - ca)$$
$$= a^3 + b^3 + c^3 - 3abc$$
$$= x^3 + (-2y)^3 + (-z)^3 - 3 \times x \times (-2y) \times (-z)$$
$$= x^3 - 8y^3 - z^3 - 6xyz.$$

EXAMPLE 3 *Factorise* $8x^3 + y^3 + 27x^3 - 18xyz$.

SOLUTION We have

$$8x^3 + y^3 + 27z^3 - 18xyz$$
$$= (2x)^3 + y^3 + (3z)^3 - 3 \times (2x) \times y \times (3z)$$
$$= a^3 + b^3 + c^3 - 3abc, \text{ where } 2x = a,\ y = b \text{ and } 3z = c$$
$$= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$
$$= (2x + y + 3z)(4x^2 + y^2 + 9z^2 - 2xy - 3yz - 6xz).$$

EXAMPLE 4 *Factorise* $a^3 - 8b^3 - 64c^3 - 24abc$.

SOLUTION We have

$$a^3 - 8b^3 - 64c^3 - 24abc$$

$$= (a)^3 + (-2b)^3 + (-4c)^3 - 3 \times a \times (-2b) \times (-4c)$$
$$= x^3 + y^3 + z^3 - 3xyz, \text{ where } a = x, -2b = y \text{ and } -4c = z$$
$$= (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$$
$$= (a - 2b - 4c)(a^2 + 4b^2 + 16c^2 + 2ab - 8bc + 4ac).$$

EXAMPLE 5 *Factorise* $2\sqrt{2}a^3 + 8b^3 - 27c^3 + 18\sqrt{2}abc$.

SOLUTION We have

$$2\sqrt{2}a^3 + 8b^3 - 27c^3 + 18\sqrt{2}abc$$
$$= (\sqrt{2}a)^3 + (2b)^3 + (-3c)^3 - 3 \times (\sqrt{2}a) \times (2b) \times (-3c)$$
$$= x^3 + y^3 + z^3 - 3xyz, \text{ where } \sqrt{2}a = x, (2b) = y \text{ and } (-3c) = z$$
$$= (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$$
$$= (\sqrt{2}a + 2b - 3c)(2a^2 + 4b^2 + 9c^2 - 2\sqrt{2}ab + 6bc + 3\sqrt{2}ca).$$

EXAMPLE 6 *Factorise* $a^3 - b^3 + 1 + 3ab$.

SOLUTION We have

$$a^3 - b^3 + 1 + 3ab$$
$$= a^3 + (-b)^3 + (1)^3 - 3 \times a \times (-b) \times 1$$
$$= x^3 + y^3 + z^3 - 3xyz, \text{ where } a = x, (-b) = y \text{ and } 1 = z$$
$$= (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$$
$$= (a - b + 1)(a^2 + b^2 + 1 + ab + b - a)$$
$$= (a - b + 1)(a^2 + b^2 + ab - a + b + 1).$$

EXAMPLE 7 *If* $a + b + c = 5$ *and* $ab + bc + ca = 10$ *then prove that*

$$a^3 + b^3 + c^3 - 3abc = -25.$$

SOLUTION We have

$$(a^3 + b^3 + c^3 - 3abc) = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$
$$= (a + b + c)[(a + b + c)^2 - 3(ab + bc + ca)]$$
$$= 5 \times [(5)^2 - (3 \times 10)]$$
$$= 5 \times (25 - 30) = 5 \times (-5) = -25.$$

Hence, $(a^3 + b^3 + c^3 - 3abc) = -25$.

EXAMPLE 8 *Factorise* $(x - 2y)^3 + (2y - 3z)^3 + (3z - x)^3$.

SOLUTION Putting $(x - 2y) = a$, $(2y - 3z) = b$ and $(3z - x) = c$, we get

$$(x - 2y)^3 + (2y - 3z)^3 + (3z - x)^3$$
$$= a^3 + b^3 + c^3, \text{ where } a + b + c = (x - 2y) + (2y - 3z) + (3z - x) = 0$$

$= 3abc$ $[\because\ a+b+c=0 \Rightarrow a^3+b^3+c^3=3abc]$

$= 3(x-2y)(2y-3z)(3z-x)$.

$\therefore\quad (x-2y)^3+(2y-3z)^3+(3z-x)^3 = 3(x-2y)(2y-3z)(3z-x)$.

EXAMPLE 9 *Factorise* $(p-q)^3+(q-r)^3+(r-p)^3$.

SOLUTION Putting $(p-q)=x$, $(q-r)=y$ and $(r-p)=z$, we get

$(p-q)^3+(q-r)^3+(r-p)^3$

$= x^3+y^3+z^3$, where $(x+y+z)=(p-q)+(q-r)+(r-p)=0$

$= 3xyz$ $[\because\ (x+y+z)=0 \Rightarrow (x^3+y^3+z^3)=3xyz]$

$= 3(p-q)(q-r)(r-p)$.

EXAMPLE 10 *Find the value of*

(i) $\left(\frac{1}{2}\right)^3+\left(\frac{1}{3}\right)^3-\left(\frac{5}{6}\right)^3$ *(ii)* $(0.2)^3-(0.3)^3+(0.1)^3$

SOLUTION We have

(i) $\left(\frac{1}{2}\right)^3+\left(\frac{1}{3}\right)^3-\left(\frac{5}{6}\right)^3$

$= \left(\frac{1}{2}\right)^3+\left(\frac{1}{3}\right)^3+\left(\frac{-5}{6}\right)^3$

$= a^3+b^3+c^3$, where $a+b+c=\frac{1}{2}+\frac{1}{3}+\left(\frac{-5}{6}\right)=0$

$= 3abc$ $[\because\ a+b+c=0 \Rightarrow a^3+b^3+c^3=3abc]$

$= 3\times\frac{1}{2}\times\frac{1}{3}\times\frac{(-5)}{6}=\frac{-5}{12}$.

(ii) $(0.2)^3-(0.3)^3+(0.1)^3$

$= (0.2)^3+(-0.3)^3+(0.1)^3$

$= a^3+b^3+c^3$, where $a+b+c=0.2+(-0.3)+0.1=0$

$= 3abc$ $[\because\ a+b+c=0 \Rightarrow a^3+b^3+c^3=3abc]$

$= 3\times 0.2\times(-0.3)\times 0.1=-0.018$.

EXAMPLE 11 *Find the value of* $x^3+y^3-12xy+64$, *when* $x+y=-4$.

SOLUTION We have

$x^3+y^3-12xy+64$

$= x^3+y^3+4^3-3\times x\times y\times 4$

$= x^3+y^3+z^3-3xyz$, where $4=z$

$= (x+y+z)(x^2+y^2+z^2-xy-yz-zx)$

$= (x+y+4)(x^2+y^2+16-xy-4y-4x)$ $[\because\ z=4]$

$= (-4+4)(x^2+y^2+16-xy-4y-4x)$ $[\because\ (x+y)=-4]$

$= 0\times(x^2+y^2+16-xy-4y-4x)=0.$

Hence, $x^3+y^3-12xy+64=0$.

EXAMPLE 12 *Find the value of* $x^3-8y^3-36xy-216$, *when* $x=2y+6$.

SOLUTION We have

$x^3-8y^3-36xy-216$

$= x^3+(-8y^3)+(-216)-36xy$

$= x^3+(-2y)^3+(-6)^3-3\times x\times(-2y)\times(-6)$

$= a^3+b^3+c^3-3abc$, where $a=x$, $b=-2y$ and $c=-6$

$= (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

$= (x-2y-6)(x^2+4y^2+36+2xy-12y+6x)$

$= 0\times(x^2+4y^2+36+2xy-12y+6x)=0$

$[\because\ x-2y-6=0$ (given)].

Hence, $x^3-8y^3-36xy-216=0$.

EXAMPLE 13 *If* $p=2-a$, *prove that* $a^3+6ap+p^3-8=0$.

SOLUTION We have

$p=2-a$

$\Rightarrow\ a+p-2=0$

$\Rightarrow\ x+y+z=0$, where $a=x$, $p=y$ and $(-2)=z$

$\Rightarrow\ x^3+y^3+z^3=3xyz$ $[\because\ x+y+z=0\ \Rightarrow\ x^3+y^3+z^3=3xyz]$

$\Rightarrow\ a^3+p^3+(-2)^3=3\times a\times p\times(-2)$

$\Rightarrow\ a^3+6ap+p^3-8=0.$

Hence, $a^3+6ap+p^3-8=0$.

EXAMPLE 14 *Prove that*

$$a^3+b^3+c^3-3abc=\frac{1}{2}(a+b+c)\times[(a-b)^2+(b-c)^2+(c-a)^2].$$

SOLUTION We have

$a^3+b^3+c^3-3abc$

$= (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

$= \frac{1}{2}(a+b+c)\times[2a^2+2b^2+2c^2-2ab-2bc-2ca]$

$$= \frac{1}{2}(a+b+c) \times [(a^2 - 2ab + b^2) + (b^2 - 2bc + c^2) + (c^2 - 2ca + a^2)]$$

$$= \frac{1}{2}(a+b+c) \times [(a-b)^2 + (b-c)^2 + (c-a)^2].$$

$$\therefore \quad a^3 + b^3 + c^3 - 3abc = \frac{1}{2}(a+b+c) \times [(a-b)^2 + (b-c)^2 + (c-a)^2].$$

EXAMPLE 15 *Simplify* $\dfrac{(a^2-b^2)^3 + (b^2-c^2)^3 + (c^2-a^2)^3}{(a-b)^3 + (b-c)^3 + (c-a)^3}$.

SOLUTION Putting $a^2 - b^2 = x$, $b^2 - c^2 = y$ and $c^2 - a^2 = z$, we get

$$x + y + z = (a^2 - b^2) + (b^2 - c^2) + (c^2 - a^2) = 0$$

$$\Rightarrow \quad x^3 + y^3 + z^3 = 3xyz \qquad [\because\ x + y + z = 0 \Rightarrow x^3 + y^3 + z^3 = 3xyz]$$

$$\Rightarrow \quad (a^2 - b^2)^3 + (b^2 - c^2)^3 + (c^2 - a^2)^3 = 3(a^2 - b^2)(b^2 - c^2)(c^2 - a^2). \qquad \ldots \text{(i)}$$

Again, putting $a - b = p$, $b - c = q$ and $c - a = r$, we get

$$(p + q + r) = (a - b) + (b - c) + (c - a) = 0$$

$$\Rightarrow \quad p^3 + q^3 + r^3 = 3pqr \qquad [\because\ p + q + r = 0 \Rightarrow p^3 + q^3 + r^3 = 3pqr]$$

$$\Rightarrow \quad (a - b)^3 + (b - c)^3 + (c - a)^3 = 3(a - b)(b - c)(c - a). \qquad \ldots \text{(ii)}$$

From (i) and (ii), we get

$$\frac{(a^2-b^2)^3 + (b^2-c^2)^3 + (c^2-a^2)^3}{(a-b)^3 + (b-c)^3 + (c-a)^3} = \frac{3(a^2-b^2)(b^2-c^2)(c^2-a^2)}{3(a-b)(b-c)(c-a)}$$

$$= (a+b)(b+c)(c+a).$$

EXERCISE 3G

Find the product.

1. $(x + y - z)(x^2 + y^2 + z^2 - xy + yz + zx)$
2. $(x - y - z)(x^2 + y^2 + z^2 + xy - yz + xz)$
3. $(x - 2y + 3)(x^2 + 4y^2 + 2xy + 6y - 3x + 9)$
4. $(3x - 5y + 4)(9x^2 + 25y^2 + 15xy - 20y + 12x + 16)$

Factorise:

5. $125a^3 + b^3 + 64c^3 - 60abc$
6. $a^3 + 8b^3 + 64c^3 - 24abc$
7. $1 + b^3 + 8c^3 - 6bc$
8. $216 + 27b^3 + 8c^3 - 108bc$
9. $27a^3 - b^3 + 8c^3 + 18abc$
10. $8a^3 + 125b^3 - 64c^3 + 120abc$
11. $8 - 27b^3 - 343c^3 - 126bc$
12. $125 - 8x^3 - 27y^3 - 90xy$

13. $2\sqrt{2}a^3+16\sqrt{2}b^3+c^3-12abc$ **14.** $27x^3-y^3-z^3-9xyz$

15. $2\sqrt{2}a^3+3\sqrt{3}b^3+c^3-3\sqrt{6}abc$ **16.** $3\sqrt{3}a^3-b^3-5\sqrt{5}c^3-3\sqrt{15}abc$

17. $(a-b)^3+(b-c)^3+(c-a)^3$ **18.** $(a-3b)^3+(3b-c)^3+(c-a)^3$

19. $(3a-2b)^3+(2b-5c)^3+(5c-3a)^3$

20. $(5a-7b)^3+(7b-9c)^3+(9c-5a)^3$

21. $a^3(b-c)^3+b^3(c-a)^3+c^3(a-b)^3$

22. Evaluate

(i) $(-12)^3+7^3+5^3$ (ii) $(28)^3+(-15)^3+(-13)^3$

23. Prove that $(a+b+c)^3-a^3-b^3-c^3=3(a+b)(b+c)(c+a)$.

24. If a, b, c are all nonzero and $a+b+c=0$, prove that

$$\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}=3.$$

25. If $a+b+c=9$ and $a^2+b^2+c^2=35$, find the value of $(a^3+b^3+c^3-3abc)$.

ANSWERS (EXERCISE 3G)

1. $x^3+y^3-z^3+3xyz$ **2.** $x^3-y^3-z^3-3xyz$

3. $x^3-8y^3+27+18xy$ **4.** $27x^3-125y^3+64+180xy$

5. $(5a+b+4c)(25a^2+b^2+16c^2-5ab-4bc-20ca)$

6. $(a+2b+4c)(a^2+4b^2+16c^2-2ab-8bc-4ca)$

7. $(1+b+2c)(1+b^2+4c^2-b-2bc-2c)$

8. $(6+3b+2c)(36+9b^2+4c^2-18b-6bc-12c)$

9. $(3a-b+2c)(9a^2+b^2+4c^2+3ab+2bc-6ca)$

10. $(2a+5b-4c)(4a^2+25b^2+16c^2-10ab+20bc+8ca)$

11. $(2-3b-7c)(4+9b^2+49c^2+6b-21bc+14c)$

12. $(5-2x-3y)(25+4x^2+9y^2+10x-6xy+15y)$

13. $(\sqrt{2}a+2\sqrt{2}b+c)(2a^2+8b^2+c^2-4ab-2\sqrt{2}bc-\sqrt{2}ac)$

14. $(3x-y-z)(9x^2+y^2+z^2+3xy-yz+3zx)$

15. $(\sqrt{2}a+\sqrt{3}b+c)(2a^2+3b^2+c^2-\sqrt{6}ab-\sqrt{3}bc-\sqrt{2}ac)$

16. $(\sqrt{3}a-b-\sqrt{5}c)(3a^2+b^2+5c^2+\sqrt{3}ab-\sqrt{5}bc+\sqrt{15}ac)$

17. $3(a-b)(b-c)(c-a)$ **18.** $3(a-3b)(3b-c)(c-a)$

19. $3(3a-2b)(2b-5c)(5c-3a)$ **20.** $3(5a-7b)(7b-9c)(9c-5a)$

21. $3abc(b-c)(c-a)(a-b)$ **22.** (i) –1260 (ii) 16380

25. 108

HINTS TO SOME SELECTED QUESTIONS

3. Given product $= [x + (-2y) + 3] \times [x^2 + (-2y)^2 + 3^2 - x \times (-2y) - (-2y) \times 3 - 3x]$
$= x^3 + (-2y)^3 + 3^3 - 3 \times x \times (-2y) \times 3$
$= x^3 - 8y^3 + 27 + 18xy.$

21. $a(b-c) + b(c-a) + c(a-b) = 0$
$\Rightarrow \quad a^3(b-c)^3 + b^3(c-a)^3 + c^3(a-b)^3 = 3abc(b-c)(c-a)(a-b).$

22. (ii) Let $a = 28$, $b = -15$ and $c = -13$. Then,
$a + b + c = 0 \Rightarrow a^3 + b^3 + c^3 = 3abc = 3 \times 28 \times (-15) \times (-13) = 16380.$

23. $(a+b+c)^3 = [(a+b)+c]^3$
$= (a+b)^3 + c^3 + 3(a+b)c \times [(a+b)+c]$ $\quad [\because (x+y)^3 = x^3 + y^3 + 3xy(x+y)]$
$= a^3 + b^3 + c^3 + 3ab(a+b) + 3(a+b) \times [(a+b)c + c^2]$
$= a^3 + b^3 + c^3 + 3(a+b)[ab + ac + bc + c^2]$
$= a^3 + b^3 + c^3 + 3(a+b)[a(b+c) + c(b+c)]$
$= a^3 + b^3 + c^3 + 3(a+b)(b+c)(c+a).$
$\therefore \quad (a+b+c)^3 - a^3 - b^3 - c^3 = 3(a+b)(b+c)(c+a).$

24. $a + b + c = 0 \Rightarrow a^3 + b^3 + c^3 = 3abc$
$\Rightarrow \dfrac{a^2}{bc} + \dfrac{b^2}{ca} + \dfrac{c^2}{ab} = 3$ $\quad$ [on dividing both sides by abc].

25. $(a+b+c) = 9 \Rightarrow (a+b+c)^2 = 81$
$\Rightarrow (a^2 + b^2 + c^2) + 2(ab + bc + ca) = 81$
$\Rightarrow 35 + 2(ab + bc + ca) = 81 \Rightarrow (ab + bc + ca) = 23.$
$\therefore \quad (a^3 + b^3 + c^3 - 3abc) = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$
$= (a+b+c)[(a+b+c)^2 - 3(ab + bc + ca)].$

MULTIPLE-CHOICE QUESTIONS (MCQ)

Choose the correct answer in each of the following questions.

1. If $(x + 1)$ is a factor of the polynomial $(2x^2 + kx)$ then the value of k is
(a) –2 (b) –3 (c) 2 (d) 3

2. The value of $(249)^2 - (248)^2$ is
(a) 1^2 (b) 477 (c) 487 (d) 497

3. If $\dfrac{x}{y} + \dfrac{y}{x} = -1$, where $x, y \neq 0$ then the value of $(x^3 - y^3)$ is
(a) 1 (b) –1 (c) 0 (d) $\dfrac{1}{2}$

4. If $a + b + c = 0$ then $(a^3 + b^3 + c^3)$ is
(a) 0 (b) abc (c) $2abc$ (d) $3abc$

5. If $\left(3x+\frac{1}{2}\right)\left(3x-\frac{1}{2}\right)=9x^2-p$ then the value of p is

(a) 0 (b) $-\frac{1}{4}$ (c) $\frac{1}{4}$ (d) $\frac{1}{2}$

6. The coefficient of x in the expansion of $(x+3)^3$ is

(a) 1 (b) 9 (c) 18 (d) 27

7. Which of the following is a factor of $(x+y)^3-(x^3+y^3)$?

(a) x^2+y^2+2xy (b) x^2+y^2-xy (c) xy^2 (d) $3xy$

8. One of the factors of $(25x^2-1)+(1+5x)^2$ is

(a) $5+x$ (b) $5-x$ (c) $5x-1$ (d) $10x$

9. If $(x+5)$ is a factor of $p(x)=x^3-20x+5k$ then $k=?$

(a) –5 (b) 5 (c) 3 (d) –3

10. If $(x+2)$ and $(x-1)$ are factors of (x^3+10x^2+mx+n) then

(a) $m=5, n=-3$ (b) $m=7, n=-18$

(c) $m=17, n=-8$ (d) $m=23, n=-19$

11. $(104\times 96)=?$

(a) 9894 (b) 9984 (c) 9684 (d) 9884

12. $(305\times 308)=?$

(a) 94940 (b) 93840 (c) 93940 (d) 94840

13. $(207\times 193)=?$

(a) 39851 (b) 39951 (c) 39961 (d) 38951

14. $4a^2+b^2+4ab+8a+4b+4=?$

(a) $(2a+b+2)^2$ (b) $(2a-b+2)^2$

(c) $(a+2b+2)^2$ (d) None of these

15. $(x^2-4x-21)=?$

(a) $(x-7)(x-3)$ (b) $(x+7)(x-3)$

(c) $(x-7)(x+3)$ (d) None of these

16. $(4x^2+4x-3)=?$

(a) $(2x-1)(2x-3)$ (b) $(2x+1)(2x-3)$

(c) $(2x+3)(2x-1)$ (d) None of these

17. $6x^2+17x+5=?$

(a) $(2x+1)(3x+5)$ (b) $(2x+5)(3x+1)$

(c) $(6x+5)(x+1)$ (d) None of these

18. $(x+1)$ is a factor of the polynomial

(a) x^3-2x^2+x+2 (b) x^3+2x^2+x-2

(c) x^3+2x^2-x-2 (d) x^3+2x^2-x+2

19. $3x^3+2x^2+3x+2=?$

(a) $(3x-2)(x^2-1)$ (b) $(3x-2)(x^2+1)$
(c) $(3x+2)(x^2-1)$ (d) $(3x+2)(x^2+1)$

20. If $a+b+c=0$ then $\left(\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}\right)=?$

(a) 1 (b) 0 (c) –1 (d) 3

21. If $x+y+z=9$ and $xy+yz+zx=23$, the value of $(x^3+y^3+z^3-3xyz)=?$

(a) 108 (b) 207 (c) 669 (d) 729

22. If $\frac{a}{b}+\frac{b}{a}=-1$ then $(a^3-b^3)=?$

(a) –3 (b) –2 (c) –1 (d) 0

ANSWERS (MCQ)

1. (c)	**2.** (d)	**3.** (c)	**4.** (d)	**5.** (c)	**6.** (d)	**7.** (d)	**8.** (d)
9. (b)	**10.** (b)	**11.** (b)	**12.** (c)	**13.** (b)	**14.** (a)	**15.** (c)	**16.** (c)
17. (b)	**18.** (c)	**19.** (d)	**20.** (d)	**21.** (a)	**22.** (d)		

HINTS TO SOME SELECTED QUESTIONS

1. Let $p(x)=2x^2+kx$. Then, $p(-1)=0$.

$\therefore\quad 2\times(-1)^2+k\times(-1)=0 \Rightarrow k=2.$

2. $(249)^2-(248)^2=(249-248)\times(249+248)=1\times 497=497.$

3. $\frac{x}{y}+\frac{y}{x}=-1 \Rightarrow x^2+y^2+xy=0$

$$\Rightarrow (x-y)(x^2+y^2+xy)=0 \Rightarrow x^3-y^3=0.$$

4. $a+b+c=0 \Rightarrow a^3+b^3+c^3-3abc=0 \Rightarrow a^3+b^3+c^3=3abc.$

5. $\left(3x+\frac{1}{2}\right)\left(3x-\frac{1}{2}\right)=(3x)^2-\left(\frac{1}{2}\right)^2=\left(9x^2-\frac{1}{4}\right)\cdot$ Hence, $a=\frac{1}{4}\cdot$

6. $(x+3)^3=x^3+3^3+3\times x\times 3\times(x+3)=x^3+9x^2+27x+27.$

So, the coefficient of x is 27.

7. $(x+y)^3-(x^3+y^3)=3xy(x+y)$.

$\therefore\quad 3xy$ is a factor of $(x+y)^3-(x^3+y^3)$.

8. $(25x^2-1)+(1+5x)^2=(5x+1)(5x-1)+(5x+1)^2$

$$=(5x+1)[(5x-1)+(5x+1)]=(5x+1)\times 10x.$$

$\therefore\quad 10x$ is a factor of $\{(25x^2-1)+(1+5x)\}$.

9. Let $p(x)=x^3-20x+5k$.

Since $(x+5)$ is a factor of $p(x)$, we have $p(-5)=0$.

$\therefore\quad (-5)^3-20\times(-5)+5k=0 \Rightarrow 5k=25 \Rightarrow k=5.$

10. Let $p(x) = x^3 + 10x^2 + mx + n$. Then, $p(-2) = 0$ and $p(1) = 0$.

$(-2)^3 + 10 \times (-2)^2 + m \times (-2) + n = 0 \Rightarrow n - 2m = 32.$ ... (i)

$(1)^3 + 10 \times (1)^2 + m \times 1 + n = 0 \Rightarrow n + m = -11.$ ... (ii)

$\therefore \quad m = 7$ and $n = -18$.

11. $(104 \times 96) = (100 + 4) \times (100 - 4) = (100)^2 - 4^2 = 10000 - 16 = 9984.$

12. $(305 \times 308) = 305 \times (300 + 8)$

$= (305 \times 300) + (305 \times 8) = (91500 + 2440) = 93940.$

13. $(207 \times 193) = 207 \times (200 - 7)$

$= (207 \times 200) - (207 \times 7) = (41400 - 1449) = 39951.$

14. Given expression $= (2a)^2 + b^2 + 2^2 + 2 \times [(2a) \times b + (b \times 2) + (2a \times 2)] = (2a + b + 2)^2$.

15. $(x^2 - 4x - 21) = (x^2 - 7x + 3x - 21)$

$= x(x - 7) + 3(x - 7) = (x - 7)(x + 3)$.

16. $(4x^2 + 4x - 3) = (4x^2 + 6x - 2x - 3) = 2x(2x + 3) - (2x + 3) = (2x + 3)(2x - 1)$.

18. $p(x) = x^3 + 2x^2 - x - 2 \Rightarrow p(-1) = (-1 + 2 + 1 - 2) = 0.$

$\therefore \quad (x + 1)$ is a factor of $(x^3 + 2x^2 - x - 2)$.

20. $a + b + c = 0 \Rightarrow a^3 + b^3 + c^3 - 3abc = 0 \Rightarrow a^3 + b^3 + c^3 = 3abc.$

Now, $\left(\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab}\right) = \frac{a^3 + b^3 + c^3}{abc} = \frac{3abc}{abc} = 3.$

21. $(x^3 + y^3 + z^3 - 3xyz) = (x + y + z)[x^2 + y^2 + z^2 - xy - yz - zx]$

$= (x + y + z)[(x + y + z)^2 - 3(xy + yz + zx)]$

$= 9 \times (81 - 3 \times 23) = 9 \times (81 - 69) = (9 \times 12) = 108.$

22. $\frac{a}{b} + \frac{b}{a} = -1 \Rightarrow a^2 + b^2 + ab = 0$

$\Rightarrow (a - b)(a^2 + b^2 + ab) = 0 \Rightarrow a^3 - b^3 = 0.$

✿

4 Linear Equations in Two Variables

INTRODUCTION

In earlier classes we have learnt that a linear equation in one variable x is of the form $ax + b = 0$, where a and b are real numbers and $a \neq 0$.

The value of x which satisfies a given linear equation, is called its solution or root.

$$ax + b = 0 \Rightarrow ax = -b \Rightarrow x = \frac{-b}{a}.$$

Thus, a given linear equation in one variable can be solved.

In this chapter, we shall introduce the notion of a linear equation in two variables.

LINEAR EQUATIONS IN TWO VARIABLES

An equation of the form $ax + by + c = 0$, where a, b, c are real numbers such that $a \neq 0$ and $b \neq 0$, is called a linear equation in two variables x and y.

Examples Each of the equations

(i) $2x + 3y - 12 = 0$, (ii) $4x - y + 5 = 0$ and (iii) $\sqrt{2}x - \sqrt{3}y - 9 = 0$

is a linear equation in two variables x and y.

ILLUSTRATIVE EXAMPLES

EXAMPLE 1 *Write each of the following equations in the form $ax + by + c = 0$ and indicate the values of a, b, c in each case.*

(i) $3 = 2x + y$ *(ii)* $3x - 8 = 5y$ *(iii)* $x = 4y$

(iv) $\frac{x}{3} - \frac{y}{2} = 5$ *(v)* $4y - 3 = \sqrt{2}x$ *(vi)* $\pi x + y = 6$

SOLUTION We have

(i) $3 = 2x + y \Rightarrow 2x + y - 3 = 0.$

This is of the form $ax + by + c = 0$, where $a = 2$, $b = 1$ and $c = -3$.

(ii) $3x - 8 = 5y \Rightarrow 3x = 5y + 8$

$\Rightarrow 3x - 5y - 8 = 0.$

This is of the form $ax + by + c = 0$, where $a = 3$, $b = -5$ and $c = -8$.

(iii) $x = 4y \Rightarrow x - 4y = 0$

$\Rightarrow x - 4y + 0 = 0.$

This is of the form $ax + by + c = 0$, where $a = 1$, $b = -4$ and $c = 0$.

(iv) $\frac{x}{3} - \frac{y}{2} = 5 \Rightarrow 2x - 3y = 30.$

This is of the form $ax + by + c = 0$, where $a = 2$, $b = -3$ and $c = -30$.

(v) $4y - 3 = \sqrt{2}x \Rightarrow \sqrt{2}x - 4y + 3 = 0.$

This is of the form $ax + by + c = 0$, where $a = \sqrt{2}$, $b = -4$ and $c = 3$.

(vi) $\pi x + y = 6 \Rightarrow \pi x + y - 6 = 0.$

This is of the form $ax + by + c = 0$, where $a = \pi$, $b = 1$ and $c = -6$.

EXAMPLE 2 *Write each of the following as an equation of the form $ax + by + c = 0$ and write the values of a, b, c in each case.*

(i) $x = -3$ *(ii)* $y = 5$ *(iii)* $3x = 2$ *(iv)* $5y = 4$

SOLUTION We have

(i) $x = -3 \Rightarrow x + 3 = 0$

$\Rightarrow 1 \cdot x + 0 \cdot y + 3 = 0.$

This is of the form $ax + by + c = 0$, where $a = 1$, $b = 0$ and $c = 3$.

(ii) $y = 5 \Rightarrow y - 5 = 0$

$\Rightarrow 0 \cdot x + 1 \cdot y - 5 = 0.$

This is of the form $ax + by + c = 0$, where $a = 0$, $b = 1$ and $c = -5$.

(iii) $3x = 2 \Rightarrow 3x - 2 = 0$

$\Rightarrow 3x + 0 \cdot y - 2 = 0.$

This is of the form $ax + by + c = 0$, where $a = 3$, $b = 0$ and $c = -2$.

(iv) $5y = 4 \Rightarrow 5y - 4 = 0$

$\Rightarrow 0 \cdot x + 5y - 4 = 0.$

This is of the form $ax + by + c = 0$, where $a = 0$, $b = 5$ and $c = -4$.

SOLUTION OF A LINEAR EQUATION

Let $ax + by + c = 0$ be a given linear equation in x and y.

Then, a pair of values, one for x and one for y, which satisfy the given equation, is called its solution.

If $x = \alpha$ and $y = \beta$ satisfy the equation $ax + by + c = 0$ then we say that the ordered pair (α, β) is its solution.

EXAMPLE 3 *Show that $x = 3$ and $y = 2$ is a solution of the linear equation $2x + 3y = 12$.*

SOLUTION The given linear equation is $2x + 3y - 12 = 0$. ... (i)

Substituting $x = 3$ and $y = 2$ in (i), we get

$$\text{LHS} = (2 \times 3 + 3 \times 2 - 12) = (6 + 6 - 12) = 0 = \text{RHS}.$$

$\therefore$ $x = 3$ and $y = 2$ is a solution of the equation $2x + 3y = 12$.

NOTE We may say that (3, 2) is a solution of the equation $2x + 3y = 12$.

EXAMPLE 4 *Check which of the following are solutions of the equation $x - 2y = 4$ and which are not.*

(i) (2, 0) *(ii)* (0, −2) *(iii)* $(4\sqrt{2}, \sqrt{2})$ *(iv)* (4, 0)

SOLUTION The given equation may be written as $x - 2y - 4 = 0$. ... (A)

(i) Putting $x = 2$ and $y = 0$ in (A), we get

$$\text{LHS} = (2 - 2 \times 0 - 4) = (2 - 0 - 4) = -2 \neq 0 = \text{RHS}.$$

$\therefore$ (2, 0) is not a solution of $x - 2y = 4$.

(ii) Putting $x = 0$ and $y = -2$ in (A), we get

$$\text{LHS} = \{0 - 2 \times (-2) - 4\} = \{0 + 4 - 4\} = 0 = \text{RHS}.$$

$\therefore$ (0, −2) is a solution of $x - 2y = 4$.

(iii) Putting $x = 4\sqrt{2}$ and $y = \sqrt{2}$ in (A), we get

$$\text{LHS} = (4\sqrt{2} - 2 \times \sqrt{2} - 4) = (4\sqrt{2} - 2\sqrt{2} - 4)$$
$$= (2\sqrt{2} - 4) \neq 0 = \text{RHS}.$$

$\therefore$ $(4\sqrt{2}, \sqrt{2})$ is not a solution of $x - 2y = 4$.

(iv) Putting $x = 4$ and $y = 0$ in (A), we get

$$\text{LHS} = (4 - 2 \times 0 - 4) = (4 - 0 - 4) = 0 = \text{RHS}.$$

$\therefore$ (4, 0) is a solution of $x - 2y = 4$.

AN IMPORTANT FACT A linear equation in two variables has infinitely many solutions.

EXAMPLE 5 *Find six different solutions of the equation $x + 2y = 6$.*

SOLUTION We have

$$x + 2y = 6 \Rightarrow 2y = (6 - x)$$

$$\Rightarrow y = \frac{1}{2}(6 - x). \qquad \dots \text{(i)}$$

Putting $x = 2$ in (i), we get $y = \frac{1}{2}(6-2) = \left(\frac{1}{2} \times 4\right) = 2.$

Putting $x = 4$ in (i), we get $y = \frac{1}{2}(6-4) = \left(\frac{1}{2} \times 2\right) = 1.$

Putting $x = 6$ in (i), we get $y = \frac{1}{2}(6-6) = \left(\frac{1}{2} \times 0\right) = 0.$

Putting $x = -2$ in (i), we get $y = \frac{1}{2}\{6-(-2)\} = \left(\frac{1}{2} \times 8\right) = 4.$

Putting $x = -4$ in (i), we get $y = \frac{1}{2}\{6-(-4)\} = \left(\frac{1}{2} \times 10\right) = 5.$

Putting $x = -6$ in (i), we get $y = \frac{1}{2}\{6-(-6)\} = \left(\frac{1}{2} \times 12\right) = 6.$

So, the required solutions are given below.

x	2	4	6	–2	–4	–6
y	2	1	0	4	5	6

EXAMPLE 6 *If $x = 2$, $y = 1$ is a solution of the equation $2x + 3y = k$, find the value of k.*

SOLUTION The given equation is $2x + 3y - k = 0$. … (i)

Since $x = 2$, $y = 1$ is a solution of (i), we have

$$2 \times 2 + 3 \times 1 - k = 0 \Rightarrow 4 + 3 - k = 0 \Rightarrow k = 7.$$

Hence, $k = 7$.

EXAMPLE 7 *If $x = 2k - 1$ and $y = k$ is a solution of the equation $3x - 5y - 7 = 0$, find the value of k.*

SOLUTION The given equation is $3x - 5y - 7 = 0$. … (i)

Since $x = 2k - 1$ and $y = k$ is solution of (i), we have

$$3 \times (2k - 1) - 5k - 7 = 0$$

$$\Rightarrow 6k - 3 - 5k - 7 = 0 \Rightarrow k - 10 = 0 \Rightarrow k = 10.$$

Hence, $k = 10$.

EXERCISE 4A

1. Express each of the following equations in the form $ax + by + c = 0$ and indicate the values of a, b, c in each case.

(i) $3x+5y=7.5$ (ii) $2x-\frac{y}{5}+6=0$ (iii) $3y-2x=6$

(iv) $4x=5y$ (v) $\frac{x}{5}-\frac{y}{6}=1$ (vi) $\sqrt{2}x+\sqrt{3}y=5$

2. Express each of the following equations in the form $ax+by+c=0$ and indicate the values of a, b, c in each case.

(i) $x=6$ (ii) $3x-y=x-1$ (iii) $2x+9=0$

(iv) $4y=7$ (v) $x+y=4$ (vi) $\frac{x}{2}-\frac{y}{3}=\frac{1}{6}+y$

3. Check which of the following are the solutions of the equation $5x-4y=20$.

(i) $(4, 0)$ (ii) $(0, 5)$ (iii) $\left(-2, \frac{5}{2}\right)$

(iv) $(0, -5)$ (v) $\left(2, \frac{-5}{2}\right)$

4. Find five different solutions of each of the following equations:

(a) $2x-3y=6$ (b) $\frac{2x}{5}+\frac{3y}{10}=3$ (c) $3y=4x$

5. If $x=3$ and $y=4$ is a solution of the equation $5x-3y=k$, find the value of k.

6. If $x=3k+2$ and $y=2k-1$ is a solution of the equation $4x-3y+1=0$, find the value of k.

7. The cost of 5 pencils is equal to the cost of 2 ballpoints. Write a linear equation in two variables to represent this statement. (Take the cost of a pencil to be ₹ x and that of a ballpoint to be ₹ y).

ANSWERS (EXERCISE 4A)

1. (i) $6x+10y-15=0$, $(a=6, b=10, c=-15)$

(ii) $10x-y+30=0$, $(a=10, b=-1, c=30)$

(iii) $-2x+3y-6=0$, $(a=-2, b=3, c=-6)$

(iv) $4x-5y=0$, $(a=4, b=-5, c=0)$

(v) $6x-5y-30=0$, $(a=6, b=-5, c=-30)$

(vi) $\sqrt{2}x+\sqrt{3}y-5=0$, $(a=\sqrt{2}, b=\sqrt{3}, c=-5)$

2. (i) $x+0\cdot y-6=0$, $(a=1, b=0, c=-6)$

(ii) $2x-y+1=0$, $(a=2, b=-1, c=1)$

(iii) $2x+0\cdot y+9=0$, $(a=2, b=0, c=9)$

(iv) $0\cdot x+4y-7=0$, $(a=0, b=4, c=-7)$

(v) $x+y-4=0$, $(a=1, b=1, c=-4)$

(vi) $3x-8y-1=0$, $(a=3, b=-8, c=-1)$

3. On by (4, 0), (0, −5), $\left(2, \frac{-5}{2}\right)$ are the solutions.

4. (a)

x	0	3	6	9	–3
y	–2	0	2	4	–4

(b)

x	0	3	–3	6	9
y	10	6	14	2	–2

(c)

x	3	–3	–6	6	9
y	4	–4	–8	8	12

5. $k = 3$ **6.** $k = -2$ **7.** $5x - 2y = 0$

GRAPH OF A LINEAR EQUATION IN TWO VARIABLES

METHOD OF DRAWING A GRAPH OF $ax + by + c = 0, a \neq 0, b \neq 0$

Step 1. Express y in terms of x.

Step 2. Choose some convenient values of x and find the corresponding values of y satisfying the given equation.

Step 3. Write down these values of x and y in the form of a table.

Step 4. Plot the ordered pairs (x, y) from the table on a graph paper.

Step 5. Join these points by a straight line and extend it in both the directions.

This line is the required graph of the equation $ax + by + c = 0$.

SOLVED EXAMPLES

EXAMPLE 1 *Draw the graph of the equation* $2x - y + 3 = 0$. *Using the graph, find the value of* y *when (a)* $x = 2$, *(b)* $x = -3$.

SOLUTION We have

$$2x - y + 3 = 0 \Rightarrow y = 2x + 3. \qquad \ldots \text{(i)}$$

Putting $x = 0$ in (i), we get $y = (2 \times 0 + 3) = 3$.

Putting $x = 1$ in (i), we get $y = (2 \times 1 + 3) = 5$.

Putting $x = -1$ in (i), we get $y = [2 \times (-1) + 3] = 1$.

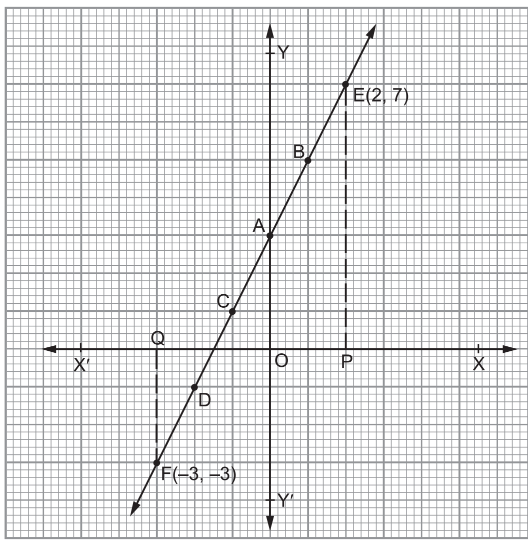
Putting $x = -2$ in (i), we get $y = [2 \times (-2) + 3] = -1$.

Thus, we have the following table:

x	0	1	–1	–2
y	3	5	1	–1

On a graph paper, draw lines $X'OX$ and YOY' representing the x-axis and the y-axis respectively.

On this graph paper, plot the points $A(0, 3)$, $B(1, 5)$, $C(-1, 1)$ and $D(-2, -1)$. Join AB, AC and CD to get a straight line $BACD$. Produce it in both ways to get the required graph.



(a) On the x-axis, we take a point P for which $x = 2$, i.e., $OP = 2$.

From P, draw $PE \perp X'OX$, meeting the graph line $BACD$ produced at E. Then, $E(2, 7)$ shows that

$(x = 2 \Rightarrow y = 7)$, i.e., when $x = 2$, then $y = 7$.

(b) On the x-axis, we take a point Q for which $x = -3$, i.e., $OQ = 3$ units.

From Q, draw $QF \perp X'OX$, meeting the graph line $BACD$ produced at F. Then, $F(-3, -3)$ shows that

$(x = -3 \Rightarrow y = -3)$, i.e., when $x = -3$, then $y = -3$.

EXAMPLE 2 *Draw the graph of the equation* $2x + 3y = 11$. *From your graph, find the value of* y *when (a)* $x = 7$, *(b)* $x = -8$.

SOLUTION We have

$$2x + 3y = 11 \Rightarrow y = \frac{(11 - 2x)}{3}. \quad \ldots \text{(i)}$$

Putting $x = 1$, we get $y = \dfrac{(11 - 2 \times 1)}{3} = \dfrac{9}{3} = 3.$

Putting $x = 4$, we get $y = \dfrac{(11 - 2 \times 4)}{3} = \dfrac{3}{3} = 1.$

Putting $x = -2$, we get $y = \dfrac{\{11 - 2 \times (-2)\}}{3} = \dfrac{15}{3} = 5.$

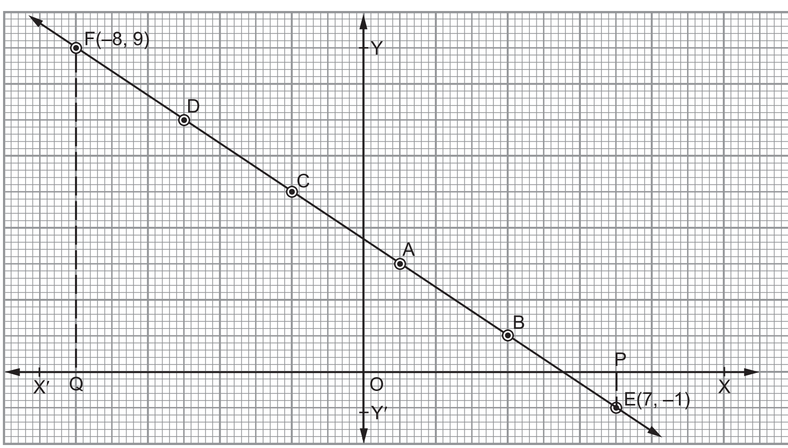
Putting $x = -5$, we get $y = \dfrac{\{11 - 2 \times (-5)\}}{3} = \dfrac{21}{3} = 7.$

Thus, we have the following table:

x	1	4	–2	–5
y	3	1	5	7

On a graph paper, draw lines $X'OX$ and YOY' as the x-axis and the y-axis respectively.

On this graph paper, plot the points $A(1, 3)$, $B(4, 1)$, $C(–2, 5)$ and $D(–5, 7)$. Join AB, AC and CD to get the straight line $BACD$. Produce it in both ways to get the required graph.



(a) On the x-axis, we take a point P for which $x = 7$, i.e., $OP = 7$.

From P, draw $PE \perp X'OX$, meeting $DCAB$ produced at $E(7, -1)$.

$\therefore$ $(x = 7 \Rightarrow y = -1)$. Thus, when $x = 7$, then $y = -1$.

(b) On the x-axis, we take a point Q on LHS of the y-axis, such that $OQ = x = -8$.

From Q, draw $QF \perp X'OX$, meeting $BACD$ produced above at $F(-8, 9)$.

$\therefore$ $(x = -8 \Rightarrow y = 9)$. Thus, when $x = -8$, then $y = 9$.

EXAMPLE 3 *Draw the graph of the equation* $2x + 3y = 6$. *From the graph, find the value of y, when*

(i) $x = \frac{3}{2}$ *(ii)* $x = -3$.

SOLUTION We have

$$2x + 3y = 6 \Rightarrow 3y = 6 - 2x \Rightarrow y = \frac{6-2x}{3}. \quad \ldots \text{(i)}$$

Putting $x = 0$ in (i), we get $y = \frac{6-(2\times 0)}{3} = \frac{6-0}{3} = \frac{6}{3} = 2$.

Putting $y = 0$ in (i), we get $(6 - 2x) = 3 \times 0 = 0 \Rightarrow 2x = 6 \Rightarrow x = 3$.

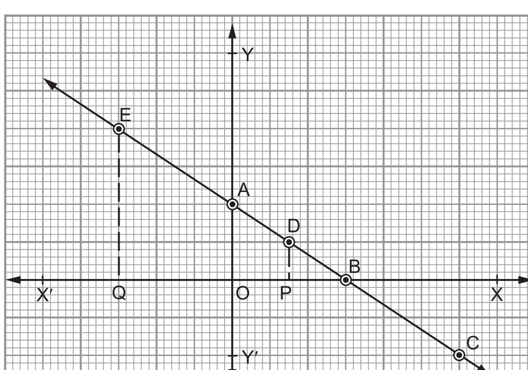
Putting $x = 6$ in (i), we get $y = \frac{(6-2\times 6)}{3} = \frac{(6-12)}{3} = \frac{-6}{3} = -2$.

Thus, we have the following table:

x	0	3	6
y	2	0	–2

On a graph paper, draw lines $X'OX$ and YOY' as the x-axis and the y-axis respectively.

Plot the points $A(0, 2)$, $B(3, 0)$ and $C(6, –2)$ on this graph paper. Join AB and BC to get the line ABC. Extend it in both the directions to get the required graph.



(a) On the x-axis, take a point P such that $OP = x = \frac{3}{2}\cdot$

Draw $PD \perp X'OX$, meeting the graph line at $D\left(\frac{3}{2}, 1\right)\cdot$

Thus, $\left(x = \frac{3}{2} \Rightarrow y = 1\right)\cdot$ Thus, when $x = \frac{3}{2}$, then $y = 1$.

(b) On the x-axis, take a point Q such that $OQ = x = -3$.

Draw $QE \perp X'OX$, meeting the graph line at $E(-3, 4)$.

Thus, $(x = -3 \Rightarrow y = 4)$, i.e., when $x = -3$, then $y = 4$.

EXAMPLE 4 *A taxi charges ₹ 20 for the first kilometre and @ ₹ 12 per* km *for subsequent distance covered. Taking the total distance covered as x* km *and total fare ₹ y, write a linear equation depicting the relation between x and y. Draw the graph between x and y.*

From your graph, find the taxi charges for covering
(a) 12 km *and (b)* 20 km.

SOLUTION The required linear equation is given by

$$y = 20 + 12(x - 1) \Rightarrow y = 8 + 12x. \qquad \dots \text{(i)}$$

Putting $x = 1$ in (i), we get $y = (8 + 12 \times 1) = 20$.

Putting $x = 6$ in (i), we get $y = (8 + 12 \times 6) = 80$.

Putting $x = 10$ in (i), we get $y = (8 + 12 \times 10) = 128$.

Putting $x = 15$ in (i), we get $y = (8 + 12 \times 15) = 188$.

Thus, we have the following table.

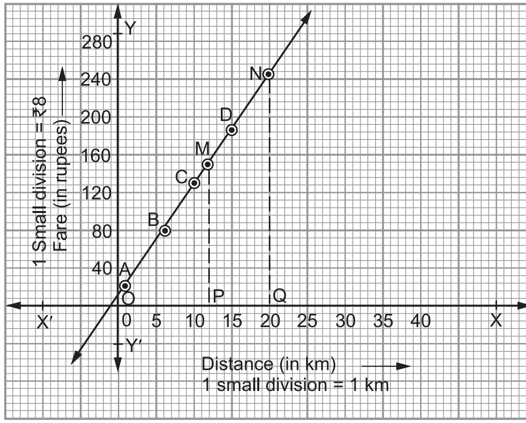
x	1	6	10	15
y	20	80	128	188

On a graph paper, draw lines $X'OX$ and YOY' as the x-axis and the y-axis respectively. Choose the scale.

Along the x-axis: 1 small division = 1 km.

Along the y-axis: 5 small divisions = ₹ 40.

Now, the plot the points $A(1, 20)$, $B(6, 80)$, $C(10, 128)$ and $D(15, 188)$. Join AB, BC and CD to get the single graph line AD, as shown below.



(a) On the x-axis, take a point P such that $OP = 12$ km.

Draw $PM \perp X'OX$, meeting the graph line at $M(12, 152)$.

Thus, ($x = 12 \Rightarrow y = 152$), i.e., when distance covered is 12 km then the taxi-fare is ₹ 152.

(b) On the x-axis, take a point Q such that $OQ = 20$ km.

Draw $QN \perp X'OX$, meeting the graph line at $N(20, 248)$.

Thus, ($x = 20 \Rightarrow y = 248$), i.e., when distance covered is 20 km then the taxi-fare is ₹ 248.

EXAMPLE 5 *There are two scales of measuring the temperature, namely degree Fahrenheit (°F) and degree Celsius (°C). The relation between the two scales is given by,* $F = \frac{9}{5}C + 32$.

(i) *If the temperature is* $0°C$, *what is the temperature in Fahrenheit?*

(ii) *If the temperature is* $50°C$, *what is the temperature in Fahrenheit?*

(iii) *If the temperature is* $86°F$, *what is the temperature in Celsius?*

(iv) *If the temperature is* $0°F$, *what is the temperature in Celsius?*

(v) *Find the numerical value of the temperature which is the same in both the scales.*

(vi) *Draw the graph of the linear equation* $F = \frac{9}{5}C + 32$, *taking C along the x-axis and F along the y-axis.*

(vii) *Using this graph, fill in the blanks given below:*

$-5°C = (\ldots\ldots)°F$ *and* $14°F = (\ldots\ldots)°C$.

SOLUTION The given relation is $F = \frac{9}{5}C + 32$. … (A)

(i) Putting $C = 0$ in (A), we get

$$F = \left(\frac{9}{5} \times 0\right) + 32 = (0 + 32) = 32.$$

$\therefore \quad C = 0 \Rightarrow F = 32.$

Hence, 0°C = 32°F.

(ii) Putting $C = 50$ in (A), we get

$$F = \left(\frac{9}{5} \times 50\right) + 32 = (90 + 32) = 122.$$

$\therefore \quad C = 50 \Rightarrow F = 122.$

Hence, 50°C = 122°F.

(iii) Putting $F = 86$ in (A), we get

$$\frac{9}{5}C + 32 = 86 \Rightarrow \frac{9}{5}C = (86 - 32) = 54$$

$$\Rightarrow C = \left(54 \times \frac{5}{9}\right) = 30.$$

$\therefore \quad F = 86 \Rightarrow C = 30.$

Hence, 86°F = 30°C.

(iv) Putting $F = 0$ in (A), we get

$$\frac{9}{5}C + 32 = 0 \Rightarrow \frac{9}{5}C = -32$$

$$\Rightarrow C = \left(-32 \times \frac{5}{9}\right) = \frac{-160}{9} = -17.8.$$

$\therefore \quad F = 0 \Rightarrow C = -17.8.$

Hence, 0°F = −17.8°C.

(v) Let $C = F$. Then, (A) becomes

$$F = \frac{9}{5}F + 32 \Rightarrow \left(\frac{9}{5} - 1\right)F = -32$$

$$\Rightarrow \frac{4}{5}F = -32 \Rightarrow F = \left(-32 \times \frac{5}{4}\right) = -40.$$

$\therefore \quad C = F = -40.$

Hence, −40°C = −40°F.

(vi) We have, $F = \frac{9}{5}C + 32$. … (A)

Putting $C = 0$ in (A), we get

$$F = \left(\frac{9}{5} \times 0\right) + 32 = (0 + 32) = 32.$$

Putting $C = 10$ in (A), we get

$$F = \left(\frac{9}{5} \times 10\right) + 32 = (18 + 32) = 50.$$

Putting $C = 20$ in (A), we get

$$F = \left(\frac{9}{5} \times 20\right) + 32 = (36 + 32) = 68.$$

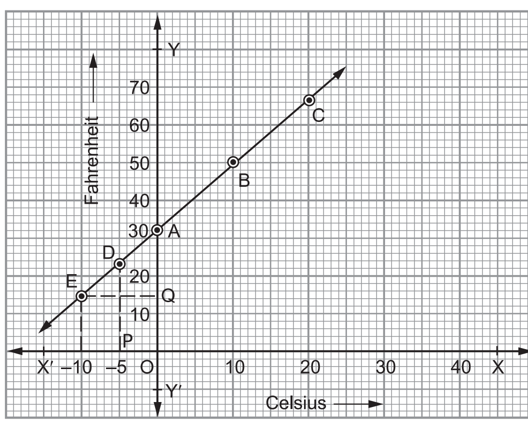
Thus, we have the following table.

C	0	10	20
F	32	50	68

On a graph paper, we take C along the x-axis and the corresponding values of F along the y-axis.

We plot the points $A(0, 32)$, $B(10, 50)$ and $C(20, 68)$ on this graph paper.

Now, we join the points A, B, and B, C to get a line ABC, which we extend in both the directions to get the required graph of the given equation, as shown below.



(vii) On the x-axis, we take a point P at which $C = -5$.

From P, draw $PD \perp X'OX$, meeting the line ABC at a point $D(-5, 23)$.

$\therefore$ $-5°C = 23°F$.

On the y-axis, take a point Q at which F = 14.

From Q, draw $QE \parallel x$-axis, meeting the line $ABCD$ at the point $E(-10, 14)$.

Hence, $14°F = -10°C$.

EXAMPLE 6 *There are two scales of measuring the temperature of a liquid, namely Kelvin* (K) *and Fahrenheit* (°F). *The relation between the two scales is given by* $F = \frac{9}{5}(K - 273) + 32.$

(i) *Find the temperature of the liquid in Fahrenheit, when it is* 313 K *in Kelvin.*

(ii) *If the temperature of the liquid is* 158°F *then express it in Kelvin.*

(iii) *Draw the graph of the linear equation,* $F = \frac{9}{5}(K - 273) + 32.$

(iv) *Using the graph, fill in the blanks given below:*

95°F = (……) K *and* 300 K = (……)° F.

SOLUTION The given relation is, $F = \frac{9}{5}(K - 273) + 32.$ … (A)

(i) Putting $K = 313$ in (A), we get

$$F = \frac{9}{5} \times (313 - 273) + 32$$

$$\Rightarrow \quad F = \left(\frac{9}{5} \times 40\right) + 32 = (72 + 32) = 104.$$

$$\therefore \quad K = 313 \;\Rightarrow\; F = 104.$$

Hence, 313 K = 104°F.

(ii) Putting $F = 158$ in (A), we get

$$\frac{9}{5}(K - 273) + 32 = 158$$

$$\Rightarrow \quad \frac{9}{5}(K - 273) = (158 - 32) = 126$$

$$\Rightarrow \quad (K - 273) = \left(126 \times \frac{5}{9}\right) = 70$$

$$\Rightarrow \quad K = (70 + 273) = 343.$$

$$\therefore \quad F = 158 \;\Rightarrow\; K = 343.$$

Hence, 158°F = 343 K.

(iii) We have, $F = \frac{9}{5}(K - 273) + 32.$ … (A)

Putting $K = 273$ in (A), we get

$$F = \frac{9}{5}(273 - 273) + 32 = (0 + 32) = 32.$$

Putting $K = 283$ in (A), we get

$$F = \frac{9}{5}(283 - 273) + 32 = (18 + 32) = 50.$$

Putting $K = 293$ in (A), we get

$$F = \frac{9}{5}(293 - 273) + 32 = (36 + 32) = 68.$$

Putting $K = 298$ in (A), we get

$$F = \frac{9}{5}(298 - 273) + 32 = (45 + 32) = 77.$$

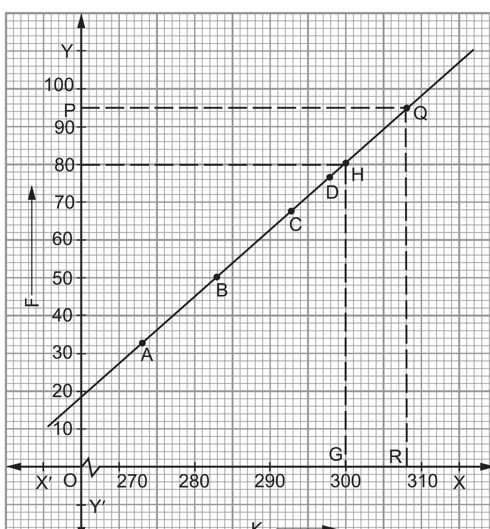
Thus, we have the following table:

K	273	283	293	298
F	32	50	68	77

On a graph paper, we draw $X'OX$ and YOY' as the x-axis and the y-axis respectively. We take the values of K along the x-axis and those of F along the y-axis.

As we start the values of K from 270, we make a kink (—/\/—) at the origin and start with 270.

On the above graph, we plot the points $A(273, 32)$, $B(283, 50)$, $C(293, 68)$ and $D(298, 77)$. Join AB, BC and CD to get the graph line $ABCD$, which is extended in both the directions, as shown below.



(iv) Along the y-axis, take a point P, showing 95°F. Draw $PQ \parallel X'OX$, meeting the graph line $ABCD$ at Q. Draw

$QR \perp X'OX$, meeting the x-axis at R. Clearly, R represents 308 K.

Hence, 95°F = 308 K.

Further, let G be a point representing 300 K.

Draw $GH \perp X'OX$, meeting $ABCD$ at the point $H(300, 80)$.

Hence, 300 K = 80°F.

EXAMPLE 7 *If the work done by a force applied on a body is directly proportional to the distance travelled by the body, express this in the form of an equation in two variables, taking the constant force as* 3 *units. Also, read from the graph the work done when the distance travelled by the body is (i)* 0 *unit, (ii)* 2 *units.*

SOLUTION Let W be the work done by a force on a body and let x be the distance travelled by the body. Then,

$W \propto x$

$\Rightarrow$ $W = Fx$, where F is the constant force applied on the body

$\Rightarrow$ $W = 3x$ [$\because$ $F = 3$ units (given)].

Thus, the required equation in two variables is $W = 3x$. ... (A)

Putting $x = 1$ in (A), we get $W = (3 \times 1) = 3$.

Putting $x = 3$ in (A), we get $W = (3 \times 3) = 9$.

Putting $x = 4$ in (A), we get $W = (3 \times 4) = 12$.

Putting $x = 5$ in (A), we get $W = (3 \times 5) = 15$.

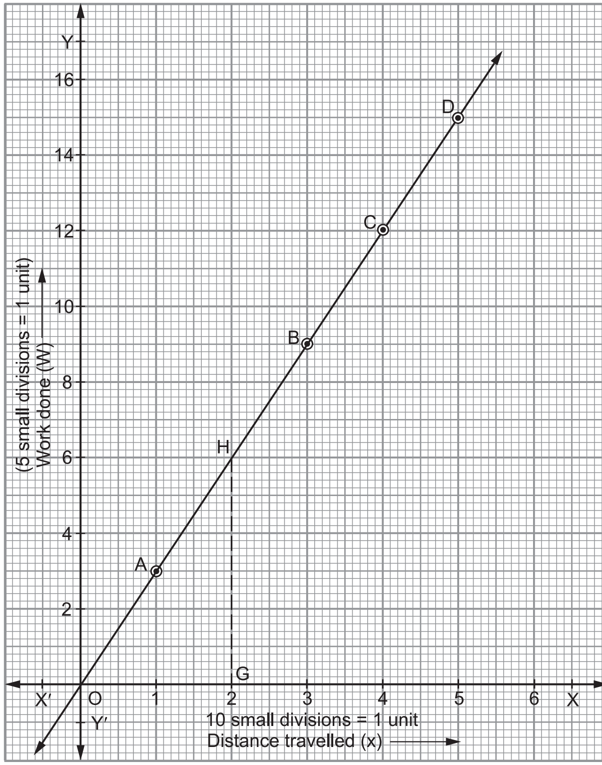
Thus, we have the following table.

x	1	3	4	5
W	3	9	12	15

On a graph paper draw mutually perpendicular lines $X'OX$ and YOY' as the x-axis and the y-axis respectively.

Along the x-axis, take 10 small divisions = 1 unit and along the y-axis, take 5 small divisions = 1 unit.

On this graph paper, plot the points $A(1, 3)$, $B(3, 9)$, $C(4, 12)$ and $D(5, 15)$. Join AB, BC and CD to obtain the graph line $ABCD$, shown below.



Produce the graph line $ABCD$ in both the directions.

(i) Clearly, the graph line passes through the point $O(0, 0)$.

$\therefore \quad x = 0 \Rightarrow W = 0.$

Thus, when the body travels 0 distance, the work done is 0.

(ii) Take a point G on the x-axis such that $x = 2$.

Draw $GH \perp x$-axis, cutting $ABCD$ at $H(2, 6)$.

$\therefore \quad x = 2 \Rightarrow W = 6.$

Thus, when the body travels 2 units, the work done is 6 units.

EXAMPLE 8 *The force exerted to pull a cart is directly proportional to the acceleration produced in the body. Express the statement as a linear equation of two variables and draw the graph of the same by taking the constant mass equal to* 6 kg. *Read from the graph the force required when the acceleration produced is (i)* $5\ \mathrm{m/s^2}$, *(ii)* $6\ \mathrm{m/s^2}$.

SOLUTION When a force F is applied on a body, let an acceleration a be produced. Then,

$F \propto a \Rightarrow F = ma$, where m is the mass of the body

$\therefore \quad F = 6a \qquad \ldots$ (i) $\quad [\because \; m = 6$ kg (given)].

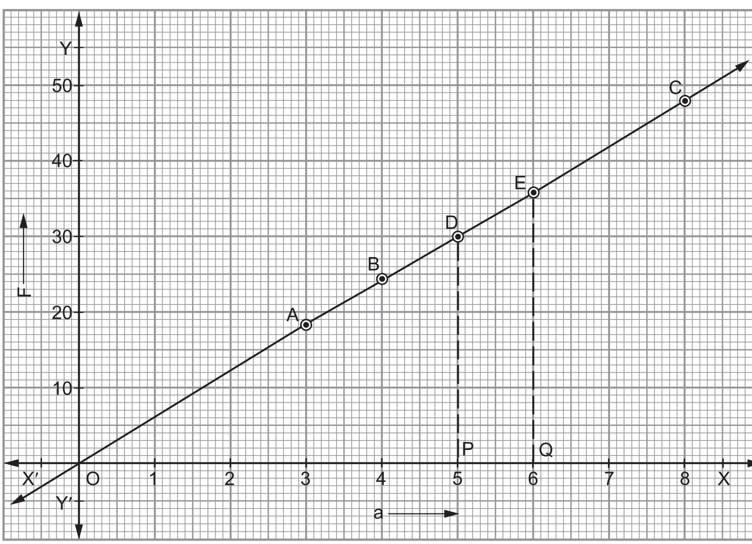
On a graph paper, let us draw two mutually perpendicular lines $X'OX$ and YOY' as the x-axis and the y-axis respectively.

On this graph paper, we take the values of a along the x-axis and the corresponding values of F along the y-axis.

Some of the values of a and F are given below.

a	0	3	4	8
F	0	18	24	48

On the graph paper, we plot the points $O(0, 0)$, $A(3, 18)$, $B(4, 24)$ and $C(8, 48)$. Join OA, AB and BC to get the graph line $OABC$.



(i) On the graph paper, take a point P along OX such that $OP = a = 5$. Draw $PD \perp X'OX$, meeting the line $OABC$ at D. Clearly, $D(5, 30)$ cuts $OABC$ at D.

So, when $a = 5$, we have $F = 30$.

(ii) On the graph paper, take a point Q along OX such that $OQ = a = 6$. Draw $QE \perp X'OX$, meeting $OABC$ at E.

Clearly, $E(6, 36)$ cuts $OABC$ at E. Hence, $a = 6 \Rightarrow F = 36$.

SOME SIMPLE GRAPHS

(i) The graph of the equation $x = 0$ is the y-axis.

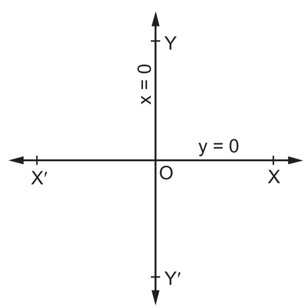
(ii) The graph of the equation $y = 0$ is the x-axis.

(iii) The graph of the equation $x = 3$ is a line parallel to the y-axis at a distance of 3 units to its right.

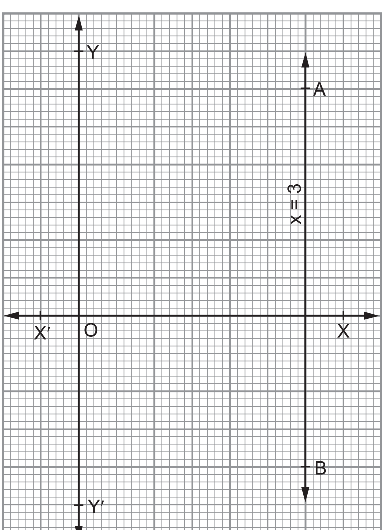
(iv) The graph of the equation $x = -3$ is a line parallel to the y-axis at a distance of 3 units to its left.

(v) The graph of the equation $y = 2$ is a line parallel to the x-axis at a distance of 2 units above the x-axis.

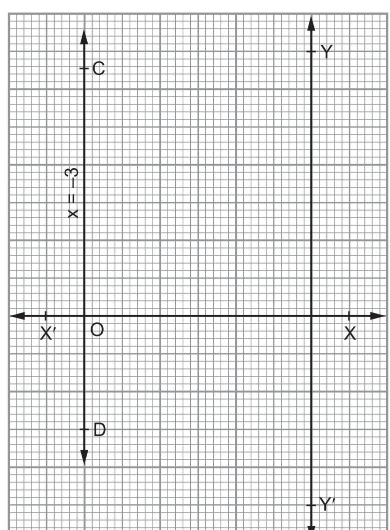
(vi) The graph of the equation $y = -2$ is a line parallel to the x-axis at a distance of 2 units below the x-axis.



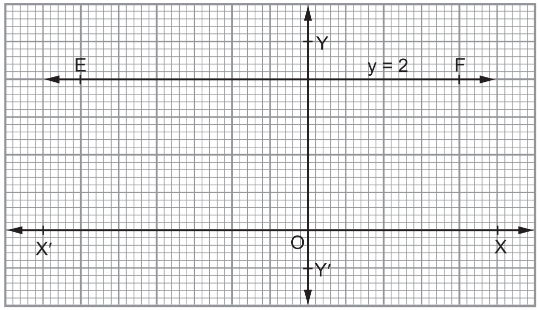
(i) and (ii)



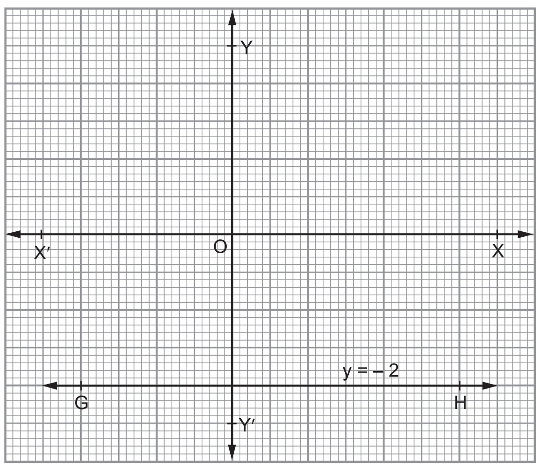
(iii) The equation of a line *AB* is $x = 3$.



(iv) The equation of a line *CD* is $x = –3$.



(v) The equation of a line EF is $y = 2$.



(vi) The equation of a line GH is $y = -2$.

EXERCISE 4B

1. Draw the graph of each of the following equations.

(i) $x = 4$ (ii) $x + 4 = 0$ (iii) $y = 3$ (iv) $y = -3$

(v) $x = -2$ (vi) $x = 5$ (vii) $y + 5 = 0$ (viii) $y = 4$

2. Draw the graph of the equation $y = 3x$.

From your graph, find the value of y when (i) $x = 2$ (ii) $x = -2$.

3. Draw the graph of the equation $x + 2y - 3 = 0$.

From your graph, find the value of y when (i) $x = 5$ (ii) $x = -5$.

4. Draw the graph of the equation, $2x - 3y = 5$.

From your graph, find (i) the value of y when $x = 4$ and (ii) the value of x when $y = 3$.

5. Draw the graph of the equation, $2x + y = 6$.
Find the coordinates of the point where the graph cuts the x-axis.

6. Draw the graph of the equation, $3x + 2y = 6$.
Find the coordinates of the point where the graph cuts the y-axis.

7. Draw the graphs of the equations $3x - 2y = 4$ and $x + y - 3 = 0$.
On the same graph paper, find the coordinates of the point where the two graph lines intersect.

8. Draw the graph of the line $4x + 3y = 24$.
   (i) Write the coordinates of the points where this line intersects the x-axis and the y-axis.
   (ii) Use this graph to find the area of the triangle formed by the graph line and the coordinate axes.

9. Draw the graphs of the lines $2x + y = 6$ and $2x - y + 2 = 0$. Shade the region bounded by these lines and the x-axis. Find the area of the shaded region.

10. Draw the graphs of the lines $x - y = 1$ and $2x + y = 8$. Shade the area formed by these two lines and the y-axis. Also, find this area.

11. Draw the graph for each of the equations $x + y = 6$ and $x - y = 2$ on the same graph paper and find the coordinates of the point where the two straight lines intersect.

12. Two students A and B contributed ₹ 100 towards the Prime Minister's Relief Fund to help the earthquake victims. Write a linear equation to satisfy the above data and draw its graph.

ANSWERS (EXERCISE 4B)

2. (i) $y = 6$ (ii) $y = -6$ **3.** (i) $y = 1$ (ii) $y = 4$
4. (i) $y = 1$ (ii) $x = 7$ **5.** (3, 0) **6.** (0, 3) **7.** (2, 1)
8. (i) $A(6, 0)$ and $B(0, 8)$ (ii) 24 sq units **9.** 8 sq units
10. 13.5 sq units **11.** (4, 1) **12.** $x + y = 100$

MULTIPLE-CHOICE QUESTIONS (MCQ)

Choose the correct answer in each of the following questions:

1. The equation of the x-axis is
   (a) $x = 0$ (b) $y = 0$ (c) $x = y$ (d) $x + y = 0$

2. The equation of the y-axis is
   (a) $x = 0$ (b) $y = 0$ (c) $x = y$ (d) $x + y = 0$

3. The point of the form (a, a), $a \neq 0$ lies on

(a) the x-axis (b) the y-axis

(c) the line $y = x$ (d) the line $x + y = 0$

4. The point of the form $(a, -a)$, $a \neq 0$ lies on

(a) the x-axis (b) the y-axis

(c) the line $y = x$ (d) the line $x + y = 0$

5. The linear equation $3x - 5y = 15$ has

(a) a unique solution (b) two solutions

(c) infinitely many solutions (d) no solution

6. The equation $2x + 5y = 7$ has a unique solution, if x and y are

(a) natural numbers (b) rational numbers

(c) positive real numbers (d) real numbers

7. The graph of $y = 5$ is a line

(a) making an intercept 5 on the x-axis

(b) making an intercept 5 on the y-axis

(c) parallel to the x-axis at a distance of 5 units from the origin

(d) parallel to the y-axis at a distance of 5 units from the origin

8. The graph of $x = 4$ is a line

(a) making an intercept 4 on the x-axis

(b) making an intercept 4 on the y-axis

(c) parallel to the x-axis at a distance of 4 units from the origin

(d) parallel to the y-axis at a distance of 4 units from the origin

9. The graph of $x + 3 = 0$ is a line

(a) making an intercept –3 on the x-axis

(b) making an intercept –3 on the y-axis

(c) parallel to the y-axis at a distance of 3 units to the left of y-axis

(d) parallel to the x-axis at a distance of 3 units below the x-axis

10. The graph of $y + 2 = 0$ is a line

(a) making an intercept of –2 on the x-axis

(b) making an intercept of –2 on the y-axis

(c) parallel to the x-axis at a distance of 2 units below the x-axis

(d) parallel to the y-axis at a distance of 2 units to the left of y-axis

11. The graph of the linear equation $2x + 3y = 6$ meets the y-axis at the point

(a) (2, 0) (b) (3, 0) (c) (0, 2) (d) (0, 3)

12. The graph of the linear equation $2x + 5y = 10$ meets the x-axis at the point

(a) (0, 2) (b) (2, 0) (c) (5, 0) (d) (0, 5)

13. The graph of the line $x = 3$ passes through the point

(a) (0, 3) (b) (2, 3) (c) (3, 2) (d) none of these

14. The graph of the line $y = 3$ passes through the point

(a) (3, 0) (b) (3, 2) (c) (2, 3) (d) none of these

15. The graph of the line $y = -3$ does not pass through the point

(a) (2, –3) (b) (3, –3) (c) (0, –3) (d) (–3, 2)

16. The graph of the line $x - y = 0$ passes through the point

(a) $\left(\frac{-1}{2}, \frac{1}{2}\right)$ (b) $\left(\frac{3}{2}, \frac{-3}{2}\right)$ (c) (0, –1) (d) (1, 1)

17. Each of the points (–2, 2), (0, 0), (2, –2) satisfies the linear equation

(a) $x - y = 0$ (b) $x + y = 0$

(c) $-x + 2y = 0$ (d) $x - 2y = 0$

18. How many linear equations can be satisfied by $x = 2$ and $y = 3$?

(a) only one (b) only two

(c) only three (d) infinitely many

19. A linear equation in two variables x and y is of the form $ax + by + c = 0$, where

(a) $a \neq 0, b \neq 0$ (b) $a \neq 0, b = 0$

(c) $a = 0, b \neq 0$ (d) $a = 0, c = 0$

20. If (2, 0) is a solution of the linear equation $2x + 3y = k$ then the value of k is

(a) 6 (b) 5 (c) 2 (d) 4

21. Any point on the x-axis is of the form

(a) (x, y) (b) $(0, y)$ (c) $(x, 0)$ (d) (x, x)

22. Any point on the y-axis is of the form

(a) (x, y) (b) $(0, y)$ (c) $(x, 0)$ (d) (y, y)

23. $x = 5, y = 2$ is a solution of the linear equation

(a) $x + 2y = 7$ (b) $5x + 2y = 7$

(c) $x + y = 7$ (d) $5x + y = 7$

24. If the point (3, 4) lies on the graph of $3y = ax + 7$ then the value of a is

(a) $\frac{2}{5}$ (b) $\frac{5}{3}$ (c) $\frac{3}{5}$ (d) $\frac{2}{7}$

ANSWERS (MCQ)

1. (b)	**2.** (a)	**3.** (c)	**4.** (d)	**5.** (c)	**6.** (a)	**7.** (c)	**8.** (d)
9. (c)	**10.** (c)	**11.** (c)	**12.** (c)	**13.** (c)	**14.** (c)	**15.** (d)	**16.** (d)
17. (b)	**18.** (d)	**19.** (a)	**20.** (d)	**21.** (c)	**22.** (b)	**23.** (c)	**24.** (b)

HINTS TO SOME SELECTED QUESTIONS

3. The point (a, a), $a \neq 0$ lies on the line $y = x$.
4. The point $(a, -a)$, $a \neq 0$ lies on the line $x + y = 0$.
5. The linear equation $3x - 5y = 15$ has infinitely many solutions.
6. $2x + 5y = 7$ has a unique solution only when x, y are natural numbers.
7. The graph of $y = 5$ is a line parallel to the x-axis at a distance of 5 units from the origin.
8. The graph of $x = 4$ is a line parallel to the y-axis at a distance of 4 units from the origin.
9. The graph of $x + 3 = 0$ is a line parallel to the y-axis at a distance of 3 units to the left of y-axis.
10. The graph of $y + 2 = 0$ is a line parallel to the x-axis at a distance of 2 units below the x-axis.
11. The graph of $2x + 3y = 6$ cuts the y-axis at the point, where $x = 0$, i.e., at the point (0, 2).
12. The graph of $2x + 5y = 10$ cuts the x-axis at the point where $y = 0$, i.e., at the point (5, 0).
13. The graph of the line $x = 3$ passes through the point (3, 2).
14. The graph of the line $y = 3$ passes through the point (5, 3).
15. The line $y = -3$ does not pass through the point (–3, 2).
16. The graph of the line $x - y = 0$, passes through the point (1, 1).
17. Each of the points (–2, 2), (0, 0) and (2, –2) satisfies the linear equation $x + y = 0$.
18. Infinitely many equations can be satisfied by $x = 2$ and $y = 3$.
19. A linear equation in two variables is of the form $ax + by + c = 0$, where $a \neq 0, b \neq 0$.
20. Putting $x = 2$ and $y = 0$ in $2x + 3y = k$, we get $k = (2 \times 2 + 3 \times 0) = (4 + 0) = 4$.
21. Any point on the x-axis is of the form $(x, 0)$.
22. Any point on the y-axis is of the form $(0, y)$.
23. Clearly, $x + y = 7$ is satisfied by $x = 5$, $y = 2$.
24. Since $3y = ax + 7$ is satisfied by (3, 4), we have

$$3 \times 4 = a \times 3 + 7 \Rightarrow 3a + 7 = 12 \Rightarrow 3a = 5 \Rightarrow a = \frac{5}{3}.$$

REVIEW OF FACTS AND FORMULAE

1. LINEAR EQUATION *An equation of the form* $ax + by + c = 0$, *where a, b, c are real numbers such that* $a \neq 0$ *and* $b \neq 0$, *is called a linear equation in two variables x and y.*

2. (i) SOLUTION OF A LINEAR EQUATION *We say that* $x = \alpha$ *and* $y = \beta$ *is a solution of* $ax + by + c = 0$ *if* $a\alpha + b\beta + c = 0$.

(ii) The equation $ax + by + c = 0$ has infinitely many solutions.

(iii) The graph of the equation $ax + by + c = 0$ is a straight line.

3. (i) The equation of x-axis is $y = 0$.

(ii) The equation of y-axis is $x = 0$.

4. (i) The equation $x = 5$ may be written as $1 \cdot x + 0 \cdot y - 5 = 0$.

(ii) The equation $y = -3$ may be written as $0 \cdot x + 1 \cdot y + 3 = 0$.

✿

5 Coordinate Geometry

INTRODUCTION

The French mathematician René Descartes in 1637 introduced the Cartesian system of coordinates for describing the position of a point in a plane. This idea has given rise to an important branch of mathematics, known as Coordinate Geometry.

CARTESIAN COORDINATES

ORDERED PAIR *A pair of numbers a and b listed in a specific order with a at the first place and b at the second place is called an ordered pair* (a, b).

Note that $(a, b) \neq (b, a)$.

Thus, (2, 3) is one ordered pair and (3, 2) is another ordered pair.

WHAT IS COORDINATE GEOMETRY?

We represent each point in a plane by means of an ordered pair of real numbers, called the *coordinates* of that point.

The branch of mathematics in which geometric problems are solved through algebra by using the coordinate system is known as *coordinate geometry*.

COORDINATE SYSTEM

COORDINATE AXES *The position of a point in a plane is determined with reference to two fixed mutually perpendicular lines, called the coordinate axes.*

Let us draw two lines $X'OX$ and YOY', which are perpendicular to each other and intersect at the point O.

These lines are called the *coordinate axes or the axes of reference*.

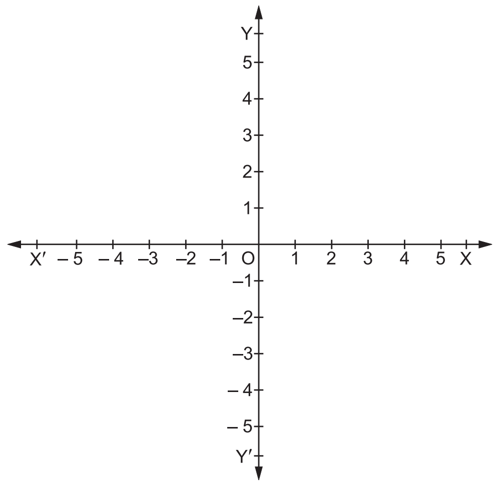
The horizontal line $X'OX$ is called the *x-axis*.

The vertical line YOY' is called the *y-axis*.

The point O is called the *origin*.

We can fix a convenient unit of length and taking the origin as zero, mark equal distances on the x-axis as well as on the y-axis.

CONVENTION OF SIGNS The distances measured along OX and OY are taken as positive and those along OX' and OY' are taken as negative, as shown in the figure given below.



COORDINATES OF A POINT IN A PLANE

Let P be a point in a plane.

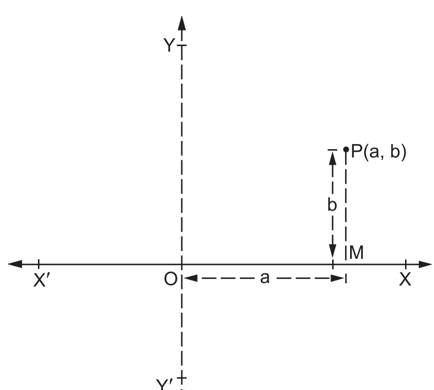
Let the distance of P from the y-axis = a units.

And, the distance of P from the x-axis = b units.

Then, we say that the *coordinates of P* are (a, b).

a is called the *x-coordinate*, or *abscissa* of P.

b is called the *y-coordinate*, or *ordinate* of P.

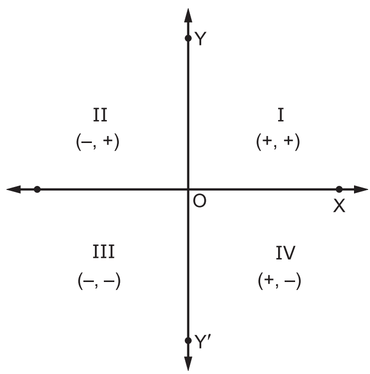


QUADRANTS Let $X'OX$ and YOY' be the coordinate axes.

These axes divide the plane of the paper into four regions, called *quadrants*. The regions $XOY, YOX', X'OY'$ and $Y'OX$ are respectively known as the first, second, third and fourth quadrants.

Using the convention of signs, we have the signs of the coordinates in various quadrants as given below.

Region	**Quadrant**	**Nature of x and y**	**Signs of coordinates**
XOY	I	$x > 0, y > 0$	(+, +)
YOX'	II	$x < 0, y > 0$	(–, +)
$X'OY'$	III	$x < 0, y < 0$	(–, –)
$Y'OX$	IV	$x > 0, y < 0$	(+, –)



NOTE Any point lying on the x-axis or y-axis does not lie in any quadrant.

EXAMPLE 1 *Draw the lines $X'OX$ and YOY' as axes on the plane of a graph paper and plot the points given below.*

(i) $A(5, 3)$ *(ii)* $B(-3, 2)$

(iii) $C(-5, -4)$ *(iv)* $D(2, -6)$

SOLUTION Let $X'OX$ and YOY' be the coordinate axes.

Fix a convenient unit of length and starting from O, mark equal distances on OX, OX', OY and OY'. Use the convention of signs.

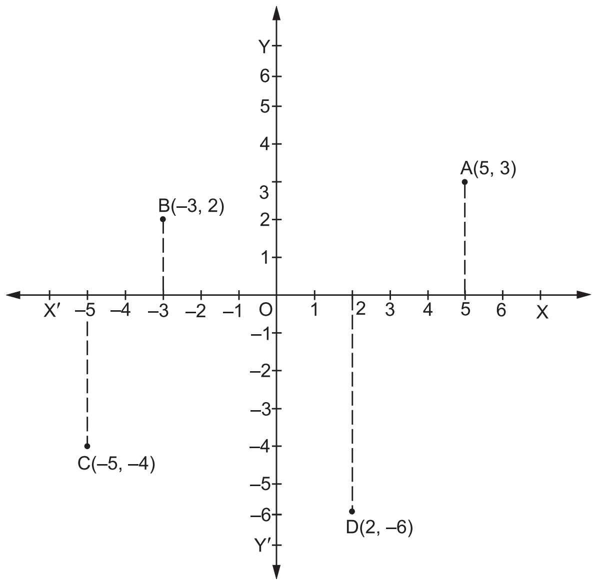
(i) Starting from O, take +5 units on the x-axis and then +3 units on the y-axis to obtain the point $A(5, 3)$.

(ii) Starting from O, take –3 units on the x-axis and then +2 units on the y-axis to obtain the point $B(–3, 2)$.

(iii) Starting from O, take –5 units on the x-axis and then –4 units on the y-axis to obtain the point $C(–5, –4)$.

(iv) Starting from O, take 2 units on the x-axis and then –6 units on the y-axis to obtain the point $D(2, -6)$.

These points are shown below.



EXAMPLE 2 *In which quadrants do the given points lie?*

(i) $(4, -2)$ *(ii)* $(-3, 7)$ *(iii)* $(-1, -2)$ *(iv)* $(3, 6)$

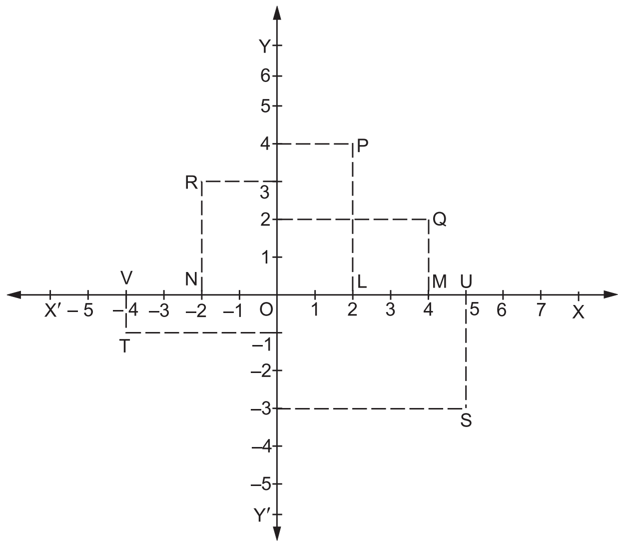
SOLUTION (i) Points of the type (+, –) lie in the 4th quadrant.
Hence, the point $(4, -2)$ lies in quadrant IV.

(ii) Points of the type (–, +) lie in the 2nd quadrant.
Hence, the point $(-3, 7)$ lies in quadrant II.

(iii) Points of the type (–, –) lie in the 3rd quadrant.
Hence, the point $(-1, -2)$ lies in quadrant III.

(iv) Points of the type (+, +) lie in the 1st quadrant.
Hence, the point (3, 6) lies in quadrant I.

EXAMPLE 3 *Write down the coordinates of each of the points P, Q, R, S and T, as shown in the figure given on next page.*



SOLUTION Draw perpendiculars *PL, QM, RN, SU* and *TV* on the x-axis.

(i) The distance of P from the y-axis = OL = 2 units.

The distance of P from the x-axis = LP = 4 units.

Hence, the coordinates of P are (2, 4).

(ii) The distance of Q from the y-axis = OM = 4 units.

The distance of Q from the x-axis = MQ = 2 units.

Hence, the coordinates of Q are (4, 2).

(iii) The distance of R from the y-axis = ON = −2 units.

The distance of R from the x-axis = NR = 3 units.

Hence, the coordinates of R are $(-2, 3)$.

(iv) The distance of S from the y-axis = OU = 5 units.

The distance of S from the x-axis = US = −3 units.

Hence, the coordinates of S are $(5, -3)$.

(v) The distance of T from the y-axis = OV = −4 units.

The distance of T from the x-axis = VT = −1 unit.

Hence, the coordinates of T are $(-4, -1)$.

POINTS ON COORDINATE AXES

(i) COORDINATES OF A POINT ON THE *x*-AXIS Every point on the x-axis is at a distance of 0 unit from the x-axis. So, its ordinate is 0.

Thus, *the coordinates of every point on the x-axis are of the form* $(x, 0)$.

(ii) COORDINATES OF A POINT ON THE *y*-AXIS Every point on the y-axis is at a distance of 0 unit from the y-axis. So, its abscissa is 0.

Thus, *the coordinates of every point on the y-axis are of the form* $(0, y)$.

REMARK The coordinates of the origin are (0, 0).

EXAMPLE 4 *On which axes do the given points lie?*

(i) $(7, 0)$ *(ii)* $(0, -3)$ *(iii)* $(0, 6)$ *(iv)* $(-5, 0)$

SOLUTION (i) In (7, 0), we have the ordinate = 0.

$\therefore$ (7, 0) lies on the x-axis.

(ii) In (0, −3), we have the abscissa = 0.

$\therefore$ (0, −3) lies on the y-axis.

(iii) In (0, 6), we have the abscissa = 0.

$\therefore$ (0, 6) lies on the y-axis.

(iv) In (−5, 0), we have the ordinate = 0.

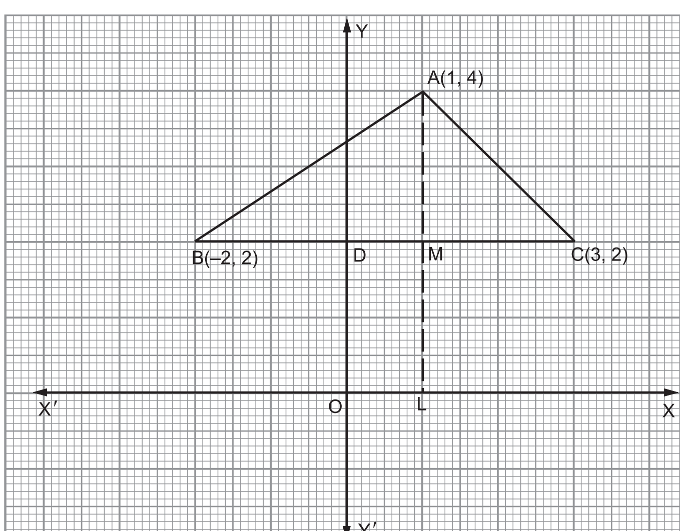
$\therefore$ (−5, 0) lies on the x-axis.

EXAMPLE 5 *The three vertices of a* $\triangle ABC$ *are* $A(1, 4)$, $B(-2, 2)$ *and* $C(3, 2)$. *Plot these points on a graph paper and calculate the area of* $\triangle ABC$.

SOLUTION Let $X'OX$ and YOY' be the x-axis and the y-axis respectively, drawn on a graph paper.

Let $A(1, 4)$, $B(-2, 2)$ and $C(3, 2)$ be the vertices of $\triangle ABC$.

Draw $AL \perp X'OX$, meeting BC at M.



$BC = (BD + DC) = (2 + 3)$ units $= 5$ units

$AM = (AL - LM) = (4 - 2)$ units $= 2$ units

$\therefore \quad \text{ar}(\triangle ABC) = \left(\frac{1}{2} \times BC \times AM\right)$ sq units

$= \left(\frac{1}{2} \times 5 \times 2\right)$ sq units $= 5$ sq units.

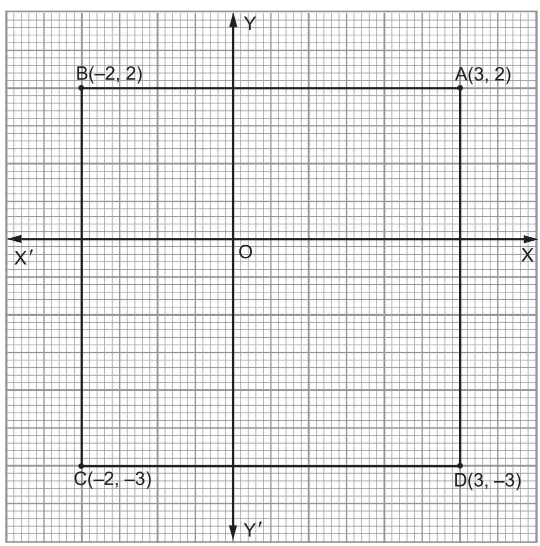
EXAMPLE 6 *The three vertices of a square ABCD are* $A(3, 2)$, $B(-2, 2)$ *and* $D(-3, 3)$. *Plot these points on a graph paper and hence, find the coordinates of C. Also, find the area of square ABCD.*

SOLUTION Let $X'OX$ and YOY' be the x-axis and y-axis respectively, drawn on a graph paper.

Let $A(3, 2)$, $B(-2, 2)$ and $D(3, -3)$ be the three vertices of square $ABCD$.

Abscissa of C = abscissa of $B = -2$.

Ordinate of C = ordinate of $D = -3$.



Thus, the coordinates of C are $(-2, -3)$.

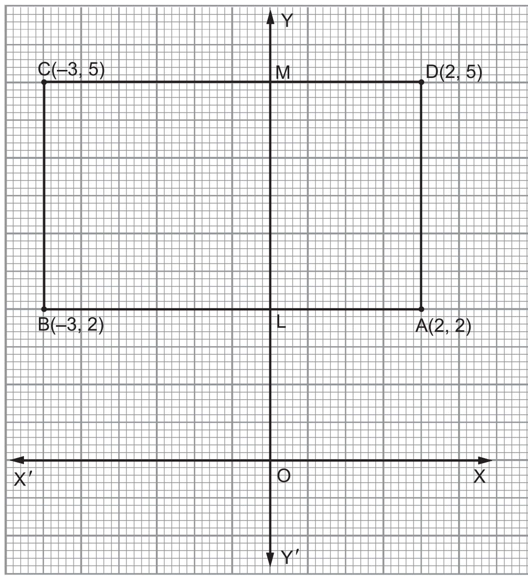
Side of sq $ABCD = CD = (2 + 3)$ units $= 5$ units.

$\therefore \quad \text{ar}(\text{sq } ABCD) = (5 \times 5)$ sq units $= 25$ sq units.

EXAMPLE 7 *The three vertices of a rectangle ABCD are* $A(2, 2)$, $B(-3, 2)$ *and* $C(-3, 5)$. *Plot these points on a graph paper and find the coordinates of D. Also, find the area of rectangle ABCD.*

SOLUTION Let $A(2, 2)$, $B(-3, 2)$ and $C(-3, 5)$ be the three vertices of a rectangle $ABCD$.

Let the y-axis cut the rectangle $ABCD$ at the points L and M respectively.



Abscissa of D = Abscissa of $A = 2$.

Ordinate of D = Ordinate of $C = 5$.

$\therefore$ coordinates of D are (2, 5).

$AB = (BL + LA) = (3 + 2)$ units = 5 units.

$BC = (5 - 2)$ units = 3 units.

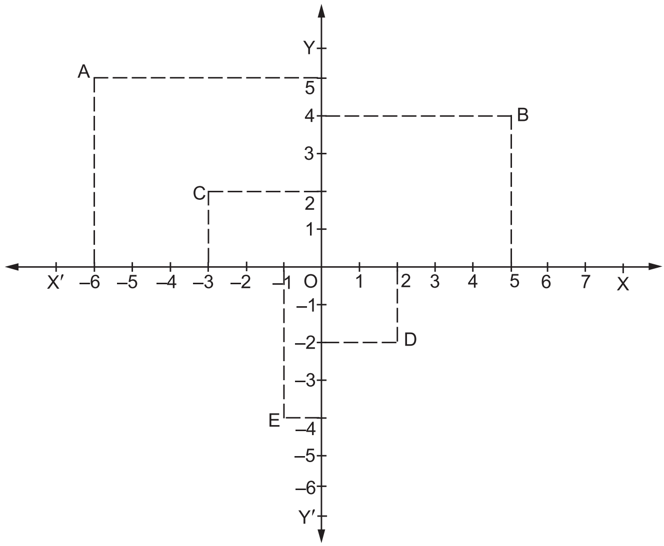
$$\begin{aligned} \text{ar(rectangle } ABCD) &= (AB \times BC) \\ &= (5 \times 3) \text{ sq units} \\ &= 15 \text{ sq units.} \end{aligned}$$

EXERCISE 5

1. On the plane of a graph paper draw $X'OX$ and YOY' as coordinate axes and plot each of the following points.

(i) $A(5, 3)$	(ii) $B(6, 2)$	(iii) $C(-5, 3)$	(iv) $D(4, -6)$
(v) $E(-3, -2)$	(vi) $F(-4, 4)$	(vii) $G(3, -4)$	(viii) $H(5, 0)$
(ix) $I(0, 6)$	(x) $J(-3, 0)$	(xi) $K(0, -2)$	(xii) $O(0, 0)$

2. Write down the coordinates of each of the following points *A, B, C, D* and *E*.



3. For each of the following points, write the quadrant in which it lies.

(i) (–6, 3) (ii) (–5, –3) (iii) (11, 6) (iv) (1, –4)
(v) (–7, –4) (vi) (4, –1) (vii) (–3, 8) (viii) (3, –8)

4. Write the axis on which the given point lies.

(i) (2, 0) (ii) (0, –5) (iii) (–4, 0) (d) (0, –1)

5. Which of the following points lie on the x-axis?

(i) $A(0, 8)$ (ii) $B(4, 0)$ (iii) $C(0, -3)$ (iv) $D(-6, 0)$
(v) $E(2, 1)$ (vi) $F(-2, -1)$ (vii) $G(-1, 0)$ (viii) $H(0, -2)$

6. Plot the points $A(2, 5)$, $B(-2, 2)$ and $C(4, 2)$ on a graph paper. Join AB, BC and AC. Calculate the area of $\triangle ABC$.

7. Three vertices of a rectangle $ABCD$ are $A(3, 1)$, $B(-3, 1)$ and $C(-3, 3)$. Plot these points on a graph paper and find the coordinates of the fourth vertex D. Also, find the area of rectangle $ABCD$.

HINT Coordinates of D are (3, 3).

ANSWERS (EXERCISE 5)

2. $A(-6, 5)$, $B(5, 4)$, $C(-3, 2)$, $D(2, -2)$ and $E(-1, -4)$.

3. (i) II (ii) III (iii) I (iv) IV
(v) III (vi) IV (vii) II (viii) IV

4. (i) x-axis (ii) y-axis (iii) x-axis (iv) y-axis

5. (ii) $B(4, 0)$ (iv) $D(-6, 0)$ (vii) $G(-1, 0)$

6. 9 sq units **7.** $D(3, 3)$, 6 sq units

MULTIPLE-CHOICE QUESTIONS (MCQ)

Choose the correct answer in each of the following questions:

1. In which quadrant does the point (–7, –4) lie?

(a) IV (b) II (c) III (d) None of these

2. If $x > 0$ and $y < 0$ then the point (x, y) lies in quadrant

(a) I (b) III (c) II (d) IV

3. If $a < 0$ and $b > 0$ then the point (a, b) lies in quadrant

(a) IV (b) II (c) III (d) none of these

4. A point both of whose coordinates are negative lies in quadrant

(a) I (b) II (c) III (iv) IV

5. The point (other than origin) for which abscissa is equal to the ordinate will lie in the quadrant

(a) I only (b) I or II (c) I or III (d) II or IV

6. The points (–5, 3) and (3, –5) lie in the

(a) same quadrant

(b) II and III quadrants respectively

(c) II and IV quadrants respectively

(d) IV and II quadrants respectively

7. Points (1, –1), (2, –2), (–3, –4), (4, –5)

(a) all lie in the II quadrant (b) all lie in the III quadrant

(c) all lie in the IV quadrant (d) do not lie in the same quadrant

8. Point (0, –8) lies

(a) in the II quadrant (b) in the IV quadrant

(c) on the x-axis (d) on the y-axis

9. Point (–7, 0) lies

(a) on the negative direction of the x-axis

(b) on the negative direction of the y-axis

(c) in the III quadrant

(d) in the IV quadrant

10. The point which lies on the y-axis at a distance of 5 units in the negative direction of the y-axis is

(a) $(-5, 0)$ (b) $(0, -5)$ (c) $(5, 0)$ (d) $(0, 5)$

11. The ordinate of every point on the x-axis is

(a) 1 (b) -1
(c) 0 (d) any real number

12. If the y-coordinate of a point is zero then this point always lies

(a) on the y-axis (b) on the x-axis
(c) in the I quadrant (d) in the IV quadrant

13. If $O(0, 0)$, $A(3, 0)$, $B(3, 4)$, $C(0, 4)$ are four given points then the figure $OABC$ is a

(a) square (b) rectangle (c) trapezium (d) rhombus

14. If $A(-2, 3)$ and $B(-3, 5)$ are two given points then (abscissa of A) – (abscissa of B) = ?

(a) -2 (b) 1 (c) -1 (d) 2

15. The perpendicular distance of the point $A(3, 4)$ from the y-axis is

(a) 3 (b) 4 (c) 5 (d) 7

16. Abscissa of a point is positive in

(a) I and II quadrants (b) I and IV quadrants
(c) I quadrant only (d) II quadrant only

17. The point at which the two coordinate axes meet is called the

(a) abscissa (b) ordinate (c) origin (d) quadrant

18. The point whose ordinate is 3 and which lies on the y-axis is

(a) $(3, 0)$ (b) $(0, 3)$ (c) $(3, 3)$ (d) $(1, 3)$

19. Which of the following points lies on the line $y = 2x + 3$?

(a) $(2, 8)$ (b) $(3, 9)$ (c) $(4, 12)$ (d) $(5, 15)$

20. Which of the following points does not lie on the line $y = 3x + 4$?

(a) $(1, 7)$ (b) $(2, 10)$ (c) $(-1, 1)$ (d) $(4, 12)$

21. Which of the following points does not lie in any quadrant?

(a) $(3, -6)$ (b) $(-3, 4)$ (c) $(5, 7)$ (d) $(0, 3)$

22. The area of $\triangle AOB$ having vertices $A(0, 6)$, $O(0, 0)$ and $B(6, 0)$ is

(a) 12 sq units (b) 36 sq units (c) 18 sq units (d) 24 sq units

ANSWERS (MCQ)

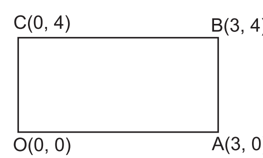
1. (c) **2.** (d) **3.** (b) **4.** (c) **5.** (c) **6.** (c) **7.** (d) **8.** (d)

9. (a) **10.** (b) **11.** (c) **12.** (b) **13.** (b) **14.** (b) **15.** (a) **16.** (b)
17. (c) **18.** (b) **19.** (b) **20.** (d) **21.** (d) **22.** (c)

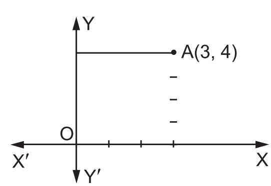
HINTS TO SOME SELECTED QUESTIONS

1. The point $(-7, -4)$ lies in III quadrant.
2. If $x > 0$ and $y < 0$ then the point (x, y) lies in IV quadrant.
3. If $a < 0$ and $b > 0$ then the point (a, b) lies in II quadrant.
4. A point both of whose coordinates are negative lies in III quadrant.
5. If $a > 0$ then (a, a) lies in I quadrant and if $a < 0$ then (a, a) lies in III quadrant.
6. $(-5, 3)$ and $(3, -5)$ lie in II and IV quadrants respectively.
7. $(1, -1)$, $(2, -2)$, $(-3, -4)$, $(4, -5)$ do not lie in the same quadrant.
8. $(0, -8)$ lies on the y-axis.
9. $(-7, 0)$ lies on the negative direction of the x-axis.
10. The required point is $(0, -5)$.
11. Every point on the x-axis is of the form $(x, 0)$ and therefore its ordinate is 0.
12. Any point of the form $(x, 0)$ lies on the x-axis.
13. Clearly, $OA = CB = 3$ units and $OC = AB = 4$ units.

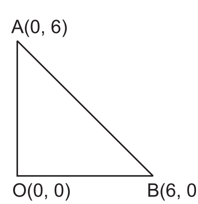
   $\therefore$ $OABC$ is a rectangle.



14. (abscissa of A) – (abscissa of B) $= (-2) - (-3) = (-2 + 3) = 1$.
15. Perpendicular distance of $A(3, 4)$ from the y-axis is 3.



16. Abscissa of a point is positive in I and IV quadrants.
17. The two coordinate axes meet at the origin.
18. A point lying on the y-axis with ordinate 3 is $(0, 3)$.
19. Clearly, $(3, 9)$ satisfies $y = 2x + 3$.
20. Clearly, $(4, 12)$ does not satisfy $y = 3x + 4$.
21. The point $(0, 3)$ does not lie in any quadrant.
22. $\text{ar}(\triangle AOB) = \left(\frac{1}{2} \times 6 \times 6\right)$ sq units $= 18$ sq units.



✿

6 Introduction to Euclid's Geometry

HISTORY

The word 'geometry' comes from the Greek words '*geo*' meaning the '*earth*' and '*metrein*' meaning '*to measure*'. Geometry appears to have originated from the need for measuring lands. This branch of mathematics was studied in various forms in ancient civilizations.

The geometry of the Vedic Period (800 BC to 500 BC) originated with the construction of altars (or *Vedis*) for performing Vedic rites. Square and circular altars were used for household rituals while altars having shapes as combinations of rectangles, triangles and trapeziums were used for public rituals (worship). The *Sriyantra* consists of **nine interwoven isosceles triangles.**

The Greeks developed geometry in a very systematic manner. Three Greek mathematicians, namely, Thales (600 BC), Pythagoras (Thales' Pupil, 572 BC) and Euclid (300 BC) contributed a lot on geometry.

Thales proved that a circle is bisected by its diameter. Pythagoras developed the theory of geometry to a great extent. Euclid introduced the method of proving mathematical results by using deductive logical reasoning and the previously proved results. The geometry of plane figures is known as **Euclidean geometry.** His famous book '*Elements*', divided into thirteen chapters, contains wonderful results on geometry.

In Indus Valley Civilization, the size of bricks was taken as $(4 \times 2 \times 1)$ inches.

Three Indian mathematicians Aryabhata (born AD 476), Brahmagupta (born AD 598) and Bhaskara II (born AD 1114) are the main contributors in geometry.

Aryabhata worked out the area of an isosceles triangle, the volume of a pyramid and approximate value of π.

Brahmagupta discovered the formula for finding the area of a cyclic quadrilateral.

Bhaskara II gave a dissection proof of Pythagoras' theorem.

AXIOMS AND POSTULATES

Self-evident true statements used throughout mathematics and not specifically linked to geometry are called *axioms* while those specific to geometry are known as *postulates*. However, nowadays, we do not distinguish between axioms and postulates.

EUCLID's AXIOMS

1. Things which are equal to the same thing are equal to one another.
2. If equals are added to equals, the wholes are equal.
3. If equals are subtracted from equals, the remainders are equal.
4. Things which coincide with one another are equal to one another.
5. The whole is greater than the part.
6. Things which are double of the same thing are equal to one another.
7. Things which are halves of the same thing are equal to one another.

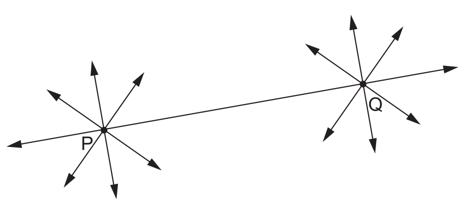
ILLUSTRATIONS

1. If the area of a triangle equals the area of a rectangle and the area of the rectangle equals that of a square, then the area of the triangle also equals the area of the square.
2. Magnitudes of the same kind can be compared and added (or subtracted). But, magnitudes of different kinds cannot be compared.
3. Everything equals itself.
4. If $A > B$, then there exists a quantity C such that $A = B + C$.

EUCLID'S FIVE POSTULATES

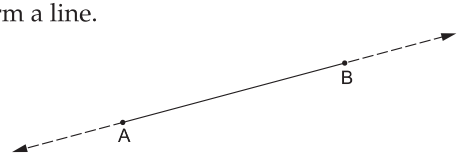
Postulate 1 *A straight line may be drawn from any one point to any other point.*

Euclid assumed that there is a unique line joining two distinct points.



Postulate 2 *A terminated line can be produced indefinitely.*

In present-day term, we say that a line segment can be extended on either side to form a line.



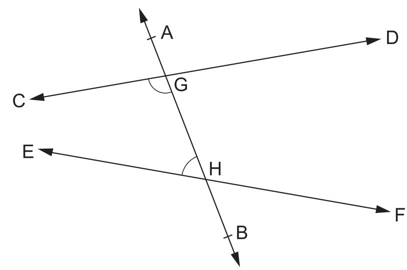
Postulate 3 *A circle can be drawn with any centre and any radius.*

Postulate 4 *All right angles are equal to one another.*

Postulate 5 (Playfair's axiom) *If a straight line falling on two straight lines makes the interior angles on the same side of it taken together less than two right angles, then the two straight lines, if produced indefinitely, meet on that side on which the sum of the angles is less than two right angles.*

In the given figure, the line AB falls on two straight lines CD and EF cutting them at the points G and H such that $\angle CGH + \angle EHG < 180°$ on the left side of AB.

So, the straight lines CD and EF when produced on the left-hand side will meet at a point.



Later on, the fifth postulate was modified as under:

'For every line L and for every point P not lying on L, there exists a unique line M passing through P and parallel to L.'

An Important Result *Two distinct intersecting lines cannot be parallel to the same line.*

STATEMENTS *A sentence which can be judged to be true or false is called a statement.*

Examples (i) The sum of the angles of a triangle is 180°, is a true statement.

(ii) The sum of the angles of a quadrilateral is 180°, is a false statement.

(iii) $x + 10 > 15$ is a sentence but not a statement.

THEOREMS *A statement that requires a proof is called a theorem.*

Establishing the truth of a theorem is known as proving the theorem.

Examples (i) The sum of all the angles around a point is 360°.

(ii) The sum of the angles of a triangle is 180°.

COROLLARY *A statement whose truth can easily be deduced from a theorem, is called its corollary.*

SOME TERMS RELATED TO GEOMETRY

POINT *A point is an exact location.*

A fine dot represents a point.

We denote a point by a capital letter—*A*, *B*, *P*, *Q*, etc.

In the given figure, *P* is a point.

LINE SEGMENT *The straight path between two points A and B is called the line segment* $\overline{AB}$.

The points *A* and *B* are called the end points of the line segment $\overline{AB}$.

A line segment has a definite length.

The distance between two points *A* and *B* is equal to the length of the line segment $\overline{AB}$.

Clearly, $\overline{AB}$ and $\overline{BA}$ denote the same line segment.

RAY *A line segment* $\overline{AB}$ *when extended indefinitely in one direction is the ray* $\overrightarrow{AB}$.

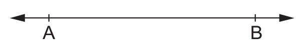
Ray $\overrightarrow{AB}$ has one end point *A*.

A ray has no definite length.

A ray cannot be drawn, it can simply be represented on the plane of a paper.

To draw a ray would mean to represent it.

LINE *A line segment* $\overline{AB}$ *when extended indefinitely in both the directions is called the line* $\overleftrightarrow{AB}$.

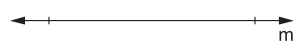


A line has no end points.

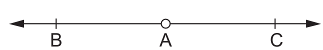
A line has no definite length.

A line cannot be drawn, it can simply be represented on the plane of a paper.

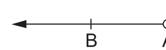
To draw a line would mean to represent it. Sometimes, we label lines by small letters *l*, *m*, *n*, etc.



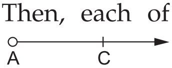
HALF-LINE Let *A*, *B* and *C* be any three points on a given line *l* such that *A* lies between *B* and *C* and we delete the point *A*.



Then, each of the two remaining parts, namely



and

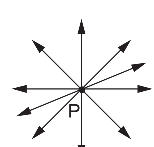


are called half-lines.

INCIDENCE AXIOMS ON LINES

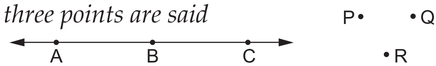
(i) A line contains infinitely many points.

(ii) Through a given point, infinitely many lines can be drawn.



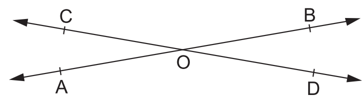
(iii) One and only one line can be drawn to pass through two given points *A* and *B*.

COLLINEAR POINTS *Three or more than three points are said to be collinear, if there is a line which contains them all.*



In the given figure A, B, C are collinear points, while P, Q, R are noncollinear.

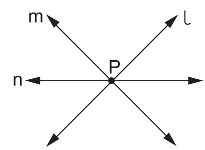
INTERSECTING LINES *Two lines having a common point are called intersecting lines.*



The point common to two given lines is called their *point of intersection*.

In the given figure, the lines AB and CD intersect at a point O.

CONCURRENT LINES *Three or more lines intersecting at the same point are said to be concurrent.*

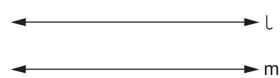


In the given figure, lines l, m, n pass through the same point P and therefore, they are concurrent.

PLANE *A plane is a surface such that every point of the line joining any two points on it, lies on it.*

Examples The surface of a smooth wall; the surface of the top of the table; the surface of a smooth blackboard; the surface of a sheet of paper, etc., are close examples of a plane. These surfaces are limited in extent but the geometrical plane extends endlessly in all directions.

PARALLEL LINES *Two lines l and m in a plane are said to be parallel, if they have no point in common and we write, $l \parallel m$.*



The distance between two parallel lines always remains the same.

NUMBER OF LINES PASSING THROUGH TWO GIVEN POINTS

An Important Axiom *Given two distinct points, there is a unique line that passes through them.*

Let A and B be two given points.

How many lines passing through A, will pass through B?

Clearly, only one, namely the line AB.

How many lines passing through B, will pass through A?

Clearly only one, namely the line AB.

Thus, there is one and only one line, passing through two given points.

Hence, the given axiom is true.

SOME RESULTS ON LINES

THEOREM 1 *Prove that two distinct lines cannot have more than one point in common.*

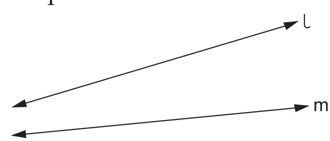
GIVEN Two distinct lines l and m.

TO PROVE l and m cannot have more than one point in common.

PROOF If $l \parallel m$, then clearly there is no point common to both l and m.

So, let us consider the case when l is not parallel to m.

If possible, let P and Q be two points common to both l and m.



Then, l contains both the points P and Q.

And, m contains both the points P and Q.

But, we know that there is only one line passing through two distinct points P and Q.

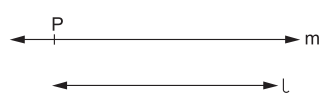
$\therefore \quad l = m.$

This contradicts the hypothesis that l and m are distinct.

Thus, our supposition is wrong.

Hence, two distinct lines cannot have more than one point in common.

PARALLEL LINES AXIOM *If P is a point outside a given line l, then one and only one line can be drawn to pass through P and parallel to l.*



Thus, we can say that two intersecting lines cannot be both parallel to a given line.

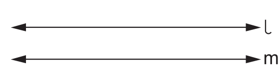
THEOREM 2 *Prove that the lines which are parallel to a given line are parallel.*

GIVEN Three lines l, m and n such that $l \parallel n$ and $m \parallel n$.

TO PROVE $l \parallel m$.

PROOF If possible, let l be not parallel to m.

Then, l and m should intersect at a unique point P.



Thus, through a point P outside n, there are two lines l and m, both parallel to n.

But, two distinct intersecting lines cannot be parallel to the same line.

So, we arrive at a contradiction.

Since the contradiction arises by assuming that l is not parallel to m, hence $l \parallel m$.

THEOREM 3 *If lines AB, AC, AD and AE are parallel to a line l, show that the points A, B, C, D, E are collinear.*

GIVEN AB, AC, AD and AE are lines, all parallel to a line l.

TO PROVE A, B, C, D, E are collinear.

PROOF Since AB, AC, AD and AE are all parallel to a line l, it follows that A is a point outside l, through which lines AB, AC, AD and AE are drawn, each parallel to l.

But, by parallel lines axiom, one and only one line can be drawn through A and parallel to l.

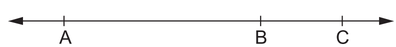
This is possible only when A, B, C, D, E all lie on the same line.

Hence, A, B, C, D, E are collinear.

SOLVED EXAMPLES

EXAMPLE 1 *If A, B and C are three points on a line and B lies between A and C (as shown in the given figure), then prove that* $AB + BC = AC$.

SOLUTION



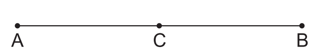
It is clear from the given figure that AC coincides with $AB + BC$.

By Euclid's Axiom 4, we know that the things that coincide with one another are equal to one another.

$\therefore \quad AB + BC = AC.$

EXAMPLE 2 *If a point C lies between two points A and B such that* $AC = BC$, *then prove that* $AC = \frac{1}{2}AB$.

SOLUTION Let us consider three points A, C and B on a line such that $AC = BC$. Then, $AC = CB$ $\quad [\because \overline{BC} = \overline{CB}]$.



By Euclid's Axiom 2, we know that if equals be added to equals, then the wholes are equal.

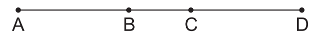
$$\therefore \quad AC = CB \Rightarrow AC + AC = AC + CB$$
$$\Rightarrow 2AC = AB$$

[by Euclid's Axiom 4, $AC + CB = AB$]

$$\Rightarrow \; AC = \frac{1}{2}AB \qquad [\because \text{ halves of equals are equal}]$$

Hence, $AC = \frac{1}{2}AB$.

EXAMPLE 3 *In the given figure, if $AC = BD$, then prove that $AB = CD$.*



SOLUTION By Euclid's Axiom 2, we know that if equals be subtracted from equals, then the remainders are equal.

$$\therefore \quad AC = BD \Rightarrow AC - BC = BD - BC$$
$$\Rightarrow AB = CD.$$

Hence, $AB = CD$.

EXAMPLE 4 *Solve the equation, $x - 5 = 15$.*

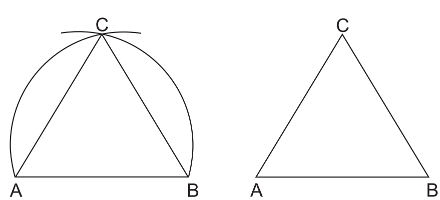
SOLUTION By Euclid's Axiom 2, we know that if equals be added to equals, then the wholes are equal.

$$\therefore \quad x - 5 = 15 \Rightarrow x - 5 + 5 = 15 + 5$$
$$\Rightarrow x = 20.$$

Hence, $x = 20$.

EXAMPLE 5 *Prove that an equilateral triangle can be constructed on any given line segment.*

SOLUTION Let AB be the line segment of given length. With A as centre and AB as the radius, draw a circle. With B as centre and BA as the radius, draw another circle, cutting the first circle at C. Join AC and BC to form $\triangle ABC$.



Now, $AB = AC$ (radii of the same circle)

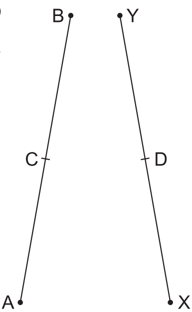
And $BA = BC$ (radii of the same circle)

$\therefore \quad AB = BC \quad [\because \overline{BA} = \overline{AB}]$.

But, by Euclid's Axiom 1, it follows that the things which are equal to the same thing are equal to one another.

$\therefore \quad AB = BC = AC$.

Hence, $\triangle ABC$ is an equilateral triangle.



EXAMPLE 6 *In the given figure, C is the midpoint of AB, D is the midpoint of XY and $AC = XD$. Using an Euclid's axiom, prove that $AB = XY$.*

SOLUTION We have

$AB = 2AC$ [$\because$ C is the midpoint of AB]

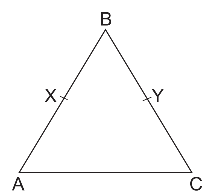
and $XY = 2XD$ [$\because$ D is the midpoint of XY].

Now, $AC = XD$ (given)

$\Rightarrow$ $2AC = 2XD$ [by Euclid's Axiom 6]

$\Rightarrow$ $AB = XY$ [by Euclid's Axiom 7]

Hence, $AB = XY$.



EXAMPLE 7 *In the given figure, a $\triangle ABC$ is given in which $AB = BC$ and $BX = BY$. Show that $AX = CY$.*

GIVEN A $\triangle ABC$ in which $AB = BC$. Also, X and Y are points on AB and BC respectively such that $BX = BY$.

TO PROVE $AX = CY$.

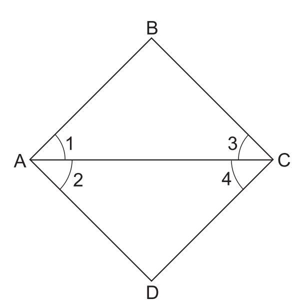
PROOF By Euclid's Axiom 3, we know that when equals are subtracted from equals, the remainders are equal.

$\therefore$ $AB = BC$ and $BX = BY$

$\Rightarrow$ $AB - BX = BC - BY$ [equals are subtracted from equals]

$\Rightarrow$ $AX = CY$.

Hence, $AX = CY$.



EXAMPLE 8 *In the given figure, we have $\angle 1 = \angle 3$ and $\angle 2 = \angle 4$. Show that $\angle A = \angle C$.*

SOLUTION In the given figure, it is given that

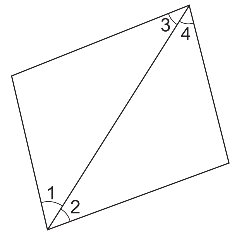
$\angle 1 = \angle 3$ and $\angle 2 = \angle 4$.

By Euclid's Axiom 2, we know that when equals are added to equals, the wholes are equal.

$\therefore$ $\angle 1 + \angle 2 = \angle 3 + \angle 4 \Rightarrow \angle A = \angle C$.

Hence, $\angle A = \angle C$.

EXAMPLE 9 *In the given figure, we have* $\angle 1 = \angle 3$, $\angle 2 = \angle 4$ *and* $\angle 3 = \angle 4$. *Prove that* $\angle 1 = \angle 2$.



SOLUTION Euclid's Axiom 1 says, the things which are equal to the same thing are equal to one another.

$\therefore$ ($\angle 1 = \angle 3$ and $\angle 3 = \angle 4$)

$\Rightarrow$ $\angle 1 = \angle 4$ [by Euclid's Axiom 1].

Now, $\angle 1 = \angle 4$ (proved) and $\angle 2 = \angle 4$ (given)

$\Rightarrow$ $\angle 1 = \angle 2$ [by Euclid's Axiom 1].

Hence, $\angle 1 = \angle 2$.

EXAMPLE 10 *Write which of the following statements are true and which are false.*

(i) A simple closed figure made up of three or more line segments is called a polygon.

(ii) Part of a line with two end points is called a line segment.

(iii) In geometry, we take a point, a line and a plane as undefined terms.

(iv) Euclid's fourth axiom says that everything equals itself.

(v) The Euclidean geometry is valid only for figures in the plane.

(vi) A figure formed by two rays with a common initial point is called an angle.

SOLUTION (i) True (ii) True (iii) True (iv) True

(v) True, as it fails on the curved surfaces (vi) True

EXERCISE 6

1. What is the difference between a theorem and an axiom?

2. Define the following terms:

(i) Line segment (ii) Ray (iii) Intersecting lines

(iv) Parallel lines (v) Half-line (vi) Concurrent lines

(vii) Collinear points (viii) Plane

3. In the adjoining figure, name:

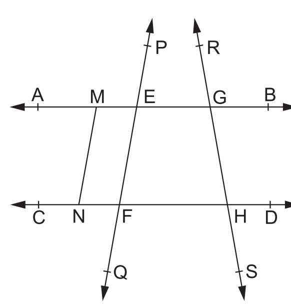
(i) Six points

(ii) Five line segments

(iii) Four rays

(iv) Four lines

(v) Four collinear points



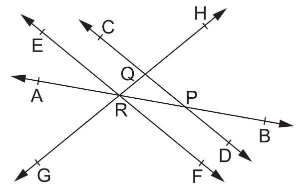
4. In the adjoining figure, name:

(i) Two pairs of intersecting lines and their corresponding points of intersection

(ii) Three concurrent lines and their points of intersection

(iii) Three rays

(iv) Two line segments

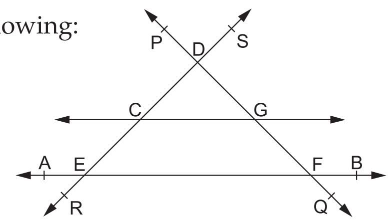


5. From the given figure, name the following:

(i) Three lines

(ii) One rectilinear figure

(iii) Four concurrent points



6. (i) How many lines can be drawn to pass through a given point?

(ii) How many lines can be drawn to pass through two given points?

(iii) In how many points can the two lines at the most intersect?

(iv) If A, B, C are three collinear points, name all the line segments determined by them.

7. Which of the following statements are true?

(i) A line segment has no definite length.

(ii) A ray has no end point.

(iii) A line has a definite length.

(iv) A line $\overleftrightarrow{AB}$ is the same as line $\overleftrightarrow{BA}$.

(v) A ray $\overrightarrow{AB}$ is the same as ray $\overrightarrow{BA}$.

(vi) Two distinct points always determine a unique line.

(vii) Three lines are concurrent if they have a common point.

(viii) Two distinct lines cannot have more than one point in common.

(ix) Two intersecting lines cannot be both parallel to the same line.

(x) Open half-line is the same thing as ray.

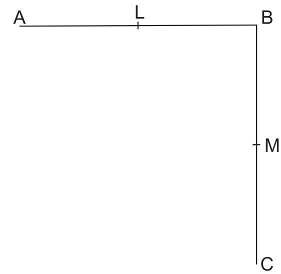
(xi) Two lines may intersect in two points.

(xii) Two lines are parallel only when they have no point in common.

8. In the given figure, L and M are the mid-points of AB and BC respectively.

(i) If $AB = BC$, prove that $AL = MC$.

(ii) If $BL = BM$, prove that $AB = BC$.



HINT (i) $AB = BC \Rightarrow \frac{1}{2}AB = \frac{1}{2}BC \Rightarrow AL = MC$.

(ii) $BL = BM \Rightarrow 2BL = 2BM \Rightarrow AB = BC$.

ANSWERS (EXERCISE 6)

3. (i) A, B, C, D, E, F (ii) $\overline{EG}, \overline{FH}, \overline{EF}, \overline{GH}, \overline{MN}$ (iii) $\overrightarrow{EP}, \overrightarrow{GR}, \overrightarrow{GB}, \overrightarrow{HD}$

(iv) $\overleftrightarrow{AB}, \overleftrightarrow{CD}, \overleftrightarrow{PQ}, \overleftrightarrow{RS}$ (v) M, E, G, B

4. (i) $\{\overleftrightarrow{EF}, \overleftrightarrow{GH}, R\}, \{\overleftrightarrow{AB}, \overleftrightarrow{CD}, P\}$ (ii) $\overleftrightarrow{AB}, \overleftrightarrow{EF}, \overleftrightarrow{GH}, R$ (iii) $\overleftrightarrow{RB}, \overleftrightarrow{RH}, \overleftrightarrow{RG}$

(iv) $\overline{RQ}, \overline{RP}$

5. (i) $\overleftrightarrow{AB}, \overleftrightarrow{PQ}, \overleftrightarrow{RS}$ (ii) $CEFG$ (iii) A, E, F, B

6. (i) infinitely many (ii) one only (iii) one point only

(iv) $\overline{AB}, \overline{BC}, \overline{AC}$

7. (iv), (vi), (vii), (viii), (ix), (x), (xii)

MULTIPLE-CHOICE QUESTIONS (MCQ)

Choose the correct answer in each of the following questions:

1. In ancient India, the shapes of altars used for household rituals were

(a) squares and rectangles (b) squares and circles

(c) triangles and rectangles (d) trapeziums and pyramids

2. In ancient India, altars with combination of shapes like rectangles, triangles and trapeziums were used for

(a) household rituals (b) public rituals

(c) both (a) and (b) (d) none of (a), (b) and (c)

3. The number of interwoven isosceles triangles in *Sriyantra* is

(a) five (b) seven (c) nine (d) eleven

4. In Indus Valley Civilization (about BC 3000), the bricks used for construction work were having dimensions in the ratio of

(a) 5 : 3 : 2 (b) 4 : 2 : 1 (c) 4 : 3 : 2 (d) 6 : 4 : 2

5. Into how many chapters was the famous treatise, '*The Elements*' divided by Euclid?

(a) 13 (b) 12 (c) 11 (d) 9

6. Euclid belongs to the country

(a) India (b) Greece (c) Japan (d) Egypt

7. Thales belongs to the country

(a) India (b) Egypt (c) Greece (d) Babylonia

8. Pythagoras was a student of

(a) Euclid (b) Thales (c) Archimedes (d) Bhaskara

9. Which of the following needs a proof?

(a) axiom (b) postulate (c) definition (d) theorem

10. The statement that *'the lines are parallel if they do not intersect'* is in the form of

(a) a definition (b) an axiom (c) a postulate (d) a theorem

11. Euclid stated that *'all right angles are equal to each other'*, in the form of

(a) a definition (b) an axiom (c) a postulate (d) a proof

12. A pyramid is a solid figure, whose base is

(a) only a triangle (b) only a square
(c) only a rectangle (d) any polygon

13. The side faces of a pyramid are

(a) triangles (b) squares (c) trapeziums (d) polygons

14. The number of dimensions of a solid are

(a) 1 (b) 2 (c) 3 (d) 5

15. The number of dimensions of a surface is

(a) 1 (b) 2 (c) 3 (d) 0

16. How many dimensions does a point have

(a) 0 (b) 1 (c) 2 (d) 3

17. Boundaries of solids are

(a) lines (b) curves (c) surfaces (d) none of these

18. Boundaries of surfaces are

(a) lines (b) curves (c) polygons (d) none of these

19. The number of planes passing through 3 noncollinear points is

(a) 4 (b) 3 (c) 2 (d) 1

20. Axioms are assumed

(a) definitions
(b) theorems
(c) universal truths specific to geometry
(d) universal truths in all branches of mathematics

21. Which of the following is a true statement?

(a) The floor and a wall of a room are parallel planes.
(b) The ceiling and a wall of a room are parallel planes.
(c) The floor and the ceiling of a room are parallel planes.
(d) Two adjacent walls of a room are parallel planes.

22. Which of the following is a true statement?

(a) Only a unique line can be drawn to pass through a given point.
(b) Infinitely many lines can be drawn to pass through two given points.
(c) If two circles are equal, then their radii are equal.
(d) A line has a definite length.

23. Which of the following is a false statement?

(a) An infinite number of lines can be drawn to pass through a given point.
(b) A unique line can be drawn to pass through two given points.
(c) Ray $\overrightarrow{AB}$ = ray $\overrightarrow{BA}$.
(d) A ray has one end point.

24. A point C is called the midpoint of a line segment $\overline{AB}$ if

(a) C is an interior point of AB
(b) $AC = CB$
(c) C is an interior point of AB such that $\overline{AC} = \overline{CB}$
(d) $AC + CB = AB$

25. A point C is said to lie between the points A and B if

(a) $AC = CB$ (b) $AC + CB = AB$
(c) points A, C and B are collinear (d) none of these

26. Euclid's which axiom illustrates the statement that when $x + y = 15$, then $x + y + z = 15 + z$?

(a) first (b) second (c) third (d) fourth

27. A is of the same age as B and C is of the same age as B. Euclid's which axiom illustrates the relative ages of A and C?

(a) First axiom (b) Second axiom
(c) Third axiom (d) Fourth axiom

ANSWERS (MCQ)

1. (b) **2.** (b) **3.** (c) **4.** (b) **5.** (a) **6.** (b) **7.** (c) **8.** (b)

9. (d)	**10.** (a)	**11.** (c)	**12.** (d)	**13.** (a)	**14.** (c)	**15.** (b)	**16.** (a)
17. (c)	**18.** (b)	**19.** (d)	**20.** (d)	**21.** (c)	**22.** (c)	**23.** (c)	**24.** (c)
25. (c)	**26.** (b)	**27.** (a)					

HINTS TO SOME SELECTED QUESTIONS

1. In ancient India, the shapes of altars used for household rituals were squares and circles.
2. In ancient India, altars with combination of rectangles, triangles and trapeziums were used for public rituals.
3. There were nine interwoven isosceles triangles in *Sriyantra*.
4. The size of bricks in Indus Valley Civilization was 4 : 2 : 1.
5. Euclid divided his book '*The Elements*' into 13 chapters.
6. Euclid belongs to Greece.
7. Thales belongs to Greece.
8. Pythagoras was a student of Thales.
9. A theorem needs a proof.
10. Lines are parallel if they do not intersect is stated as a definition.
11. Euclid stated that all right angles are equal to each other in the form of a postulate.
12. A pyramid is a solid figure whose base is any polygon.
13. The side faces of a pyramid are triangles.
14. A solid has 3 dimensions, namely length, breadth and height.
15. A surface has 2 dimensions namely, length and breadth.
16. A point has 0 dimensions.
17. Boundaries of solids are called surfaces.
18. Boundaries of surfaces are curves.
19. A unique plane passes through 3 given noncollinear points.
20. Axioms are assumed universal truths in all branches of mathematics.
21. The floor and the ceiling of a room are the parallel planes.
22. If two circles are equal, then their radii are equal.
24. A point C is called the midpoint of line segment $\overline{AB}$, if C is an interior point of $\overline{AB}$ such that $\overline{AC} = \overline{CB}$.
25. A point C is said to lie between A and B if the points A, C and B are collinear.
26. Second axiom states that if equals be added to equals, then wholes are equal.
27. Here $(A = B \text{ and } C = B) \Rightarrow A = C$.

    First axiom states that the things which are equal to the same thing are equal to one another.

REVIEW OF FACTS

1. EUCLID'S AXIOMS

(i) Things which are equal to the same thing are equal to one another.

(ii) If equals are added to equals, the wholes are equal.

(iii) If equals are subtracted from equals, the remainders are equal.

(iv) Things which coincide with one another are equal to one another.

(v) The whole is greater than the part.

(vi) Things which are double of the same thing are equal to one another.

(vii) Things which are halves of the same thing are equal to one another.

2. EUCLID'S FIVE POSTULATES

POSTULATE 1. *A straight line may be drawn from one point to any other point.*

Given two distinct points, there is a unique line that passes through them.

POSTULATE 2. *A terminated line can be produced indefinitely.*

POSTULATE 3. *A circle can be drawn with any centre and any radius.*

POSTULATE 4. *All right angles are equal to one another.*

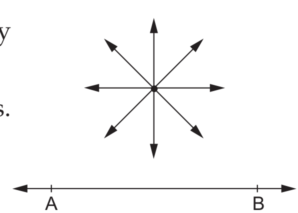
POSTULATE 5. *For every line L and for every point P not lying on L, there exists a unique line M passing through P and parallel to L.*

3. INCIDENCE AXIOMS ON LINES

(i) A line contains infinitely many points.

(ii) Through a given point, infinitely many lines can be drawn.

(iii) One and only one line can be drawn to pass through two given points A and B.



4. (i) A solid has 3 dimensions, namely length, breadth and height.

(ii) Boundaries of solids are called surfaces.

(iii) Boundaries of surfaces are curves.

(iv) A surface has 2 dimensions.

(v) A pyramid is a solid whose base is a polygon and whose side faces are triangles.

✿

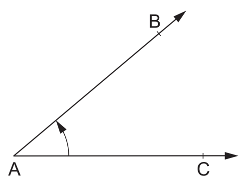
7 Lines and Angles

BASIC TERMS AND DEFINITIONS

ANGLE *When two rays originate from the same end point, then an angle is formed.*

*The rays making an angle are called the **arms** of the angle and the end point is called the **vertex** of the angle.*

In the given figure, two rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ originate from the same end point A, forming an angle denoted by $\angle BAC$ or $\angle CAB$ or $\angle A$.



Clearly, $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are the *arms* of this angle and A is its *vertex*.

We measure an angle in degrees, denoted by '°'.

If $\angle A = 60°$, we write, $m(\angle A) = 60°$.

VARIOUS TYPES OF ANGLES

(i) ACUTE ANGLE An angle whose measure is more than 0° but less than 90° is called an acute angle.

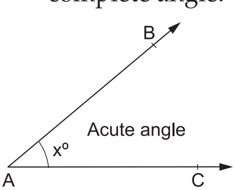
(ii) RIGHT ANGLE An angle whose measure is 90° is called a right angle.

(iii) OBTUSE ANGLE An angle whose measure is more than 90° but less than 180° is called an obtuse angle.

(iv) STRAIGHT ANGLE An angle whose measure is 180° is called a straight angle.

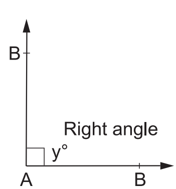
(v) REFLEX ANGLE An angle whose measure is more than 180° but less than 360° is called a reflex angle.

(vi) COMPLETE ANGLE An angle whose measure is 360° is called a complete angle.



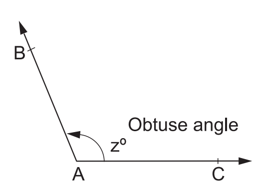
(i) Acute angle

($0° < x° < 90°$)



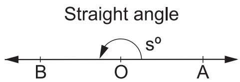
(ii) Right angle

($y° = 90°$)

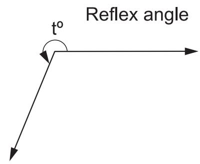


(iii) Obtuse angle

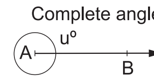
($90° < z° < 180°$)



(iv) Straight angle
($s^\circ = 180^\circ$)



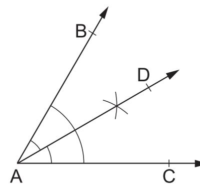
(v) Reflex angle
($180^\circ < t^\circ < 360^\circ$)



(vi) Complete angle
($u^\circ = 360^\circ$)

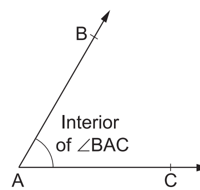
EQUAL ANGLES *Two angles are said to be equal, if they are of the same measure.*

Thus, $\angle A = \angle B \Leftrightarrow m(\angle A) = m(\angle B)$.



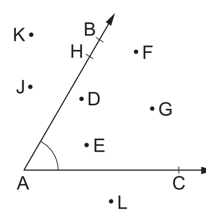
ANGLE BISECTOR *A ray* $\overrightarrow{AD}$ *is said to be the bisector of* $\angle BAC$, *if* $\angle BAD = \angle CAD$.

INTERIOR OF AN ANGLE *The interior of* $\angle BAC$ *is the set of all points in its plane which lie on the same side of AB as C and also on the same side of AC as B.*



EXTERIOR OF AN ANGLE *The exterior of* $\angle BAC$ *is the set of all points in its plane which do not lie on the angle or in its interior.*

Example In the given figure, the points D, E, F, G lie in the interior of $\angle BAC$, while A, B, C, H lie on $\angle BAC$. And, the points J, K, L lie in the exterior of $\angle BAC$.



COMPLEMENTARY ANGLES *Two angles are said to be complementary, if the sum of their measures is* 90°.

Two complementary angles are called the *complement* of each other.

Example Angles measuring 65° and 25° are complementary angles.

SUPPLEMENTARY ANGLES *Two angles are said to be supplementary if the sum of their measures is* 180°.

Two supplementary angles are called the *supplement* of each other.

Example Angles measuring 62° and 118° are supplementary angles.

SOLVED EXAMPLES

EXAMPLE 1 *Find the measure of an angle which is* 24° *more than its complement.*

SOLUTION Let the measure of the required angle be x°.

Then, measure of its complement $= (90 - x)^\circ$.

$\therefore \quad x-(90-x)=24 \Rightarrow 2x=114 \Rightarrow x=57.$

Hence, the measure of the required angle is 57°.

EXAMPLE 2 *Find the measure of an angle which is* 32° *less than its supplement.*

SOLUTION Let the measure of the required angle be $x°$.

Then, measure of its supplement = $(180-x)°$.

$\therefore \quad (180-x)-x=32 \Rightarrow 2x=148 \Rightarrow x=74.$

Hence, the measure of the required angle is 74°.

EXAMPLE 3 *Two supplementary angles are in the ratio* 3 : 2. *Find the angles.*

SOLUTION Let the measures of the given angles be $(3x)°$ and $(2x)°$ respectively.

Since the given angles are supplementary, we have

$$3x+2x=180 \Rightarrow 5x=180 \Rightarrow x=36.$$

$\therefore$ the measures of the given angles are $(3\times 36)°$ and $(2\times 36)°$, i.e, 108° and 72° respectively.

EXAMPLE 4 *The supplement of an angle is one third of the given angle. Find the measures of the given angle and its supplement.*

SOLUTION Let the measure of the given angle be $x°$.

Then, the measure of its supplement = $(180-x)°$.

$$\therefore \quad 180-x=\frac{1}{3}x \Rightarrow 3(180-x)=x$$
$$\Rightarrow 540-3x=x$$
$$\Rightarrow 4x=540 \Rightarrow x=135.$$

Hence, the measure of the given angle is 135° and the measure of its supplement is $(180-135)°=45°$.

EXAMPLE 5 *Find the measure of an angle, if seven times its complement is* 10° *less than three times its supplement.*

SOLUTION Let the measure of the required angle be $x°$.

Then, its complement = $(90-x)°$

and its supplement = $(180-x)°$.

$$\therefore \quad 7(90-x)=3(180-x)-10$$
$$\Rightarrow \quad 630-7x=540-3x-10$$
$$\Rightarrow \quad 4x=100 \Rightarrow x=25.$$

Hence, the measure of the required angle is 25°.

EXERCISE 7A

1. Define the following terms:

(i) Angle
(ii) Interior of an angle
(iii) Obtuse angle
(iv) Reflex angle
(v) Complementary angles
(vi) Supplementary angles

2. Find the complement of each of the following angles:

(i) $55°$
(ii) $16°$
(iii) $90°$
(iv) $\frac{2}{3}$ of a right angle

3. Find the supplement of each of the following angles:

(i) $42°$
(ii) $90°$
(iii) $124°$
(iv) $\frac{3}{5}$ of a right angle

4. Find the measure of an angle which is

(i) equal to its complement,
(ii) equal to its supplement.

5. Find the measure of an angle which is $36°$ more than its complement.

6. Find the measure of an angle which is $30°$ less than its supplement.

7. Find the angle which is four times its complement.

8. Find the angle which is five times its supplement.

9. Find the angle whose supplement is four times its complement.

10. Find the angle whose complement is one third of its supplement.

11. Two complementary angles are in the ratio 4 : 5. Find the angles.

12. Find the value of x for which the angles $(2x-5)°$ and $(x-10)°$ are the complementary angles.

ANSWERS (EXERCISE 7A)

2. (i) $35°$ (ii) $74°$ (iii) $0°$ (iv) $30°$
3. (i) $138°$ (ii) $90°$ (iii) $56°$ (iv) $126°$
4. (i) $45°$ (ii) $90°$ **5.** $63°$ **6.** $75°$ **7.** $72°$
8. $150°$ **9.** $60°$ **10.** $45°$ **11.** $40°, 50°$ **12.** $x = 35$

HINTS TO SOME SELECTED QUESTIONS

5. $x = (90 - x) + 36$. Find x.
6. $x = (180 - x) - 30$. Find x.

7. $x = 4(90 - x)$. Find x.

8. $x = 5(180 - x)$. Find x.

9. $(180 - x) = 4(90 - x)$. Find x.

10. $(90 - x) = \frac{1}{3}(180 - x) \Rightarrow 270 - 3x = 180 - x$. Find x.

11. $4x + 5x = 90$. Find x.

12. $(2x - 5) + (x - 10) = 90$. Find x.

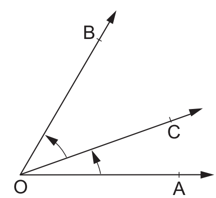
SOME ANGLE RELATIONS

ADJACENT ANGLES *Two angles are called adjacent angles, if*

(i) *they have the same vertex,*

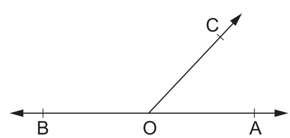
(ii) *they have a common arm and*

(iii) *their non-common arms are on either side of the common arm.*



In the given figure, $\angle AOC$ and $\angle BOC$ are adjacent angles having the same vertex O, a common arm OC and their non-common arms OA and OB on either side of OC.

LINEAR PAIR OF ANGLES *Two adjacent angles are said to form a linear pair of angles, if their non-common arms are two opposite rays.*



In the adjoining figure, $\angle AOC$ and $\angle BOC$ are two adjacent angles whose non-common arms OA and OB are two opposite rays, i.e., BOA is a line.

$\therefore$ $\angle AOC$ and $\angle BOC$ form a linear pair of angles.

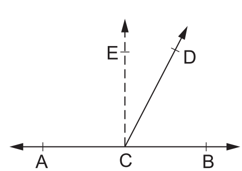
SOME RESULTS ON ANGLE RELATIONS

THEOREM 1 *If a ray stands on a line then the sum of the adjacent angles so formed is* 180°.

GIVEN A ray CD stands on a line AB such that $\angle ACD$ and $\angle BCD$ are formed.

TO PROVE $\angle ACD + \angle BCD = 180°$.

CONSTRUCTION Draw $CE \perp AB$.



PROOF $\angle ACD = \angle ACE + \angle ECD$ ... (i)

and $\angle BCD = \angle BCE - \angle ECD$. ... (ii)

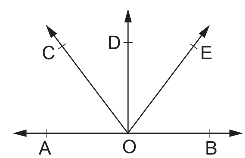
Adding (i) and (ii), we get

$$\angle ACD + \angle BCD = (\angle ACE + \angle ECD) + (\angle BCE + \angle ECD)$$
$$= \angle ACE + \angle BCE$$
$$= (90^\circ + 90^\circ) = 180^\circ \quad [\because\ \angle ACE = \angle BCE = 90^\circ].$$

Hence, $\angle ACD + \angle BCD = 180^\circ$.

REMARK We may state the above theorem as 'the sum of the angles of a linear pair is 180°'.

COROLLARY 1 *Prove that the sum of all the angles formed on the same side of a line at a given point on the line is* 180°.



GIVEN AOB is a straight line and rays OC, OD and OE stand on it, forming $\angle AOC$, $\angle COD$, $\angle DOE$ and $\angle EOB$.

TO PROVE $\angle AOC + \angle COD + \angle DOE + \angle EOB = 180^\circ$.

PROOF Ray OC stands on line AB.

$\therefore\quad \angle AOC + \angle COB = 180^\circ$

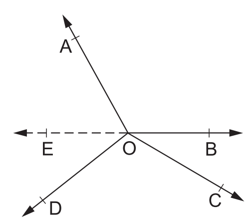
$\Rightarrow\quad \angle AOC + (\angle COD + \angle DOE + \angle EOB) = 180^\circ$

$[\because\ \angle COB = \angle COD + \angle DOE + \angle EOB] = 180^\circ$

$\Rightarrow\quad \angle AOC + \angle COD + \angle DOE + \angle EOB = 180^\circ$.

Hence, the sum of all the angles formed on the same side of line AB at a point O on it is 180°.

COROLLARY 2 *Prove that the sum of all angles around a point is* 360°.



GIVEN A point O and the rays OA, OB, OC and OD make angles around O.

TO PROVE $\angle AOB + \angle BOC + \angle COD + \angle AOD = 360^\circ$.

CONSTRUCTION Produce ray OB backwards to a point E such that EOB is a line.

PROOF Clearly, EOB is a straight line and the ray OA stands on it.

$\therefore\quad \angle AOE + \angle AOB = 180^\circ$. ... (i)

We know that the sum of all angles at a point on a given line on the same side of it, is 180°.

$\therefore\quad \angle BOC + \angle COD + \angle DOE = 180^\circ$. ... (ii)

Adding the corresponding sides of (i) and (ii), we get

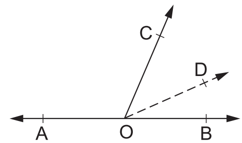
$$\angle AOE + \angle AOB + \angle BOC + \angle COD + \angle DOE = 360^\circ$$

$\Rightarrow\quad \angle AOB + \angle BOC + \angle COD + (\angle AOE + \angle DOE) = 360^\circ$

$\Rightarrow\quad \angle AOB + \angle BOC + \angle COD + \angle AOD = 360^\circ$.

Hence, the sum of all the angles around a given point is 360°.

THEOREM 2 (Converse of Theorem 1) *If the sum of two adjacent angles is* 180° *then the non-common arms of the angles are in a straight line.*



GIVEN Two adjacent angles $\angle AOC$ and $\angle BOC$ with common arm OC such that $\angle AOC + \angle BOC = 180°$.

TO PROVE OA and OB are in the same straight line, i.e., AOB is a straight line.

CONSTRUCTION If possible, let AOB be not a straight line. Then, produce AO to D so that AOD is a straight line.

PROOF By our assumption, AOD is a straight line and ray OC stands on it.

$\therefore\quad \angle AOC + \angle COD = 180°$ [linear pair].

But, $\angle AOC + \angle BOC = 180°$ [given].

$\therefore\quad \angle AOC + \angle COD = \angle AOC + \angle BOC$ [each equal to 180°].

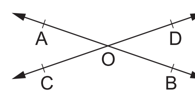
So, $\angle COD = \angle BOC$.

But, this is not true, since a part cannot be equal to whole.

$\therefore$ our supposition is wrong.

Hence, AOB is a straight line.

VERTICALLY OPPOSITE ANGLES *Two angles are called a pair of vertically opposite angles, if their arms form two pairs of opposite rays.*



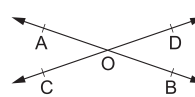
Let two lines AB and CD intersect at a point O. Then, two pairs of vertically opposite angles are formed:

(i) $\angle AOC$ and $\angle BOD$ (ii) $\angle AOD$ and $\angle BOC$.

THEOREM 3 *If two lines intersect then the vertically opposite angles are equal.*

GIVEN Two lines AB and CD intersect at a point O.

TO PROVE (i) $\angle AOC = \angle BOD$, (ii) $\angle AOD = \angle BOC$.



PROOF Since ray OA stands on line CD, we have

$\angle AOC + \angle AOD = 180°$ [linear pair].

Again, ray OD stands on line AB.

$\therefore\quad \angle AOD + \angle BOD = 180°$ [linear pair].

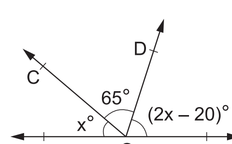
$\therefore\quad \angle AOC + \angle AOD = \angle AOD + \angle BOD$ [each equal to 180°]

$\therefore\quad \angle AOC = \angle BOD$.

Similarly, $\angle AOD = \angle BOC$.

SOLVED EXAMPLES

EXAMPLE 1 *In the adjoining figure, AOB is a straight line. Find $\angle AOC$ and $\angle BOD$.*



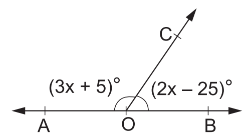
SOLUTION Since AOB is a straight line, the sum of all the angles on the same side of AOB at a point O on it, is 180°.

$\therefore \quad x + 65 + (2x - 20) = 180$

$\Rightarrow \quad 3x = 135 \;\Rightarrow\; x = 45.$

$\therefore \quad \angle AOC = x° = 45°$ and $\angle BOD = (2 \times 45 - 20)° = 70°.$

EXAMPLE 2 *In the adjoining figure, what value of x will make AOB a straight line?*



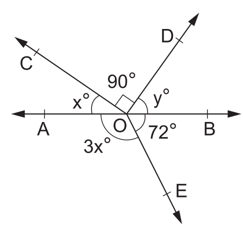
SOLUTION AOB will be a straight line, if $\angle AOC + \angle BOC = 180°$.

$\therefore \quad (3x + 5) + (2x - 25) = 180$

$\Rightarrow \quad 5x = 200 \;\Rightarrow\; x = 40.$

Hence, $x = 40$ will make AOB a straight line.

EXAMPLE 3 *Calculate $\angle AOC$, $\angle BOD$ and $\angle AOE$ in the adjoining figure, it is being given that $\angle COD = 90°$, $\angle BOE = 72°$ and AOB is a straight line.*



SOLUTION Since AOB is a straight line, the sum of all the angles on the lower side of AOB at a point O on it, is 180°.

$\therefore \quad \angle AOE + \angle BOE = 180°$

$\Rightarrow \quad 3x + 72 = 180$

$\Rightarrow \quad 3x = (180 - 72) = 108 \;\Rightarrow\; x = \dfrac{108}{3} = 36.$

Again, AOB is a straight line and O is a point on it.

So, the sum of all angles on the upper side of AOB at a point O on it, is 180°.

$\therefore \quad \angle AOC + \angle COD + \angle DOB = 180°$

$\Rightarrow \quad x + 90 + y = 180$

$[\because\; \angle AOC = x°, \angle COD = 90° \text{ and } \angle DOB = y°]$

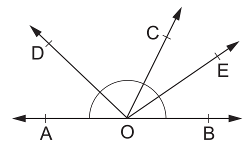
$\Rightarrow \quad 36 + 90 + y = 180 \quad [\because\; x = 36]$

$\Rightarrow \quad 126 + y = 180 \;\Rightarrow\; y = (180 - 126) = 54.$

$\therefore \quad \angle AOC = x° = 36°, \angle BOD = y° = 54°,$

$\angle AOE = (3x)° = (3 \times 36)° = 108°.$

EXAMPLE 4 *In the given figure, ray OC stands on a straight line AOB. Ray OD and ray OE are the bisectors of $\angle AOC$ and $\angle BOC$ respectively. Find the measure of $\angle DOE$.*



SOLUTION Since AOB is a straight line and the ray OC stands on it.

$\therefore \quad \angle AOC + \angle BOC = 180°$

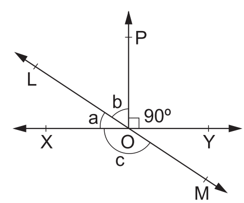
$\Rightarrow \quad \frac{1}{2}\angle AOC + \frac{1}{2}\angle BOC = 90°$

$\Rightarrow \quad \angle COD + \angle COE = 90°$

$\Rightarrow \quad \angle DOE = 90° \qquad [\because \angle COD + \angle COE = \angle DOE].$

Hence, the bisectors of a linear pair are at right angles.

EXAMPLE 5 *In the given figure, lines XY and LM intersect at O. If $\angle XOL = a$, $\angle LOP = b$, $\angle POY = 90°$, $\angle XOM = c$ and $a : b = 2 : 3$, find the value of c.*



SOLUTION It is given that $a : b = 2 : 3$.

So, let $a = (2x)°$ and $b = (3x)°$.

In the given figure, ray OP stands on line XY.

$\therefore \quad \angle XOP + \angle POY = 180°$

$\Rightarrow \quad \angle XOL + \angle LOP + 90° = 180° \qquad [\because \angle POY = 90°]$

$\Rightarrow \quad a + b + 90 = 180$

$\Rightarrow \quad a + b = 90$

$\Rightarrow \quad 2x + 3x = 90 \Rightarrow 5x = 90 \Rightarrow x = 18.$

$\therefore \quad a = (2x)° = (2 \times 18)° = 36°$ and $b = (3x)° = (3 \times 18)° = 54°$.

Again, ray OX stands on line LM.

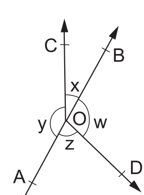
$\therefore \quad \angle LOX + \angle XOM = 180°$

$\Rightarrow \quad a + c = 180°$

$\Rightarrow \quad 36° + c = 180° \Rightarrow c = (180° - 36°) = 144°.$

Hence, $c = 144°$.

EXAMPLE 6 *In the given figure, if $x + y = z + w$ then prove that AOB is a line.*



SOLUTION We know that the sum of all the angles around a point is 360°.

$\therefore \quad x + y + z + w = 360°$

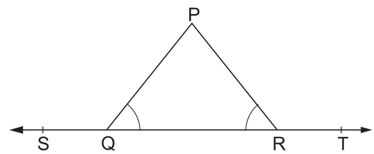
$\Rightarrow \quad (x + y) + (z + w) = 360°$

$\Rightarrow \quad (x + y) + (x + y) = 360° \qquad [\because\ z + w = x + y]$

$\Rightarrow \quad 2(x + y) = 360° \ \Rightarrow\ x + y = 180°.$

Hence, AOB is a straight line.

EXAMPLE 7 *In the given figure, if $\angle PQR = \angle PRQ$ then prove that $\angle PQS = \angle PRT$.*



SOLUTION Since ray QP stands on line ST, we have

$$\angle PQR + \angle PQS = 180°. \qquad \ldots \text{(i)}$$

Again, ray RP stands on line ST.

$$\therefore \quad \angle PRQ + \angle PRT = 180°. \qquad \ldots \text{(ii)}$$

From (i) and (ii), we get

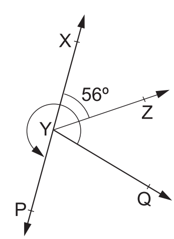
$\angle PQR + \angle PQS = \angle PRQ + \angle PRT$

$\Rightarrow \quad \angle PRQ + \angle PQS = \angle PRQ + \angle PRT \qquad [\because\ \angle PQR = \angle PRQ]$

$\Rightarrow \quad \angle PQS = \angle PRT.$

Hence, $\angle PQS = \angle PRT$.

EXAMPLE 8 *In the given figure, $\angle XYZ = 56°$ and XY is produced to a point P. If ray YQ bisects $\angle ZYP$, find $\angle XYQ$ and reflex $\angle QYP$.*



SOLUTION Since XY is produced to point P, it follows that XP is a straight line and ray YZ stands on it.

$\therefore \quad \angle XYZ + \angle ZYP = 180°$

$\Rightarrow \quad 56° + \angle ZYP = 180°$

$\Rightarrow \quad \angle ZYP = (180° - 56°) = 124°$

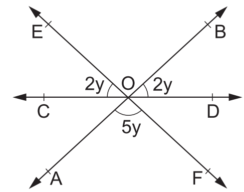
$\Rightarrow \quad \angle ZYQ = \frac{1}{2}\angle ZYP = \left(\frac{1}{2} \times 124°\right) = 62°.$

$\therefore \quad \angle XYQ = \angle XYZ + \angle ZYQ = (56° + 62°) = 118°.$

Also, $\angle QYP = \frac{1}{2}\angle ZYP = \left(\frac{1}{2} \times 124°\right) = 62°.$

$\therefore \quad \text{reflex } \angle QYP = 360° - \angle QYP = (360 - 62)° = 298°.$

EXAMPLE 9 *In the given figure, AB, CD and EF are three lines concurrent at O. Find the value of y.*



SOLUTION Lines EOF and AOB intersect each other at the point O.

$\therefore \quad \angle BOE = \angle AOF = 5y.$ ... (i)

[vertically opposite angles]

Also, the sum of all the angles formed on the upper side of COD at the point O is 180°.

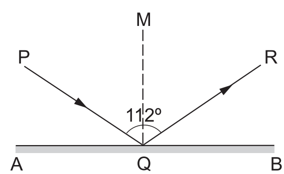
$\therefore \quad \angle DOB + \angle BOE + \angle EOC = 180^\circ$

$\Rightarrow \quad 2y + 5y + 2y = 180^\circ$ [using (i)]

$\Rightarrow \quad 9y = 180^\circ \Rightarrow y = 20^\circ.$

Hence, $y = 20^\circ$.

EXAMPLE 10 *In the given figure, AB is a mirror, PQ is the incident ray and QR, the reflected ray. If $\angle PQR = 112^\circ$, find $\angle PQA$.*



SOLUTION When a ray falls on a mirror, it is reflected and angle of incidence = angle of reflection = x° (say).

QM is drawn normal to AB and therefore,

angle of incidence = $\angle PQM$,

angle of reflection = $\angle MQR$

and $\angle AQM = 90^\circ$.

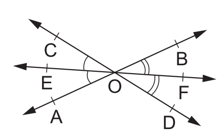
Now, $\angle PQM + \angle MQR = \angle PQR = 112^\circ$ (given)

$\therefore \quad 2\angle PQM = 112^\circ$

$\therefore \quad \angle PQM = 56^\circ.$

So, $\angle PQA = \angle AQM - \angle PQM = 90^\circ - 56^\circ = 34^\circ.$

EXAMPLE 11 *Prove that the bisectors of a pair of vertically opposite angles are in the same straight line.*



GIVEN Two lines AB and CD intersect at a point O. Also, OE and OF are the bisectors of $\angle AOC$ and $\angle BOD$ respectively.

TO PROVE EOF is a straight line.

PROOF Since the sum of all the angles around a point is 360°, we have:

$$\angle AOC + \angle BOC + \angle BOD + \angle AOD = 360^\circ$$

$\Rightarrow \quad \angle AOC + 2\angle BOC + \angle BOD = 360°$

[$\because$ $\angle AOD = \angle BOC$ (vert. opp. $\angle$s)]

$\Rightarrow \quad 2\angle EOC + 2\angle BOC + 2\angle BOF = 360°$

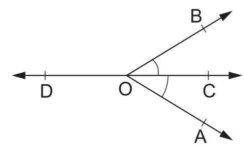
[$\because$ OE is the bisector of $\angle AOC$ and OF is the bisector of $\angle BOD$]

$\Rightarrow \quad \angle EOC + \angle BOC + \angle BOF = 180°$

$\Rightarrow \quad \angle EOF = 180°$, i.e., $\angle EOF$ is a straight angle.

Hence, EOF is a straight line.

<u>EXAMPLE 12</u> *In the given figure, ray OC is the bisector of $\angle AOB$ and OD is the ray opposite to OC. Show that $\angle AOD = \angle BOD$.*



SOLUTION Since OC is the bisector of $\angle AOB$, we have

$\angle BOC = \angle AOC.$ ... (i)

Since ray OB stands on line DOC at a point O, we have

$\angle BOC + \angle BOD = 180°.$ ... (ii)

Again, ray OA stands on line DOC at a point O, so we have

$\angle AOC + \angle AOD = 180°.$ ... (iii)

From (ii) and (iii), we get

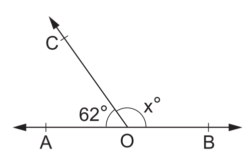
$\angle AOC + \angle AOD = \angle BOC + \angle BOD$

$\Rightarrow \quad \angle AOD = \angle BOD$ [$\because$ $\angle AOC = \angle BOC$ from (i)].

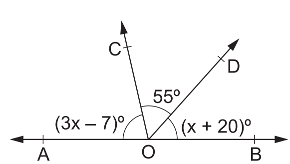
Hence, $\angle AOD = \angle BOD$.

EXERCISE 7B

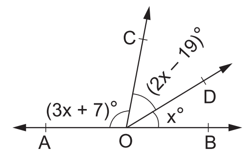
1. In the adjoining figure, AOB is a straight line. Find the value of x.



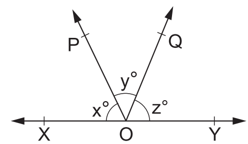
2. In the adjoining figure, AOB is a straight line. Find the value of x. Hence, find $\angle AOC$ and $\angle BOD$.



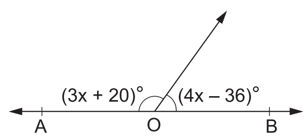
3. In the adjoining figure, AOB is a straight line. Find the value of x. Hence, find $\angle AOC$, $\angle COD$ and $\angle BOD$.



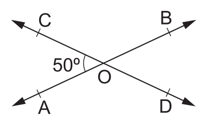
4. In the adjoining figure, $x : y : z = 5 : 4 : 6$. If XOY is a straight line, find the values of x, y and z.



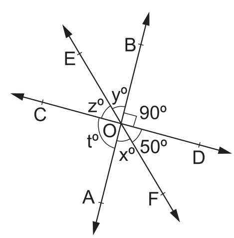
5. In the adjoining figure, what value of x will make AOB, a straight line?



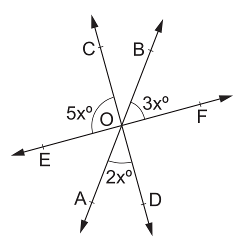
6. Two lines AB and CD intersect at O. If $\angle AOC = 50°$, find $\angle AOD$, $\angle BOD$ and $\angle BOC$.



7. In the adjoining figure, three coplanar lines AB, CD and EF intersect at a point O, forming angles as shown. Find the values of x, y, z and t.



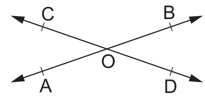
8. In the adjoining figure, three coplanar lines AB, CD and EF intersect at a point O. Find the value of x. Hence, find $\angle AOD$, $\angle COE$ and $\angle AOE$.



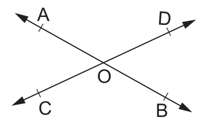
9. Two adjacent angles on a straight line are in the ratio 5 : 4. Find the measure of each one of these angles.

10. If two straight lines intersect each other in such a way that one of the angles formed measures 90°, show that each of the remaining angles measures 90°.

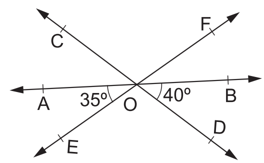
11. Two lines AB and CD intersect at a point O such that $\angle BOC + \angle AOD = 280°$, as shown in the figure. Find all the four angles.



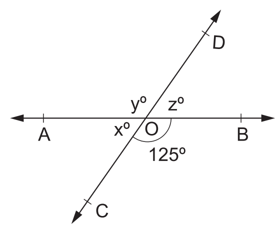
12. Two lines AB and CD intersect each other at a point O such that $\angle AOC : \angle AOD = 5:7$. Find all the angles.



13. In the given figure, three lines AB, CD and EF intersect at a point O such that $\angle AOE = 35°$ and $\angle BOD = 40°$. Find the measure of $\angle AOC$, $\angle BOF$, $\angle COF$ and $\angle DOE$.



14. In the given figure, the two lines AB and CD intersect at a point O such that $\angle BOC = 125°$. Find the values of x, y and z.



15. If two straight lines intersect each other then prove that the ray opposite to the bisector of one of the angles so formed bisects the vertically opposite angle.

16. Prove that the bisectors of two adjacent supplementary angles include a right angle.

ANSWERS (EXERCISE 7B)

1. 118 **2.** $x = 28$, $\angle AOC = 77°$, $\angle BOD = 48°$

3. $x = 32$, $\angle AOC = 103°$, $\angle COD = 45°$, $\angle BOD = 32°$

4. $x = 60$, $y = 48$, $z = 72$ **5.** $x = 28$

6. $\angle AOD = 130°$, $\angle BOC = 130°$, $\angle BOD = 50°$

7. $x = 40$, $y = 40$, $z = 50$, $t = 90$

8. $x = 18$, $\angle AOD = 36°$, $\angle COE = 90°$, $\angle AOE = 54°$ **9.** 100°, 80°

11. $\angle BOC = 140°$, $\angle AOD = 140°$, $\angle AOC = 40°$, $\angle BOD = 40°$

12. $\angle AOC = 75° = \angle BOD$, $\angle AOD = 105° = \angle BOC$

13. $\angle AOC = 40°$, $\angle BOF = 35°$, $\angle COF = 105°$, $\angle DOE = 105°$

14. $x = 55$, $y = 125$, $z = 55$

HINTS TO SOME SELECTED QUESTIONS

4. Let $x = (5t)^\circ$, $y = (4t)^\circ$ and $z = (6t)^\circ$.

$x + y + z = 180^\circ \Rightarrow 5t + 4t + 6t = 180 \Rightarrow 15t = 180 \Rightarrow t = 12.$

$\therefore \quad x = (5 \times 12)^\circ = 60^\circ,\ y = (4 \times 12)^\circ = 48^\circ$ and $z = (6 \times 12)^\circ = 72^\circ.$

8. $\angle DOF = \angle EOC = (5x)^\circ$ [vertically opposite $\measuredangle$s].

Now, EOF is a line. Sum of all angles above it is 180°.

$3x + 5x + 2x = 180 \Rightarrow 10x = 180 \Rightarrow x = 18.$

$\therefore \quad \angle BOF = (3 \times 18)^\circ = 54^\circ, \angle DOF = \angle EOC = (5 \times 18)^\circ = 90^\circ, \angle AOD = (2 \times 18)^\circ = 36^\circ.$

$\angle AOE = \angle BOF = 54^\circ, \angle COE = \angle DOF = 90^\circ.$

9. $5x + 4x = 180 \Rightarrow 9x = 180 \Rightarrow x = 20.$

So, the angles measure 100° and 80°.

11. $\angle BOC = \angle AOD$ [vertically opposite $\measuredangle$s].

$\therefore \quad \angle BOC = \angle AOD = 140^\circ.$

$\angle AOC + \angle AOD = 180^\circ \Rightarrow \angle AOC + 140^\circ = 180^\circ \Rightarrow \angle AOC = 40^\circ.$

$\angle BOD = \angle AOC = 40^\circ.$

12. Let $AOC = (5x)^\circ$ and $\angle AOD = (7x)^\circ$.

$\angle AOC + \angle AOD = 180^\circ \Rightarrow 5x + 7x = 180 \Rightarrow 12x = 180 \Rightarrow x = 15.$

$\therefore \quad \angle AOC = (5 \times 15)^\circ = 75^\circ$ and $\angle AOD = (7 \times 15)^\circ = 105^\circ.$

$\angle BOD = \angle AOC = 75^\circ$ [vertically opposite $\measuredangle$s].

$\angle BOC = \angle AOD = 105^\circ$ [vertically opposite $\measuredangle$s].

13. $\angle AOC = \angle BOD = 40^\circ$ [vertically opposite $\measuredangle$s].

$\angle BOF = \angle AOE = 35^\circ$ [vertically opposite $\measuredangle$s].

14. $y = \angle BOC = 125^\circ$ [vertically opposite $\measuredangle$s].

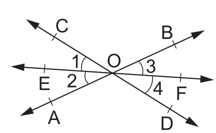
$y + z = 180^\circ \Rightarrow 125^\circ + z = 180^\circ \Rightarrow z = 55^\circ = x$ [vertically opposite $\measuredangle$s].

15. Let the ray OE bisects $\angle AOC$ and ray OF be opposite to OE.

Then, EOF is a straight line.

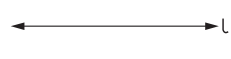
$\therefore \quad \angle 1 = \angle 4$ and $\angle 2 = \angle 3$. [vert. opp. $\measuredangle$s]

But, $\angle 1 = \angle 2 \Rightarrow \angle 4 = \angle 3.$



RESULTS ON PARALLEL LINES

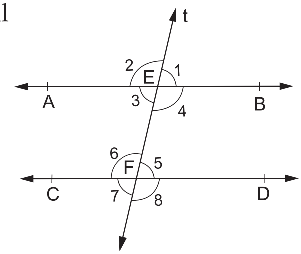
PARALLEL LINES *If two lines lie in the same plane and do not intersect when produced on either side then such lines are said to be parallel to each other.*



TRANSVERSAL *A straight line which cuts two or more straight lines at distinct points is called a transversal.*

THE ANGLES FORMED WHEN A TRANSVERSAL CUTS TWO LINES

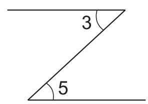
Let AB and CD be two lines, cut by a transversal t. Then, the following angles are formed.

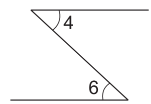


(i) Pairs of corresponding angles:

$(\angle 1 = \angle 5)$; $(\angle 4 = \angle 8)$; $(\angle 2 = \angle 6)$ and $(\angle 3 = \angle 7)$.

(ii) Pairs of alternate interior angles: $(\angle 3, \angle 5)$ and $(\angle 4 = \angle 6)$.





(iii) Pairs of consecutive interior angles (allied angles or conjoined angles): $(\angle 4 = \angle 5)$ and $(\angle 3, \angle 6)$.

REMARKS We shall abbreviate as follows:

(i) Corresponding angles as corres. $\angle$s.

(ii) Alternate interior angles as alt. int. $\angle$s.

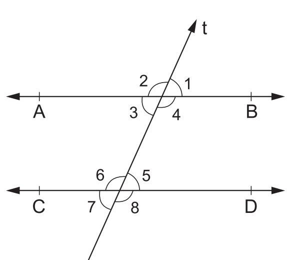
(iii) Consecutive interior angles as co. int. $\angle$s.

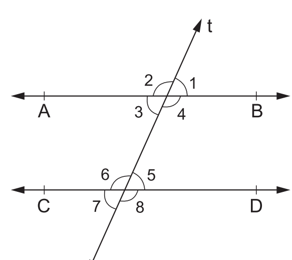
CORRESPONDING ANGLES AXIOM

If a transversal cuts two parallel lines then each pair of corresponding angles are equal.

Conversely, if a transversal cuts two lines, making a pair of corresponding angles equal, then the lines are parallel.

Thus, whenever $AB \parallel CD$ are cut by a transversal t, then $\angle 1 = \angle 5$; $\angle 4 = \angle 8$; $\angle 2 = \angle 6$ and $\angle 3 = \angle 7$.



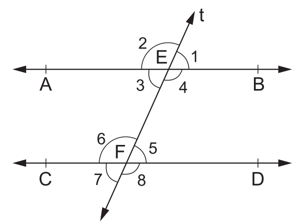


On the other hand, if a transversal t cuts two lines AB and CD such that ($\angle 1 = \angle 5$) or ($\angle 4 = \angle 8$) or ($\angle 2 = \angle 6$) or ($\angle 3 = \angle 7$) then $AB \parallel CD$.

THEOREM 1 *If a transversal intersects two parallel lines then alternate angles of each pair of interior angles are equal.*

GIVEN $AB \parallel CD$ and a transversal t cuts AB at E and CD at F, forming two pairs of alternate interior angles, namely ($\angle 3, \angle 5$) and ($\angle 4 = \angle 6$).

TO PROVE $\angle 3 = \angle 5$ and $\angle 4 = \angle 6$.



PROOF We have:

$\angle 3 = \angle 1$ (vert. opp. $\angle$s)

and $\angle 1 = \angle 5$ (corres. $\angle$s).

$\therefore$ $\angle 3 = \angle 5$.

Again, $\angle 4 = \angle 2$ (vert. opp. $\angle$s)

and $\angle 2 = \angle 6$ (corres. $\angle$s).

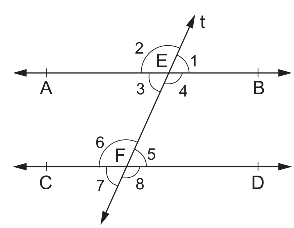
$\therefore$ $\angle 4 = \angle 6$.

Hence, $\angle 3 = \angle 5$ and $\angle 4 = \angle 6$.

THEOREM 2 *If a transversal intersects two parallel lines then each pair of consecutive interior angles are supplementary.*

GIVEN $AB \parallel CD$ and a transversal t cuts AB at E and CD at F, forming two pairs of consecutive interior angles, namely ($\angle 3, \angle 6$) and ($\angle 4, \angle 5$).

TO PROVE $\angle 3 + \angle 6 = 180°$ and $\angle 4 + \angle 5 = 180°$.



PROOF Since ray EF stands on line AB, we have: $\angle 3 + \angle 4 = 180°$ (linear pair).

But, $\angle 4 = \angle 6$ (alt. int. $\angle$s)

$\therefore$ $\angle 3 + \angle 6 = 180°$.

Again, since ray FE stands on line CD, we have: $\angle 6 + \angle 5 = 180°$.

But, $\angle 6 = \angle 4$ (alt. int. $\angle$s)

$\angle 4 + \angle 5 = 180°$.

Hence, $\angle 3 + \angle 6 = 180°$ and $\angle 4 + \angle 5 = 180°$.

THEOREM 3 (Converse of Theorem 1) *If a transversal intersects two lines, making a pair of alternate interior angles equal, then the two lines are parallel.*

GIVEN A transversal t cuts two lines AB and CD at E and F respectively such that $\angle 3 = \angle 5$.

TO PROVE $AB \parallel CD$.

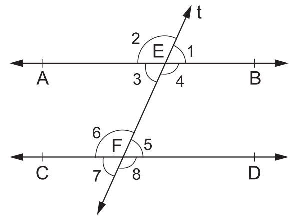
PROOF We have: $\angle 3 = \angle 5$ (given).

But, $\angle 3 = \angle 1$ (vert. opp. ∠s)

$\therefore$ $\angle 1 = \angle 5$.

But, these are corresponding angles.

$\therefore$ $AB \parallel CD$ (by corres. ∠s axiom).



THEOREM 4 (Converse of Theorem 2) *If a transversal intersects two lines in such a way that a pair of consecutive interior angles are supplementary then the two lines are parallel.*

GIVEN A transversal cuts two lines AB and CD at E and F respectively such that $\angle 4 + \angle 5 = 180°$.

TO PROVE $AB \parallel CD$.

PROOF Since ray EB stands on line t, we have:

$\angle 1 + \angle 4 = 180°$ (linear pair)

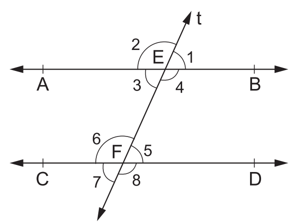
and $\angle 4 + \angle 5 = 180°$ (given).

$\therefore$ $\angle 1 + \angle 4 = \angle 4 + \angle 5$.

This gives, $\angle 1 = \angle 5$.

But, these are corresponding angles.

$\therefore$ $AB \parallel CD$ (by corres. ∠s axiom).



SOLVED EXAMPLES

EXAMPLE 1 *Prove that the two lines which are both parallel to the same line are parallel to one another.*

GIVEN Three lines l, m, n such that $l \parallel n$ and $m \parallel n$.

TO PROVE $l \parallel m$.

CONSTRUCTION Draw a transversal t, cutting l, m, n at E, F, G respectively.

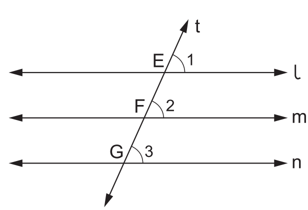
PROOF Since $l \parallel n$ and transversal t cuts them at E and G respectively, so

$\angle 1 = \angle 3$ (corres. ∠s).

Again $m \parallel n$ and transversal t cuts them at F and G respectively, so

$\angle 2 = \angle 3$ (corres. ∠s).

$\therefore$ $\angle 1 = \angle 2$ (each equal to $\angle 3$).



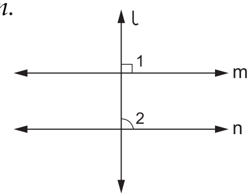
But these are corresponding angles formed when the transversal t cuts l and m at E and F respectively.

$\therefore$ $l \parallel m$ (by corres. ∠s axiom).

EXAMPLE 2 *If a line is perpendicular to one of the two given parallel lines then prove that it is also perpendicular to the other line.*

GIVEN l, m, n are three lines such that $m \parallel n$ and $l \perp m$.

TO PROVE $l \perp n$.



PROOF Since $l \perp m$, we have: $\angle 1 = 90°$.

Now, $m \parallel n$ and the transversal l cuts them.

$\therefore \quad \angle 2 = \angle 1 \quad$ (corres. $\angle$s)

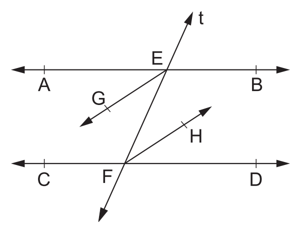
$\therefore \quad \angle 2 = 90° \quad [\because \ \angle 1 = 90°]$.

Hence, $l \perp n$.

EXAMPLE 3 *If two parallel lines are intersected by a transversal then prove that the bisectors of any pair of alternate interior angles are parallel.*

GIVEN $AB \parallel CD$ are cut by a transversal t at E and F respectively; EG and FH are the bisectors of a pair of alt. int. $\angle$s, $\angle AEF$ and $\angle EFD$ respectively.

TO PROVE $EG \parallel FH$.



PROOF Since $AB \parallel CD$ and t is a transversal, we have:

$\angle AEF = \angle EFD \quad$ [alt. int. $\angle$s]

$\Rightarrow \quad \frac{1}{2}\angle AEF = \frac{1}{2}\angle EFD$

$\Rightarrow \quad \angle GEF = \angle EFH$.

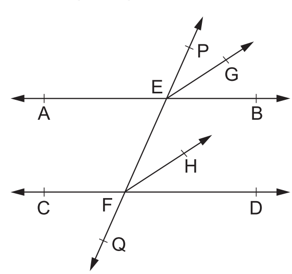
But, these are alternate interior angles formed when the transversal EF cuts EG and FH.

$\therefore \quad EG \parallel FH$.

EXAMPLE 4 *If the bisectors of a pair of corresponding angles formed by a transversal with two given lines are parallel, prove that the given lines are parallel.*

GIVEN A transversal PQ cuts two lines AB and CD at E and F respectively. EG and FH are the bisectors of a pair of corresponding angles $\angle PEB$ and $\angle EFD$ respectively such that $EG \parallel FH$.

TO PROVE $AB \parallel CD$.



PROOF $EG \parallel FH$ are cut by the transversal EF.

$\therefore \quad \angle PEG = \angle EFH \quad$ (corres. $\angle$s)

$\Rightarrow \quad \angle GEB = \angle HFD$

$\Rightarrow \quad 2\angle GEB = 2\angle HFD$

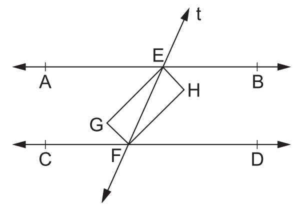
$\Rightarrow \quad \angle PEB = \angle EFD$

$\left[\because \ \angle GEB = \frac{1}{2}\angle PEB \text{ and } \angle HFD = \frac{1}{2}\angle EFD\right]$.

But, these are corresponding angles when AB and CD are cut by the transversal PQ.

$\therefore \quad AB \parallel CD$ (by corres. ∡s axiom).

EXAMPLE 5 *If two parallel lines are intersected by a transversal, prove that the bisectors of the two pairs of interior angles enclose a rectangle.*



GIVEN $AB \parallel CD$ and a transversal t cuts them at E and F respectively. EG, FG, EH and FH are the bisectors of the interior angles $\angle AEF$, $\angle CFE$, $\angle BEF$ and $\angle EFD$ respectively.

TO PROVE $EGFH$ is a rectangle.

PROOF $AB \parallel CD$ and the transversal t cuts them at E and F respectively.

$\therefore \quad \angle AEF = \angle EFD$ [alt. int. ∡s]

$\Rightarrow \quad \frac{1}{2}\angle AEF = \frac{1}{2}\angle EFD \Rightarrow \angle GEF = \angle EFH.$

But, these are alternate interior angles formed when the transversal EF cuts EG and FH.

$\therefore \quad EG \parallel FH$. Similarly, $EH \parallel FG$.

$\therefore \quad EGFH$ is a parallelogram.

Now, ray EF stands on line AB.

$\therefore \quad \angle AEF + \angle BEF = 180°$ (linear pair)

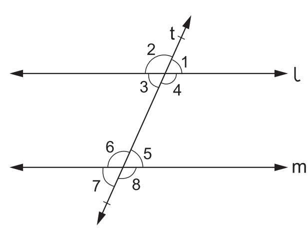
$\Rightarrow \quad \frac{1}{2}\angle AEF + \frac{1}{2}\angle BEF = 90° \Rightarrow \angle GEF + \angle HEF = 90°$

$\Rightarrow \quad \angle GEH = 90°$ [$\because \angle GEF + \angle HEF = \angle GEH$].

Thus, $EGFH$ is a parallelogram, one of whose angles is 90°.

Hence, $EGFH$ is a rectangle.

EXAMPLE 6 *In the given figure, $l \parallel m$ and a transversal t cuts them. If $\angle 1 = 70°$, find the measure of each of the remaining marked angles.*



SOLUTION Clearly, a ray t stands on line l making adjacent angles $\angle 1$ and $\angle 2$.

$\therefore \quad \angle 1 + \angle 2 = 180° \Rightarrow 70° + \angle 2 = 180°$

$\Rightarrow \angle 2 = (180° - 70°) = 110°.$

$\therefore \quad \angle 4 = \angle 2 = 110°$ [vertically opposite ∡s]

and $\angle 3 = \angle 1 = 70°$ [vertically opposite ∡s].

Now, $l \parallel m$ and t is the transversal.

$\therefore \quad \angle 5 = \angle 3 = 70°$ [alternate interior $\angle$s],

$\angle 6 = \angle 4 = 110°$ [alternate interior $\angle$s],

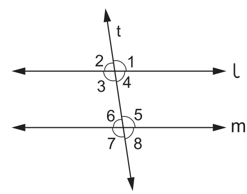
$\angle 7 = \angle 3 = 70°$ [corresponding $\angle$s]

and $\angle 8 = \angle 4 = 110°$ [corresponding $\angle$s].

$\therefore \quad \angle 2 = 110°, \angle 3 = 70°, \angle 4 = 110°, \angle 5 = 70°, \angle 6 = 110°,$

$\angle 7 = 70°$ and $\angle 8 = 110°$.

EXAMPLE 7 *In the given figure, $l \parallel m$ and a transversal t cuts them. If $\angle 1 : \angle 2 = 5 : 4$, find the measure of each of the marked angles.*



SOLUTION Let $\angle 1 = (5x)°$ and $\angle 2 = (4x)°$.

Clearly, the ray t stands on line l.

$\therefore \quad \angle 1 + \angle 2 = 180° \Rightarrow 5x + 4x = 180$

$\Rightarrow 9x = 180 \Rightarrow x = 20.$

$\therefore \quad \angle 1 = (5 \times 20)° = 100°$ and $\angle 2 = (4 \times 20)° = 80°$.

Now, $\angle 3 = \angle 1 = 100°$ [vertically opposite $\angle$s]

$\angle 4 = \angle 2 = 80°$ [vertically opposite $\angle$s].

Now, $l \parallel m$ and t is the transversal.

$\therefore \quad \angle 5 = \angle 3 = 100°$ [alternate interior $\angle$s]

$\angle 6 = \angle 4 = 80°$ [alternate interior $\angle$s]

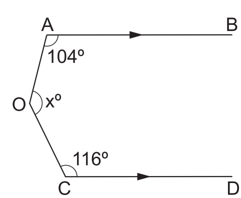
$\angle 7 = \angle 3 = 100°$ [corresponding $\angle$s]

$\angle 8 = \angle 4 = 80°$ [corresponding $\angle$s]

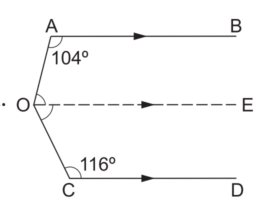
$\therefore \quad \angle 1 = 100°, \angle 2 = 80°, \angle 3 = 100°, \angle 4 = 80°$

$\angle 5 = 100°, \angle 6 = 80°, \angle 7 = 100°, \angle 8 = 80°$.

EXAMPLE 8 *In the given figure, $AB \parallel CD$ and $\angle AOC = x°$. If $\angle OAB = 104°$ and $\angle OCD = 116°$, find the value of x.*



SOLUTION Through O draw $OE \parallel AB \parallel CD$.



Then, $\angle AOE + \angle COE = x°$.

Now, $AB \parallel OE$ and AO is the transversal.

$\therefore \quad \angle OAB + \angle AOE = 180°$

$\Rightarrow \quad 104° + \angle AOE = 180°$

$\Rightarrow \quad \angle AOE = (180 - 104)^\circ = 76^\circ.$

Again, $CD \parallel OE$ and OC is the transversal.

$\therefore \quad \angle COE + \angle OCD = 180^\circ$

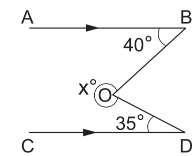
$\Rightarrow \quad \angle COE + 116^\circ = 180^\circ$

$\Rightarrow \quad \angle COE = (180^\circ - 116^\circ) = 64^\circ.$

$\therefore \quad \angle AOC = \angle AOE + \angle COE = (76^\circ + 64^\circ) = 140^\circ.$

Hence, $x = 140$.

EXAMPLE 9 *In the given figure, $AB \parallel CD$, $\angle ABO = 40^\circ$, $\angle CDO = 35^\circ$. Find the value of the reflex $\angle BOD$ and hence the value of x.*



SOLUTION Through O, draw $EO \parallel AB \parallel CD$.

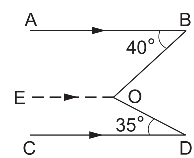
Then, $\angle EOB + \angle EOD = x^\circ$.

Now, $AB \parallel EO$ and BO is the transversal.

$\therefore \quad \angle ABO + \angle BOE = 180^\circ \quad$ [consecutive int. ∠s]

$\Rightarrow \quad 40^\circ + \angle BOE = 180^\circ$

$\Rightarrow \quad \angle BOE = (180^\circ - 40^\circ) = 140^\circ.$



Again, $CD \parallel EO$ and OD is the transversal.

$\therefore \quad \angle EOD + \angle ODC = 180^\circ$

$\Rightarrow \quad \angle EOD + 35^\circ = 180^\circ$

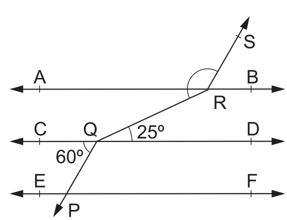
$\Rightarrow \quad \angle EOD = (180^\circ - 35^\circ) = 145^\circ.$

$\therefore \quad \text{reflex } \angle BOD = x^\circ = (\angle BOE + \angle EOD)$

$\qquad = (140^\circ + 145^\circ) = 285^\circ.$

Hence, $x = 285$.

EXAMPLE 10 *In the given figure, $AB \parallel CD \parallel EF$, $PQ \parallel RS$, $\angle RQD = 25^\circ$ and $\angle CQP = 60^\circ$. Find $\angle QRS$.*



SOLUTION CD is a line and the ray QP stands on it.

$\therefore \quad \angle CQP + \angle PQD = 180^\circ$

$\Rightarrow \quad 60^\circ + \angle PQD = 180^\circ$

$\Rightarrow \quad \angle PQD = (180^\circ - 60^\circ) = 120^\circ.$

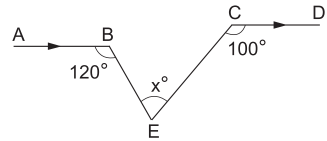
$\therefore \quad \angle PQR = \angle PQD + \angle RQD = (120^\circ + 25^\circ) = 145^\circ.$

Now, $PQ \parallel RS$ and QR is the transversal.

$\therefore \quad \angle QRS = \angle PQR = 145°$ [alternate interior $\angle$s].

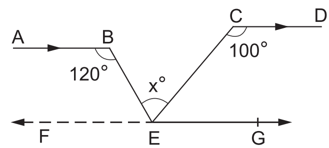
Hence, $\angle QRS = 145°$.

EXAMPLE 11 *In the given figure, $AB \parallel CD$, $\angle ABE = 120°$, $\angle ECD = 100°$ and $\angle BEC = x°$. Find the value of x.*



SOLUTION Through E, draw $FEG \parallel AB \parallel CD$.

Now, $FE \parallel AB$ and BE is the transversal.



$\therefore \quad \angle BEF + \angle ABE = 180°$ [co-interior $\angle$s]

$\Rightarrow \quad \angle BEF + 120° = 180°$

$\Rightarrow \quad \angle BEF = (180° - 120°) = 60°$.

Again, $CD \parallel EG$ and CE is the transversal.

$\therefore \quad \angle CEG + \angle ECD = 180°$ [co-interior $\angle$s]

$\Rightarrow \quad \angle CEG + 100° = 180°$

$\Rightarrow \quad \angle CEG = (180° - 100°) = 80°$.

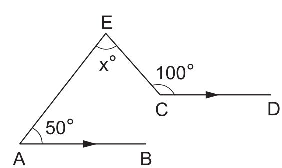
Now, FEG is a straight line.

$\therefore \quad \angle BEF + \angle BEC + \angle CEG = 180°$

$\Rightarrow \quad 60° + x° + 80° = 180° \Rightarrow x + 140 = 180 \Rightarrow x = 40$.

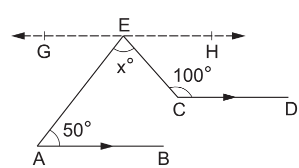
Hence, $x = 40$.

EXAMPLE 12 *In the given figure, $AB \parallel CD$, $\angle BAE = 50°$, $\angle AEC = x°$ and $\angle ECD = 100°$. Find the value of x.*



SOLUTION Through E, draw a line $GEH \parallel AB \parallel CD$.

Now, $GE \parallel AB$ and EA is the transversal.



$\therefore \quad \angle GEA = \angle EAB = 50°$ [alt. int. $\angle$s].

Again, $EH \parallel CD$ and EC is the transversal.

$\therefore \quad \angle HEC + \angle ECD = 180°$ [co. int. $\angle$s]

$\Rightarrow \quad \angle HEC + 100° = 180° \Rightarrow \angle HEC = 80°$.

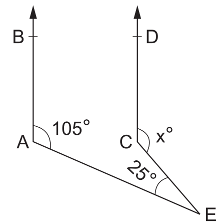
Now, GEH is a straight line.

$\therefore \quad \angle GEA + \angle AEC + \angle HEC = 180°$ [straight angle]

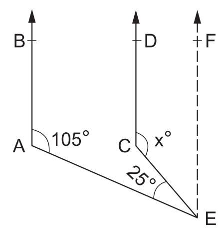
$\Rightarrow \quad 50° + x° + 80° = 180° \Rightarrow x° = 50°$.

Hence, $x = 50$.

EXAMPLE 13 *In the given figure, $AB \parallel CD$, $\angle EAB = 105°$, $\angle AEC = 25°$ and $\angle ECD = x°$. Find the value of x.*



SOLUTION From E, draw $EF \parallel AB \parallel CD$.



Now, $EF \parallel CD$ and CE is the transversal.

$\therefore \quad \angle DCE + \angle CEF = 180°$ [co. int. ∠s]

$\Rightarrow \quad x° + \angle CEF = 180°$

$\Rightarrow \quad \angle CEF = (180° - x°)$.

Again, $EF \parallel AB$ and AE is the transversal.

$\therefore \quad \angle BAE + \angle AEF = 180°$ [co. int. ∠s]

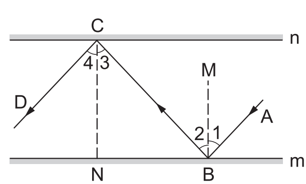
$\Rightarrow \quad 105° + \angle AEC + \angle CEF = 180°$

$\Rightarrow \quad 105° + 25° + (180° - x°) = 180°$

$\Rightarrow \quad x° = 130°$.

Hence, $x = 130$.

EXAMPLE 14 *Two plane mirrors, m and n, are placed parallel to each other as shown in the figure. A ray AB is incident on the first mirror. It is reflected twice and emerges in the direction CD. Prove that $AB \parallel CD$.*



GIVEN Two mirrors m and n such that $m \parallel n$. The paths taken by the incident ray AB after reflections at m and n are BC and CD respectively. Also, BM and CN are normals to m and n respectively.

TO PROVE $AB \parallel CD$.

PROOF Since $m \parallel n$ and $CN \perp n$, therefore, $CN \perp m$.

Now, $BM \perp m$ and $CN \perp m \Rightarrow BM \parallel CN$.

Now, $BM \parallel CN$ and BC is a transversal.

$\therefore \quad \angle 2 = \angle 3$ [alt. int. ∠s].

But, $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$ [by laws of reflection]

$\Rightarrow \quad \angle 1 + \angle 2 = 2(\angle 2)$ and $\angle 3 + \angle 4 = 2(\angle 3)$

$\Rightarrow \quad \angle 1 + \angle 2 = \angle 3 + \angle 4 \quad [\because \angle 2 = \angle 3 \Rightarrow 2(\angle 2) = 2(\angle 3)]$

$\Rightarrow \quad \angle ABC = \angle BCD.$

But, these are alternate interior angles formed when lines AB and CD are cut by the transversal BC.

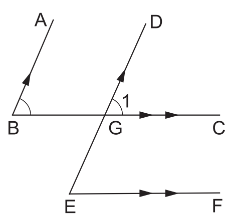
$\therefore \quad AB \parallel CD.$

EXAMPLE 15 *If the arms of one angle are respectively parallel to the arms of another angle, show that the two angles are either equal or supplementary.*

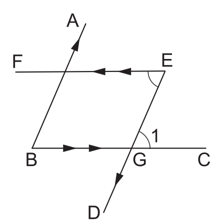
GIVEN Two angles $\angle ABC$ and $\angle DEF$ such that $BA \parallel ED$ and $BC \parallel EF$.

TO PROVE $\angle ABC = \angle DEF$ or $\angle ABC + \angle DEF = 180°$.

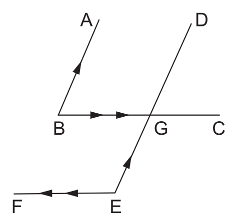
PROOF The arms of the angles may be parallel in the same sense or in the opposite sense. So, three cases arise.



(i)



(ii)



(iii)

Case I *When both pairs of arms are parallel in same sense [Fig. (i)]:*

In this case, $BA \parallel ED$ and BC is the transversal.

$\therefore \quad \angle ABC = \angle 1$ (corres. $\angle$s).

Again, $BC \parallel EF$ and DE is the transversal.

$\therefore \quad \angle 1 = \angle DEF$ (corres. $\angle$s).

Hence, $\angle ABC = \angle DEF$.

Case II *When both pairs of arms are parallel in opposite sense [Fig. (ii)]:*

In this case, $BA \parallel ED$ and BC is the transversal.

$\therefore \quad \angle ABC = \angle 1$ (corres. $\angle$s).

Again, $FE \parallel BC$ and ED is the transversal.

$\therefore \quad \angle DEF = \angle 1$ (alt. int. $\angle$s).

Hence, $\angle ABC = \angle DEF$.

Case III *When one pair of arms are parallel in same sense and other pair parallel in opposite sense: [Fig. (iii)]*

In this case, $BA \parallel ED$ and BC is the transversal.

$\therefore \quad \angle EGB = \angle ABC$ [Alt. Int. $\angle$s].

Now, $BC \parallel EF$ and DE is the transversal.

$\therefore \quad \angle DEF + \angle EGB = 180^\circ \quad$ [Co. Int. $\angle$s]

$\Rightarrow \quad \angle DEF + \angle ABC = 180^\circ \quad [\because \angle EGB = \angle ABC]$.

Hence, $\angle ABC$ and $\angle DEF$ are supplementary.

EXAMPLE 16 *Prove that the opposite angles of a parallelogram are equal.*

GIVEN A ∥gm $ABCD$ in which $AB \parallel DC$ and $AD \parallel BC$.

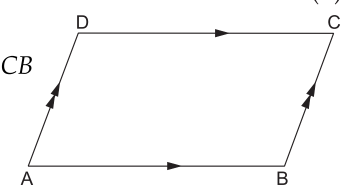
TO PROVE $\angle A = \angle C$ and $\angle B = \angle D$.

PROOF $AB \parallel DC$ and AD is a transversal,

$\therefore \quad \angle BAD + \angle ADC = 180^\circ. \quad \ldots$ (i)

Again, $AD \parallel BC$ and DC is a transversal.

$\therefore \quad \angle ADC + \angle DCB = 180^\circ. \quad \ldots$ (ii)



From (i) and (ii), we get

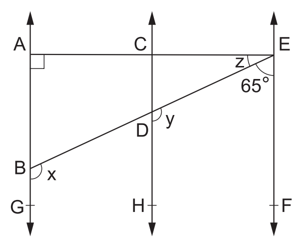
$$\angle BAD + \angle ADC = \angle ADC + \angle DCB$$

$\Rightarrow \quad \angle BAD = \angle DCB \;\Rightarrow\; \angle A = \angle C$.

Similarly, $\angle B = \angle D$.

Hence, $\angle A = \angle C$ and $\angle B = \angle D$.

EXAMPLE 17 *In the given figure, $AB \parallel CD \parallel EF$, $\angle DBG = x$, $\angle EDH = y$, $\angle AEB = z$, $\angle EAB = 90^\circ$ and $\angle BEF = 65^\circ$. Find the values of x, y and z.*



SOLUTION $EF \parallel CD$ and ED is the transversal.

$\therefore \quad \angle FED + \angle EDH = 180^\circ \quad$ [co-interior $\angle$s]

$\Rightarrow \quad 65^\circ + y = 180^\circ \;\Rightarrow\; y = (180^\circ - 65^\circ) = 115^\circ$.

Now, $CH \parallel AG$ and DB is the transversal.

$\therefore \quad x = y = 115^\circ \quad$ [corresponding $\angle$s].

Now, ABG is a straight line.

$\therefore \quad \angle ABE + \angle EBG = 180^\circ \quad$ [linear pair]

$\Rightarrow \quad \angle ABE + x = 180^\circ$

$\Rightarrow \quad \angle ABE + 115^\circ = 180^\circ$

$\Rightarrow \quad \angle ABE = (180^\circ - 115^\circ) = 65^\circ$.

We know that the sum of the angles of a triangle is 180°.

From $\triangle EAB$, we get

$$\angle EAB + \angle ABE + \angle BEA = 180^\circ$$

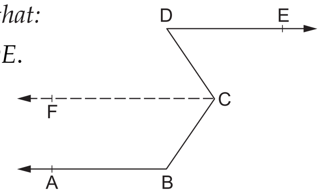
$\Rightarrow \quad 90° + 65° + z = 180°$

$\Rightarrow \quad z = (180° - 155°) = 25°.$

$\therefore \quad x = 115°, y = 115° \text{ and } z = 25°.$

EXAMPLE 18 *In the given figure, AB ∥ DE. Prove that:*

$\angle ABC + \angle BCD = 180° + \angle CDE.$



CONSTRUCTION Draw $CF \parallel BA$.

PROOF $CF \parallel BA$ and CB is the transversal.

$\therefore \quad \angle ABC + \angle BCF = 180° \quad \ldots \text{(i)} \quad$ [co-interior ∠s].

Again, $FC \parallel DE$ and DC is the transversal.

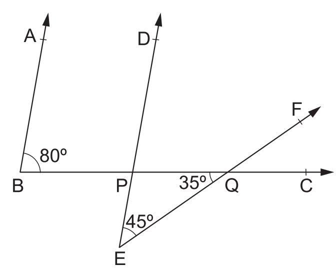
$\therefore \quad \angle FCD = \angle CDE \quad \ldots \text{(ii)} \quad$ [alternate interior ∠s].

Adding the corresponding sides of (i) and (ii), we get

$\angle ABC + (\angle BCF + \angle FCD) = 180° + \angle CDE.$

$\Rightarrow \quad \angle ABC + \angle BCD = 180° + \angle CDE.$

EXAMPLE 19 *In the given figure,* $\angle ABC = 80°$ *and* $\angle DEF = 45°$. *The arms DE and EF of* $\angle DEF$ *cut BC at P and Q respectively. Prove that* $PD \parallel BA$.



SOLUTION Let $\angle EPQ = x°$.

We know that the sum of the angles of a triangle is 180°.

$\therefore \quad \angle EPQ + \angle PEQ + \angle EQP = 180°$

$\Rightarrow \quad \angle EPQ + 45° + 35° = 180°$

$\Rightarrow \quad \angle EPQ = (180° - 80°) = 100°.$

Now, EPD is a straight line.

$\therefore \quad \angle EPQ + \angle DPQ = 180° \Rightarrow 100° + \angle DPQ = 180°$

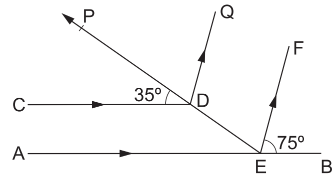
$\Rightarrow \angle DPQ = (180° - 100°) = 80°.$

Thus, $\angle ABP = \angle DPQ$ [each equal to 80°].

But, these are corresponding angles.

Hence, $PD \parallel BA$.

EXAMPLE 20 *In the given figure, $AB \parallel CD$ and $\angle CDP = 35°$. PD is produced downwards to meet AB at E and $\angle BEF = 75°$. If $DQ \parallel EF$, find $\angle AED$, $\angle DEF$ and $\angle PDQ$.*



SOLUTION $AB \parallel CD$ and PDE is the transversal.

$\therefore \quad \angle AED = \angle CDP = 35°$ [corresponding ∠s].

Now, AB is a straight line and ray PDE stands on it.

$\therefore \quad \angle AED + \angle DEF + \angle BEF = 180°$

$\Rightarrow \quad 35° + \angle DEF + 75° = 180°$

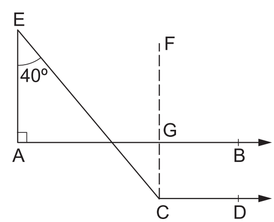
$\Rightarrow \quad \angle DEF = (180° - 110°) = 70°$.

Now, $DQ \parallel EF$ and PDE is the transversal.

$\therefore \quad \angle PDQ = \angle DEF = 70°$ [corresponding ∠s].

Thus, $\angle AED = 35°$, $\angle DEF = 70°$ and $\angle PDQ = 70°$.

EXAMPLE 21 *In the given figure, $AB \parallel CD$, $\angle A = 90°$ and $\angle AEC = 40°$. Find $\angle ECD$.*



SOLUTION Through C draw $CF \parallel AE$, cutting AB at G.

Now, $CF \parallel AE$ and EC is the transversal.

$\therefore \quad \angle ECG = \angle AEC = 40°$ [alternate int. ∠s].

Now, $CF \parallel AE$ and AGB is a transversal.

$\therefore \quad \angle FGB = \angle EAG = 90°$ [corresponding ∠s]

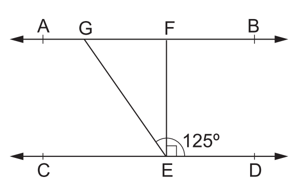
Again, $AB \parallel CD$ and GC is the transversal.

$\therefore \quad \angle GCD = \angle FGB = 90°$ [corresponding ∠s]

$\therefore \quad \angle ECD = \angle ECG + \angle GCD = (40° + 90°) = 130°$.

Hence, $\angle ECD = 130°$.

EXAMPLE 22 *In the given figure, $AB \parallel CD$, $\angle GED = 125°$ and $EF \perp CD$. Find $\angle AGE$, $\angle GEF$ and $\angle FGE$.*



SOLUTION $AB \parallel CD$ and GE is the transversal.

$\therefore \quad \angle AGE = \angle GED = 125°$

[alternate interior ∠s]

and $\angle GEF = (\angle GED - \angle FED) = (125° - 90°) = 35°$.

Now, $AB \parallel CD$ and EF is the transversal.

$\therefore \quad \angle BFE + \angle FED = 180°$ [sum of int. $\angle$s]

$\Rightarrow \quad \angle BFE + 90° = 180° \Rightarrow \angle BFE = 90°.$

Now, AB is a straight line and EF stands on it.

$\therefore \quad \angle GFE + \angle BFE = 180° \Rightarrow \angle GFE + 90° = 180°$

$\Rightarrow \angle GFE = (180° - 90°) = 90°.$

We know that the sum of the angles of a triangle is 180°.

In $\triangle GEF$, we have

$\angle GEF + \angle EFG + \angle FGE = 180°$

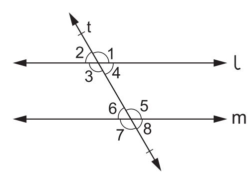
$\Rightarrow \quad 35° + 90° + \angle FGE = 180°$

$\Rightarrow \quad \angle FGE = (180° - 125°) = 55°.$

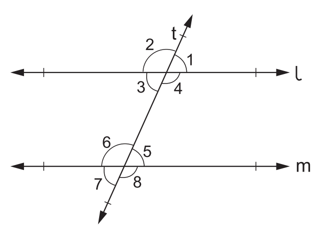
Hence, $\angle AGE = 125°$, $\angle GEF = 35°$ and $\angle FGE = 55°$.

EXERCISE 7C

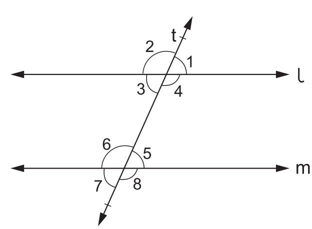
1. In the given figure, $l \parallel m$ and a transversal t cuts them. If $\angle 1 = 120°$, find the measure of each of the remaining marked angles.



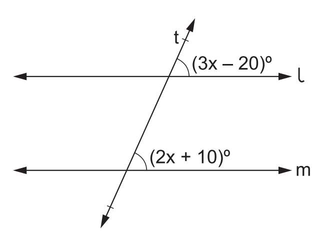
2. In the given figure, $l \parallel m$ and a transversal t cuts them. If $\angle 7 = 80°$, find the measure of each of the remaining marked angles.



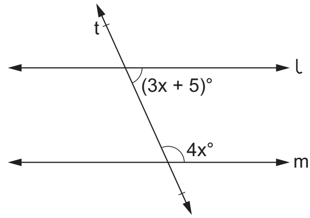
3. In the given figure, $l \parallel m$ and a transversal t cuts them. If $\angle 1 : \angle 2 = 2 : 3$, find the measure of each of the marked angles.



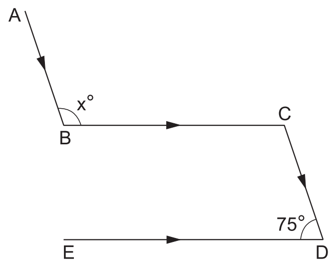
4. For what value of x will the lines l and m be parallel to each other?



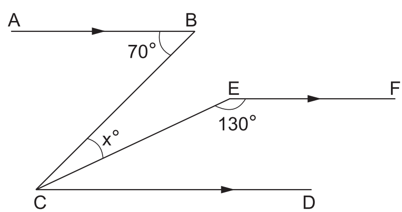
5. For what value of x will the lines l and m be parallel to each other?



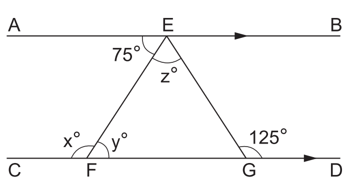
6. In the given figure, $AB \parallel CD$ and $BC \parallel ED$. Find the value of x.



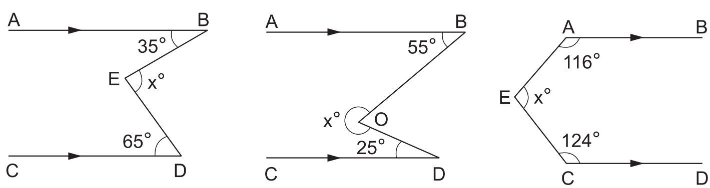
7. In the given figure, $AB \parallel CD \parallel EF$. Find the value of x.



8. In the given figure, $AB \parallel CD$. Find the values of x, y and z.

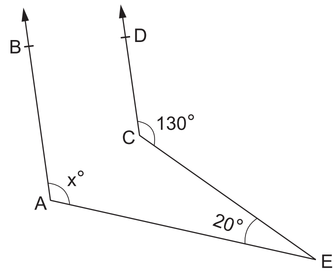


9. In each of the figures given below, $AB \parallel CD$. Find the value of x in each case.

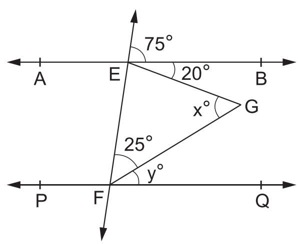


(i) (ii) (iii)

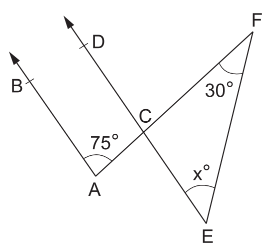
10. In the given figure, $AB \parallel CD$. Find the value of x.



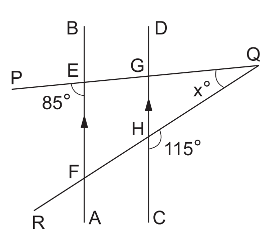
11. In the given figure, $AB \parallel PQ$. Find the values of x and y.



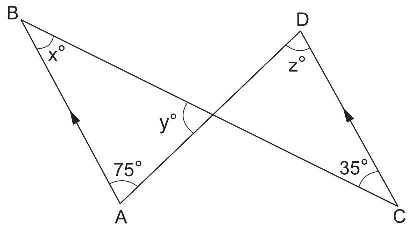
12. In the given figure, $AB \parallel CD$. Find the value of x.



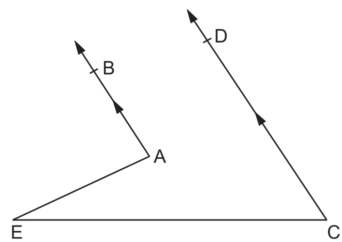
13. In the given figure, $AB \parallel CD$. Find the value of x.



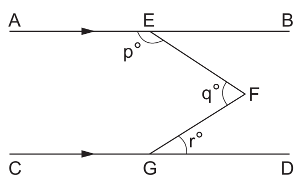
14. In the given figure, $AB \parallel CD$. Find the value of x, y and z.



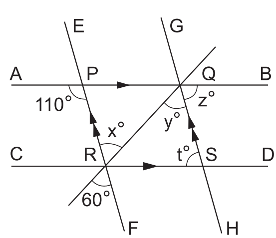
15. In the given figure, $AB \parallel CD$. Prove that $\angle BAE - \angle ECD = \angle AEC$.



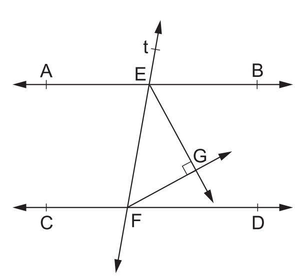
16. In the given figure, $AB \parallel CD$. Prove that $p + q - r = 180$.



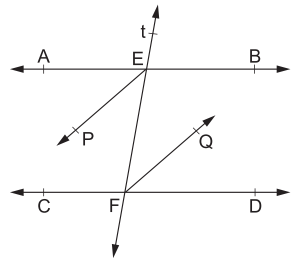
17. In the given figure, $AB \parallel CD$ and $EF \parallel GH$. Find the values of x, y, z and t.



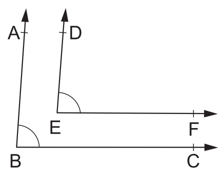
18. In the given figure, $AB \parallel CD$ and a transversal t cuts them at E and F respectively. If EG and FG are the bisectors of $\angle BEF$ and $\angle EFD$ respectively, prove that $\angle EGF = 90°$.



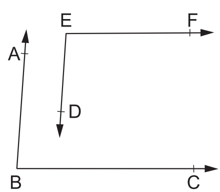
19. In the given figure, $AB \parallel CD$ and a transversal t cuts them at E and F respectively. If EP and FQ are the bisectors of $\angle AEF$ and $\angle EFD$ respectively, prove that $EP \parallel FQ$.



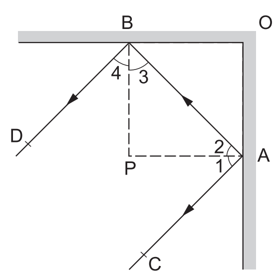
20. In the given figure, $BA \parallel ED$ and $BC \parallel EF$. Show that $\angle ABC = \angle DEF$.



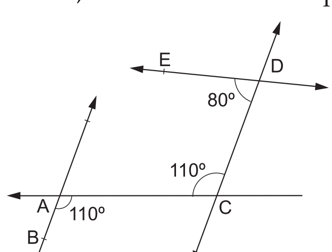
21. In the given figure, $BA \parallel ED$ and $BC \parallel EF$. Show that $\angle ABC + \angle DEF = 180°$.



22. In the given figure, m and n are two plane mirrors perpendicular to each other. Show that the incident ray CA is parallel to the reflected ray BD.



23. In the figure given below, state which lines are parallel and why?



24. Two lines are respectively perpendicular to two parallel lines. Show that they are parallel to each other.

ANSWERS (EXERCISE 7C)

1. $\angle 2 = 60°, \angle 3 = 120°, \angle 4 = 60°, \angle 5 = 120°, \angle 6 = 60°, \angle 7 = 120°, \angle 8 = 60°$
2. $\angle 1 = 80°, \angle 2 = 100°, \angle 3 = 80°, \angle 4 = 100°, \angle 5 = 80°, \angle 6 = 100°, \angle 8 = 100°$
3. $\angle 1 = 72°, \angle 2 = 108°, \angle 3 = 72°, \angle 4 = 108°, \angle 5 = 72°, \angle 6 = 108°,$
$\angle 7 = 72°, \angle 8 = 108°$
4. $x = 30$ **5.** $x = 25$ **6.** $x = 105$ **7.** $x = 20$
8. $x = 105, y = 75, z = 50$ **9.** (i) $x = 100$ (ii) $x = 280$ (iii) $x = 120$
10. $x = 110$ **11.** $x = 70, y = 50$ **12.** $x = 45$ **13.** $x = 20$
14. $x = 35, y = 70, z = 75$ **17.** $x = 60, y = 60, z = 70, t = 70$
23. $AB \parallel CD$, DE not parallel to AC

HINTS TO SOME SELECTED QUESTIONS

2. $\angle 5 = \angle 7 = 80°$ [vert. opp. $\angle$s].
$\angle 1 = \angle 5 = 80°$ [corresponding $\angle$s].
$\therefore \quad \angle 1 = 80°, \angle 2 = 100°$, and so on.

3. Let $\angle 1 = (2x)°$ and $\angle 2 = (3x)°$.
Then, $2x + 3x = 180 \Rightarrow 5x = 180 \Rightarrow x = 36$.
$\therefore \quad \angle 1 = 72°, \angle 2 = 108°$, and so on.

4. Corresponding angles are equal.
$\therefore \quad 3x - 20 = 2x + 10 \Rightarrow x = 30$.

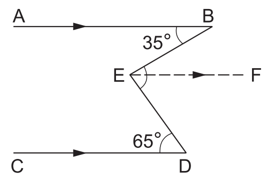
5. Sum of consecutive interior angles is 180°.
$\therefore \quad 3x + 5 + 4x = 180 \Rightarrow 7x = 175 \Rightarrow x = 25$.

6. $\angle BCD + \angle CDE = 180° \Rightarrow \angle BCD + 75° = 180° \Rightarrow \angle BCD = 105°$.
$\angle ABC = \angle BCD$ [alternate interior $\angle$s] $\Rightarrow x° = 105°$.

7. $EF \parallel CD$ and EC is the transversal.
$\therefore \quad \angle ECD + \angle CEF = 180° \Rightarrow \angle ECD + 130° = 180° \Rightarrow \angle ECD = 50°$.
$\angle ABC = \angle BCD$ [alternate int. $\angle$s] $\Rightarrow 70 = x + 50 \Rightarrow x = 20$.

8. $75° = y°$ [alternate int. $\angle$s] $\Rightarrow y = 75$.
$\angle EGF + \angle EGD = 180° \Rightarrow \angle EGF + 125° = 180° \Rightarrow \angle EGF = 55°$.
$z° + y° + 55° = 180° \Rightarrow z + 75 + 55 = 180 \Rightarrow z = 50$.
$x + y = 180° \Rightarrow x + 75 = 180 \Rightarrow x = 105$.
$\therefore \quad x = 105, y = 75$ and $z = 50$.

9. (i) Draw $EF \parallel AB \parallel CD$.

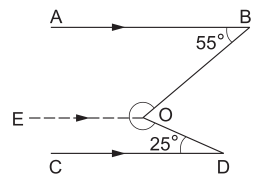


$\angle BEF = \angle ABE = 35^\circ$ [alt. int. $\angle$s].

$\angle DEF = \angle CDE = 65^\circ$ [alt. int. $\angle$s].

$\therefore \quad x^\circ = (\angle BEF + \angle DEF) = (35^\circ + 65^\circ) = 100^\circ$.

(ii) $AB \parallel EO$ and BO is the transversal.



$\therefore \quad \angle ABO + \angle EOB = 180^\circ$

$\Rightarrow \quad 55^\circ + \angle EOB = 180^\circ \Rightarrow \angle EOB = 125^\circ$.

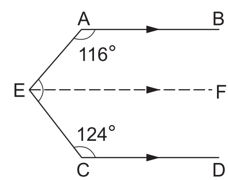
$CD \parallel EO$ and DO is the transversal.

$\therefore \quad \angle CDO + \angle EOD = 180^\circ$

$\Rightarrow \quad 25^\circ + \angle EOD = 180^\circ \Rightarrow \angle EOD = 155^\circ$

$\therefore \quad x^\circ = (125^\circ + 155^\circ) = 280^\circ$.

(iii) Draw $EF \parallel AB \parallel CD$.



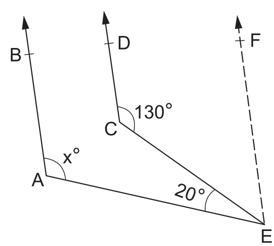
$\angle BAE + \angle AEF = 180^\circ$

$\Rightarrow \quad 116^\circ + \angle AEF = 180^\circ \Rightarrow \angle AEB = 64^\circ$.

$\angle CEF + \angle ECD = 180^\circ \Rightarrow \angle CEB = (180^\circ - 124^\circ) = 56^\circ$.

$\therefore \quad x^\circ = (64^\circ + 56^\circ) = 120^\circ$.

10. Draw $EF \parallel AB \parallel CD$.



$\angle DCB + \angle CEF = 180^\circ$

$\Rightarrow \quad 130^\circ + \angle CEF = 180^\circ \Rightarrow \angle CEF = 50^\circ$.

$AB \parallel EF$ and AE is the transversal.

$\angle BAE + \angle AEF = 180^\circ$

$\Rightarrow \quad x + (20^\circ + 50^\circ) = 180^\circ \Rightarrow x^\circ = 110^\circ$.

11. $75^\circ + 20^\circ + \angle GEF = 180^\circ \Rightarrow \angle GEF = 85^\circ$.

Sum of the angles of a triangle is 180°.

$\therefore \quad 85 + x + 25 = 180 \Rightarrow x = (180 - 110) = 70$.

$25^\circ + y^\circ = 75^\circ$ [corresponding $\angle$s] $\Rightarrow y = 50$.

12. $AB \parallel CD$ and AC cuts them.

$\therefore \quad \angle BAC + \angle ACD = 180^\circ$

$\Rightarrow \quad 75^\circ + \angle ACD = 180^\circ \Rightarrow \angle ACD = 105^\circ$.

$\therefore \quad \angle ECF = \angle ACD = 105^\circ$.

Sum of the angles of a triangle is 180°.

$\therefore \quad 105^\circ + 30^\circ + x^\circ = 180^\circ \Rightarrow x = (180 - 135) = 45$.

13. $\angle QGH = \angle GEF = (180 - 85)^\circ = 95^\circ$ [corresponding $\angle$s].

$\angle CHQ + \angle GHQ = 180^\circ \Rightarrow 115^\circ + \angle GHQ = 180^\circ$

$\Rightarrow \angle GHQ = (180^\circ - 115^\circ) = 65^\circ$.

$\therefore \quad 95 + 65 + x = 180 \Rightarrow x = (180 - 160) = 20$.

14. $AB \parallel CD$ and AC is the transversal.

$\therefore \quad x° = 35°$ [alt. int. ∠s].

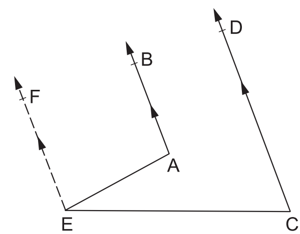
$AB \parallel CD$ and AD is the transversal.

$\therefore \quad z = 75$ [alt. int. ∠s].

$x + y + 75 = 180 \Rightarrow 35 + y + 75 = 180 \Rightarrow y = 70.$

$\therefore \quad x = 35, y = 70$ and $z = 75$.

15. Through E, draw $EF \parallel AB \parallel CD$.



$EF \parallel CD$ and EC is the transversal.

$\therefore \quad \angle CEF + \angle ECD = 180°.$

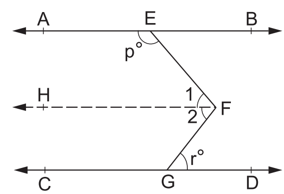
$AB \parallel EF$ and EA is the transversal.

$\therefore \quad \angle BAE + \angle AEF = 180°.$

So, $\angle CEF + \angle ECD = \angle BAE + \angle AEF$

$\Rightarrow \quad \angle BAE - \angle ECD = \angle CEF - \angle AEF = \angle AEC.$

16. Draw $HF \parallel AB \parallel CD$.



$p° + \angle 1 = 180°, (180° - r°) + \angle 2 = 180°.$

$p° + \angle 1 + \angle 2 - r° = 180°$ [adding]

$\therefore \quad p + q - r = 180 \qquad [\because \angle 1 + \angle 2 = q°].$

17. $x = 60$ [vert. opp. ∠s]

$\angle QPR = (180° - 110°) = 70° = \angle BQS = z°$ [corresponding ∠s]

$x° + \angle QRS = 110°$ [alt. int. ∠s] $\Rightarrow 60° + \angle QRS = 110°$

$\Rightarrow \angle QRS = 50°.$

$x° + \angle QRS + t° = 180°$ [consecutive int. ∠s]

$\Rightarrow \quad 110° + t° = 180° \Rightarrow t = 70.$

$y° + \angle QRS + t° = 180° \Rightarrow y° + 50° + 70° = 180° \Rightarrow y = 60.$

18. $\angle BEF + \angle DFE = 180°$

$\Rightarrow \quad \frac{1}{2}\angle BEF + \frac{1}{2}\angle DFE = 90°$

$\Rightarrow \quad \angle GEF + \angle GFE = 90°.$

In $\triangle EGF$, we have

$\angle GEF + \angle GFE + \angle EGF = 180° \Rightarrow \angle EGF = 90°.$

19. $AB \parallel CD$ and t is the transversal.

$\therefore \quad \angle AEF = \angle EFD$ [alt. int. ∠s]

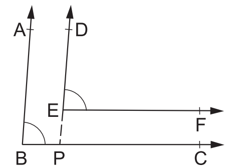
$\Rightarrow \quad \frac{1}{2}\angle AEF = \frac{1}{2}\angle EFD$

$\Rightarrow \quad \angle FEP = \angle EFQ.$

But these are alternate interior angles.

Hence, $EP \parallel FQ$.

20. Produce DE to meet BC at P.



$BA \parallel PD$ and BPC is the transversal.

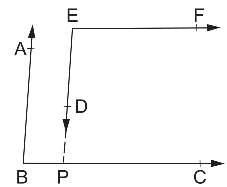
$\therefore \quad \angle ABC = \angle DPC$ [corresponding $\angle$s].

$EF \parallel BC$ and DP is the transversal.

$\therefore \quad \angle DPC = \angle DEF$ [corresponding $\angle$s].

Hence, $\angle ABC = \angle DEF$.

21. Produce ED to meet BC at P.



$AB \parallel PE$ and BPC is the transversal.

$\therefore \quad \angle ABP + \angle BPE = 180°$

$\Rightarrow \quad \angle ABC + \angle BPE = 180°.$ ... (i)

Now, $EF \parallel BPC$ and EP is the transversal.

$\therefore \quad \angle BPE = \angle PEF \Rightarrow \angle BPE = \angle DEF.$ ... (ii)

Hence, $\angle ABC + \angle DEF = 180°$ [using (ii)].

22. Let the normals at A and B meet at P.

Since angle of incidence = angle of reflection, we have

$\angle 1 = \angle 2$ and $\angle 4 = \angle 3$.

As the mirrors are perpendicular to each other, we have

$BP \parallel OA$ and $AP \parallel OB$.

Since $BP \perp PA$, we have $\angle APB = 90°$.

$\therefore \quad \angle 2 + \angle 3 = 90°.$

$\therefore \quad \angle 1 + \angle 4 = \angle 2 + \angle 3 = 90°.$

So, $\angle 1 + \angle 2 + \angle 3 + \angle 4 = 180°.$

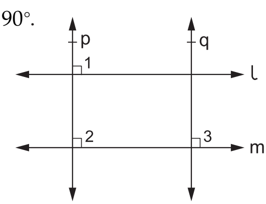
$\therefore \quad \angle CAB + \angle ABD = 180°.$

Hence, $AC \parallel BD$.

23. Since $\angle BAC = \angle ACD$, which are alternate angles, so $AB \parallel CD$. But, $\angle ACD + \angle CDE = (110° + 80°) = 190° \neq 180°.$

$\therefore \quad DE$ and AC are not parallel.

24. Let $l \parallel m$ and let $p \perp l$ and $q \perp m$. Then, $\angle 1 = 90°$ and $\angle 3 = 90°$.



Now, $l \parallel m$ and p is a transversal.

$\therefore \quad \angle 2 = \angle 1 = 90°.$

Also, $\angle 3 = 90°$ (given)

$\therefore \quad \angle 3 = \angle 2$ (each = 90°)

But these are corresponding angles.

$\therefore \quad p \parallel q.$

MULTIPLE-CHOICE QUESTIONS (MCQ)

Choose the correct option in each of the following questions:

1. If one angle of a triangle is equal to the sum of the other two angles, then the triangle is

(a) an isosceles triangle (b) an obtuse triangle
(c) an equilateral triangle (d) a right triangle

2. An exterior angle of a triangle is 110° and its two interior opposite angles are equal. Each of these equal angles is

(a) 70° (b) 55° (c) 35° (d) $27\frac{1}{2}^{\circ}$

3. The angles of a triangle are in the ratio 3 : 5 : 7. The triangle is

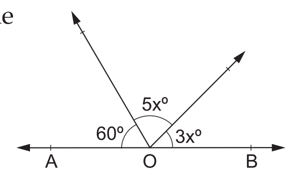
(a) acute angled (b) obtuse angled
(c) right angled (d) an isosceles triangle

4. If one of the angles of a triangle is 130° then the angle between the bisectors of the other two angles can be

(a) 50° (b) 65° (c) 90° (d) 155°

5. In the given figure, *AOB* is a straight line. The value of *x* is

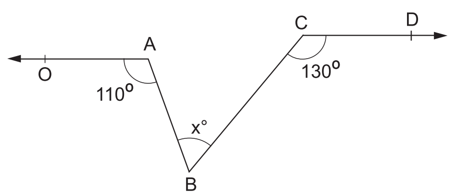
(a) 12 (b) 15
(c) 20 (d) 25



6. The angles of a triangle are in the ratio 2 : 3 : 4. The largest angle of the triangle is

(a) 120° (b) 100° (c) 80° (d) 60°

7. In the given figure, $\angle OAB = 110°$ and $\angle BCD = 130°$ then $\angle ABC$ is equal to



(a) 40° (b) 50° (c) 60° (d) 70°

8. If two angles are complements of each other then each angle is

(a) an acute angle (b) an obtuse angle
(c) a right angle (d) a reflex angle

9. An angle which measures more than 180° but less than 360°, is called

(a) an acute angle (b) an obtuse angle
(c) a straight angle (d) a reflex angle

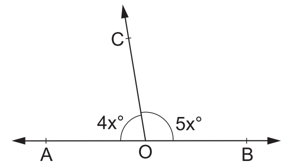
10. The measure of an angle is five times its complement. The angle measures

(a) 25° (b) 35° (c) 65° (d) 75°

11. Two complementary angles are such that twice the measure of the one is equal to three times the measure of the other. The larger of the two measures

(a) 72° (b) 54° (c) 63° (d) 36°

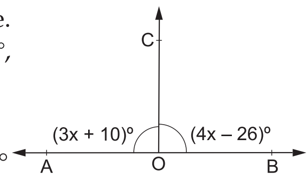
12. In the given figure, AOB is a straight line. If $\angle AOC = 4x°$ and $\angle BOC = 5x°$ then $\angle AOC = ?$



(a) 40° (b) 60°

(c) 80° (d) 100°

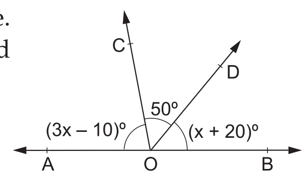
13. In the given figure, AOB is a straight line. If $\angle AOC = (3x+10)°$ and $\angle BOC = (4x-26)°$, then $\angle BOC = ?$



(a) 96° (b) 86°

(c) 76° (d) 106°

14. In the given figure, AOB is a straight line. If $\angle AOC = (3x-10)°$, $\angle COD = 50°$ and $\angle BOD = (x+20)°$ then $\angle AOC = ?$



(a) 40° (b) 60°

(c) 80° (d) 50°

15. Which of the following statements is false?

(a) Through a given point, only one straight line can be drawn.

(b) Through two given points, it is possible to draw one and only one straight line.

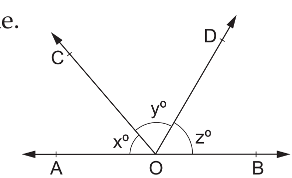
(c) Two straight lines can intersect only at one point.

(d) A line segment can be produced to any desired length.

16. An angle is one fifth of its supplement. The measure of the angle is

(a) 15° (b) 30° (c) 75° (d) 150°

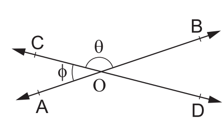
17. In the adjoining figure, AOB is a straight line. If $x : y : z = 4 : 5 : 6$, then $y = ?$



(a) 60° (b) 80°

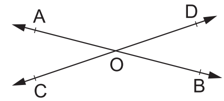
(c) 48° (d) 72°

18. In the given figure, straight lines AB and CD intersect at O. If $\angle AOC = \phi$, $\angle BOC = \theta$ and $\theta = 3\phi$, then $\phi = ?$



(a) 30° (b) 40° (c) 45° (d) 60°

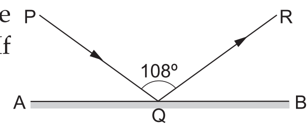
19. In the given figure, straight lines AB and CD intersect at O. If $\angle AOC + \angle BOD = 130°$ then $\angle AOD = ?$



(a) 65° (b) 115°

(c) 110° (d) 125°

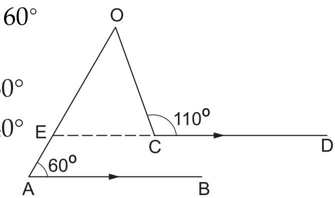
20. In the given figure; AB is a mirror, PQ is the incident ray and QR is the reflected ray. If $\angle PQR = 108°$ then $\angle AQP = ?$



(a) 72° (b) 18°

(c) 36° (d) 54°

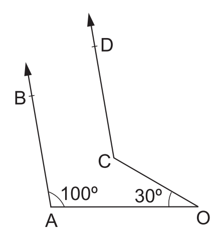
21. In the given figure, $AB \parallel CD$. If $\angle BAO = 60°$ and $\angle OCD = 110°$ then $\angle AOC = ?$



(a) 70° (b) 60°

(c) 50° (d) 40°

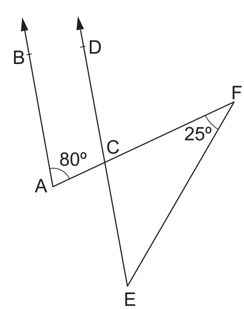
22. In the given figure, $AB \parallel CD$. If $\angle AOC = 30°$ and $\angle OAB = 100°$ then $\angle OCD = ?$



(a) 130° (b) 150°

(c) 80° (d) 100°

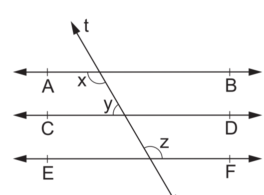
23. In the given figure, $AB \parallel CD$. If $\angle CAB = 180°$ and $\angle EFC = 25°$ then $\angle CEF = ?$



(a) 65° (b) 55°

(c) 45° (d) 75°

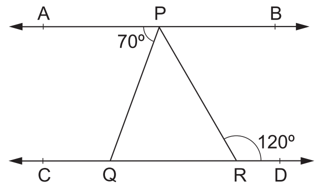
24. In the given figure $AB \parallel CD$ and $CD \parallel EF$. If $y : z = 3 : 7$ then $x = ?$



(a) 108° (b) 126°

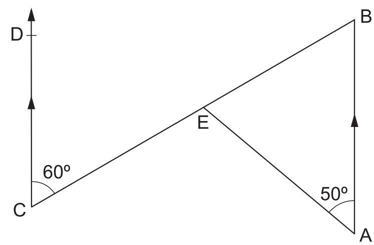
(c) 162° (d) 63°

25. In the given figure, $AB \parallel CD$. If $\angle APQ = 70°$ and $\angle PRD = 120°$ then $\angle QPR = ?$



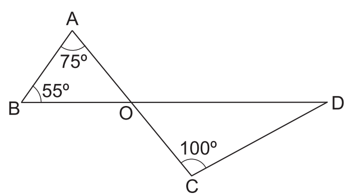
(a) 50° (b) 60°
(c) 40° (d) 35°

26. In the given figure, $AB \parallel CD$. If $\angle EAB = 50°$ and $\angle ECD = 60°$ then $\angle AEB = ?$



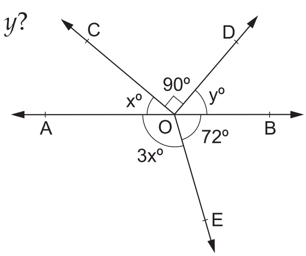
(a) 50°
(b) 60°
(c) 70°
(d) 55°

27. In the given figure, $\angle OAB = 75°$, $\angle OBA = 55°$ and $\angle OCD = 100°$. Then, $\angle ODC = ?$



(a) 20° (b) 25°
(c) 30° (d) 35°

28. In the adjoining figure, what is the value of y?

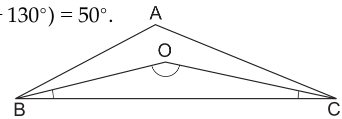


(a) 36 (b) 54
(c) 63 (d) 72

ANSWERS (MCQ)

1. (d)	**2.** (b)	**3.** (a)	**4.** (d)	**5.** (b)	**6.** (c)	**7.** (c)	**8.** (c)
9. (d)	**10.** (d)	**11.** (b)	**12.** (c)	**13.** (b)	**14.** (c)	**15.** (a)	**16.** (b)
17. (a)	**18.** (c)	**19.** (b)	**20.** (c)	**21.** (c)	**22.** (a)	**23.** (b)	**24.** (b)
25. (a)	**26.** (c)	**27.** (c)	**28.** (b)				

HINTS TO SOME SELECTED QUESTIONS



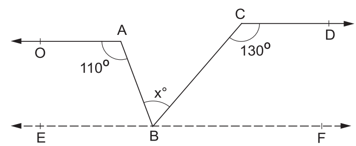
4. In $\triangle ABC$, let $\angle A = 130°$. Then, $\angle B + \angle C = (180° - 130°) = 50°$.

   $\therefore \quad \frac{1}{2}\angle B + \frac{1}{2}\angle C = \left(\frac{1}{2} \times 50°\right) = 25°.$

   Required angle $= (180° - 25°) = 155°$.

6. $2x + 3x + 4x = 180 \Rightarrow 9x = 180° \Rightarrow x = 20°$.

   Largest angle $= 4x° = 4 \times 20° = 80°$.



7. $\angle ABE + \angle OAB = 180°$

   $\Rightarrow \quad \angle ABE + 110° = 180° \Rightarrow \angle ABE = 70°$.

   And, $\angle BCD + \angle CBF = 180°$

   $\Rightarrow \quad 130° + \angle CBF = 180° \Rightarrow \angle CBF = 50°$.

   $\therefore \quad 70° + x° + 50° = 180 \Rightarrow x = 60°$.

20. Angle of incidence = angle of reflection = $x°$ (say).

    Draw QM as normal to AB at Q.

21. Produce DC to E on left-hand side.

    $\angle OEC = \angle OAB = 60°$ [corresponding $\angle$s].

    $\angle ECO = (180° - 110°) = 70°$.

    $\angle EOC + \angle OEC + \angle ECO = 180°$

    $\Rightarrow \quad \angle EOC + 60° + 70° = 180° \Rightarrow \angle EOC = 50°$.

23. $\angle ACD + 80° = 180° \Rightarrow \angle ACD = 100°$.

    $\angle ECF = \angle ACD = 100°$ [vert. opp. $\angle$s].

    $100° + 25° + \angle CEF = 180° \Rightarrow \angle CEF = 55°$.

24. $x + y = 180°$, $y + z = 180°$.

    Let $y = (3p)°$ and $z = (7p)°$. Then

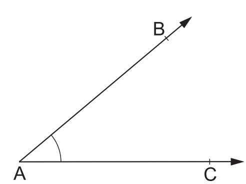
    $3p + 7p = 180 \Rightarrow p = 18$.

    $\therefore \quad y = 54°$, $z = 126°$ and $x = (180° - 54°) = 126°$.

28. $x + y = 90$, $3x + 72 = 180$.

    $\therefore \quad x = 36$ and $y = 54$.

IMPORTANT FACTS AND FORMULAE



1. ANGLE *Two rays AB and AC having a common end point A form angle BAC, written as $\angle BAC$ or $\angle A$.*

We measure angles in degrees, denoted by '°'.

2. (i) RIGHT ANGLE An angle whose measure is $90°$ is called a right angle.

(ii) ACUTE ANGLE An angle whose measure is more than $0°$ but less than $90°$ is called an acute angle.

(iii) OBTUSE ANGLE An angle whose measure is more than 90° but less than 180° is called an obtuse angle.

(iv) STRAIGHT ANGLE An angle whose measure is 180° is called a straight angle.

(v) REFLEX ANGLE An angle whose measure is more than 180° but less than 360° is called a reflex angle.

(vi) COMPLETE ANGLE An angle whose measure is 360° is called a complete angle.

3. (i) COMPLEMENTARY ANGLES Two angles are said to be complementary, if their sum is 90°.

Example Complement of $30° = (90° - 30°) = 60°$.

(ii) SUPPLEMENTARY ANGLES Two angles are said to be supplementary, if their sum is 180°.

Example Supplement of $30° = (180° - 30°) = 150°$.

4. (i) ADJACENT ANGLES Two angles are called adjacent angles, if

(a) they have the same vertex,

(b) they have a common arm, and

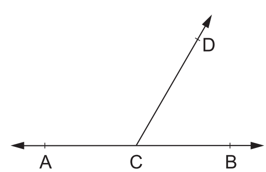
(c) their non-common arms are on either side of the common arm.

(ii) LINEAR PAIR OF ANGLES Two adjacent angles are said to form a linear pair, if their sum is 180°.

5. (i) *If a ray stands on a line, then the sum of the adjacent angles so formed is* 180°.

In the given figure, a ray CD stands on a line AB.

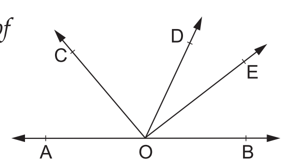
$$\angle ACD + \angle BCD = 180°.$$



(ii) *The sum of all the angles formed on the same side of a line at a given point on the line is* 180°.

In the given figure, AOB is a line.

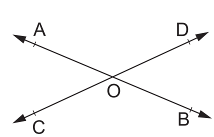
$$\therefore \quad \angle AOC + \angle COD + \angle DOE + \angle EOB = 180°.$$



(iii) *The sum of all the angles around a point is* 360°.

6. VERTICALLY OPPOSITE ANGLES *If two lines AB and CD intersect at a point O then the vertically opposite angles are equal.*

i.e., $\angle AOC = \angle BOD$ and $\angle BOC = \angle AOD$.



7. The sum of the angles of a triangle is 180°.

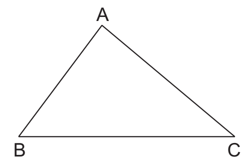
✿

8 Triangles

TRIANGLE *A plane figure bounded by three line segments is called a triangle.*

We denote a triangle by the symbol $\triangle$.

A $\triangle ABC$ has:



(i) three vertices, namely A, B and C;

(ii) three sides, namely AB, BC and CA;

(iii) three angles, namely $\angle A$, $\angle B$ and $\angle C$.

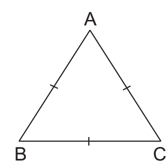
A triangle has six parts—three sides and three angles.

TYPES OF TRIANGLES ON THE BASIS OF SIDES

(i) EQUILATERAL TRIANGLE *A triangle having all sides equal is called an equilateral triangle.*

In the given figure, ABC is a triangle in which $AB = BC = CA$.

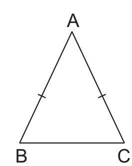
$\therefore$ $\triangle ABC$ is an equilateral triangle.



(ii) ISOSCELES TRIANGLE *A triangle having two sides equal is called an isosceles triangle.*

In the given figure, ABC is a triangle in which $AB = AC$.

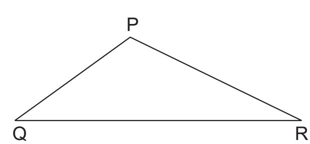
$\therefore$ $\triangle ABC$ is an isosceles triangle.



(iii) SCALENE TRIANGLE *A triangle in which all the sides are of different lengths is called a scalene triangle.*

In the given figure, PQR is a triangle in which $PQ \neq QR \neq PR$.

$\therefore$ $\triangle PQR$ is a scalene triangle.



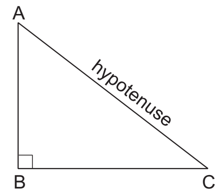
PERIMETER OF A TRIANGLE *The sum of the lengths of three sides of a triangle is called its perimeter.*

TYPES OF TRIANGLES ON THE BASIS OF ANGLES

(i) RIGHT-ANGLED TRIANGLE *A triangle in which one of the angles measures* 90° *is called a right-angled triangle or simply a right triangle.*

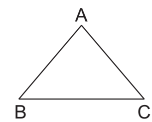
In a right-angled triangle, the side opposite to the right angle is called its *hypotenuse* and the remaining two sides are called its legs.

In the given figure, $\triangle ABC$ is a right triangle in which $\angle B = 90°$, AC is the hypotenuse, and AB, BC are its legs.



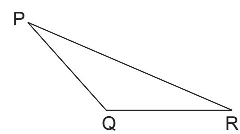
(ii) ACUTE-ANGLED TRIANGLE *A triangle in which every angle measures more than* $0°$ *but less than* $90°$ *is called an acute-angled triangle.*

In the given figure, ABC is a triangle in which every angle is an acute angle.



$\therefore$ $\triangle ABC$ is an acute-angled triangle.

(iii) OBTUSE-ANGLED TRIANGLE *A triangle in which one of the angles measures more than* $90°$ *but less than* $180°$ *is called an obtuse-angled triangle.*



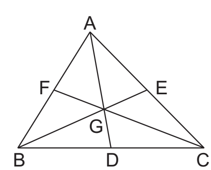
In the given figure, PQR is a triangle in which $\angle PQR$ is an obtuse angle.

$\therefore$ $\triangle PQR$ is an obtuse-angled triangle.

SOME TERMS RELATED TO A TRIANGLE

I. MEDIANS *The median of a triangle corresponding to any side is the line segment joining the midpoint of that side with the opposite vertex.*

In the given figure, D, E, F are the respective midpoints of sides BC, CA and AB of $\triangle ABC$.



$\therefore$ AD is the median, corresponding to side BC.

BE is the median, corresponding to side CA.

CF is the median, corresponding to side AB.

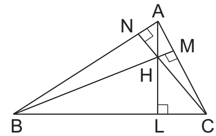
The medians of a triangle are *concurrent*, i.e., they intersect each other at the same point.

CENTROID *The point of intersection of all the three medians of a triangle is called its centroid.*

In the above figure, the medians AD, BE and CF of $\triangle ABC$ intersect at the point G.

$\therefore$ G is the centroid of $\triangle ABC$.

II. ALTITUDES *The altitude of a triangle corresponding to any side is the length of perpendicular drawn from the opposite vertex to that side.*



The side on which the perpendicular is being drawn, is called its *base*.

In the given figure, $AL \perp BC$; $BM \perp CA$ and $CN \perp AB$.

$\therefore$ AL is the altitude, corresponding to base BC.

BM is the altitude, corresponding to base CA.

CN is the altitude, corresponding to base AB.

A triangle has three altitudes.

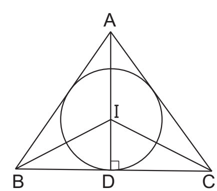
The altitudes of a triangle are concurrent.

ORTHOCENTRE *The point of intersection of all the three altitudes of a triangle is called its orthocentre.*

In the above figure, the three altitudes AL, BM and CN of $\triangle ABC$ intersect at a point H.

$\therefore$ H is the orthocentre of $\triangle ABC$.

III. INCENTRE OF A TRIANGLE *The point of intersection of the internal bisectors of the angles of a triangle is called its incentre.*

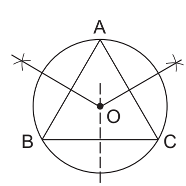


In the given figure, the internal bisectors of the angles of $\triangle ABC$ intersect at I.

$\therefore$ I is the incentre of $\triangle ABC$. Let $ID \perp BC$.

Then, a circle with centre I and radius ID is called the *incircle* of $\triangle ABC$.

IV. CIRCUMCENTRE OF A TRIANGLE *The point of intersection of the perpendicular bisectors of the sides of a triangle is called its circumcentre.*



In the given figure, the right bisectors of the sides of $\triangle ABC$ intersect at O.

$\therefore$ O is the circumcentre of $\triangle ABC$.

With O as centre and radius equal to $OA = OB = OC$, we draw a circle passing through the vertices of the given triangle.

This circle is called the *circumcircle* of $\triangle ABC$.

SUMMARY

(i) CENTROID *The point of intersection of the medians of a triangle is called its centroid.*

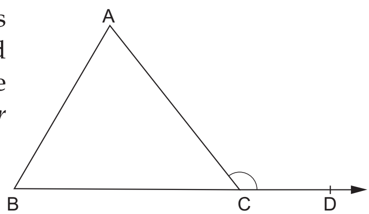
(ii) ORTHOCENTRE *The point of intersection of the altitudes of a triangle is called its orthocentre.*

(iii) INCENTRE *The point of intersection of the internal bisectors of the angles of a triangle is called its incentre.*

(iv) CIRCUMCENTRE *The point of intersection of the perpendicular bisectors of the sides of a triangle is called its circumcentre.*

EXTERIOR AND INTERIOR OPPOSITE ANGLES OF A TRIANGLE

Let ABC be a triangle, one of whose sides BC is produced to D. Then, $\angle ACD$ is called an *exterior angle* (or *remote angle*), while $\angle ABC$ and $\angle BAC$ are called its *interior opposite angles*.



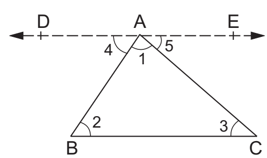
SOME RESULTS ON TRIANGLES

THEOREM 1 *The sum of the angles of a triangle is* 180.

GIVEN A $\triangle ABC$.

TO PROVE $\angle 1 + \angle 2 + \angle 3 = 180°$.

CONSTRUCTION Through A, draw a line $DAE \parallel BC$.



PROOF $DAE \parallel BC$ and AB is the transversal.

$\therefore \quad \angle 4 = \angle 2$ [alternate interior $\angle$s].

$DAE \parallel BC$ and AC is the transversal.

$\therefore \quad \angle 5 = \angle 3$ [alternate interior $\angle$s].

Now, DAE is a straight line.

$\therefore \quad \angle 4 + \angle 1 + \angle 5 = 180°$

[angles on the same side of DAE at the point A]

$\Rightarrow \quad \angle 1 + \angle 4 + \angle 5 = 180°$

$\Rightarrow \quad \angle 1 + \angle 2 + \angle 3 = 180°$ [$\because$ $\angle 4 = \angle 2$ and $\angle 5 = \angle 3$].

Hence, the sum of the angles of a triangle is 180°.

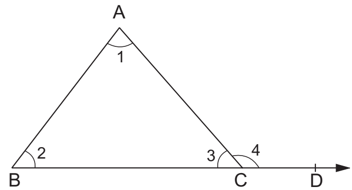
REMARKS As a consequence of the above theorem, it follows:

(i) A triangle cannot have more than one right angle.

(ii) A triangle cannot have more than one obtuse angle.

(iii) In a right triangle, the sum of two acute angles is 90°.

THEOREM 2 *If a side of a triangle is produced then the exterior angle so formed is equal to the sum of the two interior opposite angles.*



GIVEN A $\triangle ABC$ whose side BC has been produced to D forming exterior angle $\angle 4$.

TO PROVE $\angle 4 = \angle 1 + \angle 2$.

PROOF We know that the sum of the angles of a triangle is 180°.

$\therefore \quad \angle 1 + \angle 2 + \angle 3 = 180°$. ... (i)

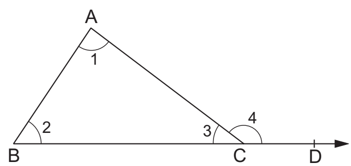
Also, $\angle 3 + \angle 4 = 180°$ ... (ii) [linear pair].

From (i) and (ii), we get

$\angle 3 + \angle 4 = \angle 1 + \angle 2 + \angle 3$.

Hence, $\angle 4 = \angle 1 + \angle 2$.

COROLLARY *An exterior angle of a triangle is greater than either of the interior opposite angles.*



GIVEN A $\triangle ABC$ whose side BC has been produced to D forming exterior angle $\angle 4$.

TO PROVE $\angle 4 > \angle 1$ and $\angle 4 > \angle 2$.

PROOF We know that an exterior angle of a triangle is equal to the sum of the interior opposite angles.

$\therefore \quad \angle 4 = \angle 1 + \angle 2$.

So, $\angle 4 > \angle 1$ and $\angle 4 > \angle 2$.

Hence, an exterior angle of a triangle is greater than each of the interior opposite angles.

SOLVED EXAMPLES

EXAMPLE 1 *The angles of a triangle are in the ratio* 4 : 5 : 6. *Find the angles.*

SOLUTION Let the required angles be $(4x)°$, $(5x)°$ and $(6x)°$.

We know that the sum of the angles of a triangle is 180°.

$$\therefore \quad 4x + 5x + 6x = 180 \Rightarrow 15x = 180$$
$$\Rightarrow x = 12.$$

So, the required angles are

$(4 \times 12)°$, $(5 \times 12)°$ and $(6 \times 12)°$, i.e., 48°, 60° and 72°.

EXAMPLE 2 *In a* $\triangle ABC$, $2\angle A = 3\angle B = 6\angle C$, *then find* $\angle A$, $\angle B$ *and* $\angle C$.

SOLUTION Let $2\angle A = 3\angle B = 6\angle C = k$ (say).

Then, $\angle A = \left(\frac{k}{2}\right)^\circ$, $\angle B = \left(\frac{k}{3}\right)^\circ$ and $\angle C = \left(\frac{k}{6}\right)^\circ$.

We know that the sum of the angles of a triangle is 180°.

$\therefore \quad \angle A + \angle B + \angle C = 180^\circ$

$\Rightarrow \quad \frac{k}{2} + \frac{k}{3} + \frac{k}{6} = 180$

$\Rightarrow \quad (3k + 2k + k) = (180 \times 6)$

$\Rightarrow \quad 6k = (180 \times 6) \;\Rightarrow\; k = 180.$

$\therefore \quad \angle A = \left(\frac{k}{2}\right)^\circ = \left(\frac{180}{2}\right)^\circ = 90^\circ, \angle B = \left(\frac{k}{3}\right)^\circ = \left(\frac{180}{3}\right)^\circ = 60^\circ$

and $\angle C = \left(\frac{k}{6}\right)^\circ = \left(\frac{180}{6}\right)^\circ = 30^\circ.$

Hence, $\angle A = 90^\circ$, $\angle B = 60^\circ$ and $\angle C = 30^\circ$.

EXAMPLE 3 *In a $\triangle ABC$, $\angle A + \angle B = 65^\circ$ and $\angle B + \angle C = 140^\circ$. Find the angles.*

SOLUTION $\angle A + \angle B = 65^\circ$ and $\angle B + \angle C = 140^\circ$

$\Rightarrow \quad \angle A + \angle B + \angle B + \angle C = (65^\circ + 140^\circ)$

$\Rightarrow \quad (\angle A + \angle B + \angle C) + \angle B = 205^\circ$

$\Rightarrow \quad 180^\circ + \angle B = 205^\circ \quad [\because \angle A + \angle B + \angle C = 180^\circ]$

$\Rightarrow \quad \angle B = (205^\circ - 180^\circ) = 25^\circ.$

$\therefore \quad \angle C = (140^\circ - \angle B) = (140^\circ - 25^\circ) = 115^\circ$

and $\angle A = (65^\circ - \angle B) = (65^\circ - 25^\circ) = 40^\circ.$

Hence, $\angle A = 40^\circ$, $\angle B = 25^\circ$ and $\angle C = 115^\circ$.

EXAMPLE 4 *In a $\triangle ABC$, $\angle A - \angle B = 33^\circ$ and $\angle B - \angle C = 18^\circ$. Find the angles of the triangle.*

SOLUTION $\angle A - \angle B = 33^\circ$ and $\angle B - \angle C = 18^\circ$

$\Rightarrow \quad \angle A = (33^\circ + \angle B)$ and $\angle C = (\angle B - 18^\circ)$. ... (i)

We know that the sum of the angles of a triangle is 180°.

$\therefore \quad \angle A + \angle B + \angle C = 180^\circ$

$\Rightarrow \quad (33^\circ + \angle B) + \angle B + (\angle B - 18^\circ) = 180^\circ \quad$ [using (i)]

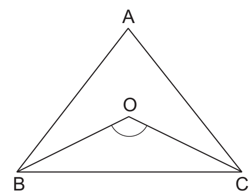
$\Rightarrow \quad 3\angle B = 165^\circ \;\Rightarrow\; \angle B = 55^\circ.$

$\therefore \quad \angle A = (33^\circ + \angle B) = (33^\circ + 55^\circ) = 88^\circ.$

and $\angle C = (\angle B - 18^\circ) = (55^\circ - 18^\circ) = 37^\circ.$

Hence, $\angle A = 88^\circ$, $\angle B = 55^\circ$ and $\angle C = 37^\circ$.

EXAMPLE 5 *In a $\triangle ABC$, the bisectors of $\angle B$ and $\angle C$ intersect each other at a point O. Prove that $\angle BOC = 90° + \frac{1}{2}\angle A$.*



GIVEN A $\triangle ABC$ in which BO and CO are the bisectors of $\angle B$ and $\angle C$ respectively.

TO PROVE $\angle BOC = \left(90° + \frac{1}{2}\angle A\right)$.

PROOF We know that the sum of the angles of a triangle is 180°.

$$\therefore \quad \angle A + \angle B + \angle C = 180°$$

$$\Rightarrow \quad \frac{1}{2}\angle A + \frac{1}{2}\angle B + \frac{1}{2}\angle C = 90°$$

$$\Rightarrow \quad \frac{1}{2}\angle A + \angle OBC + \angle OCB = 90°$$

$$\Rightarrow \quad \angle OBC + \angle OCB = \left(90° - \frac{1}{2}\angle A\right). \quad \ldots \text{(i)}$$

Now, in $\triangle OBC$, we have

$$\angle OBC + \angle OCB + \angle BOC = 180°$$

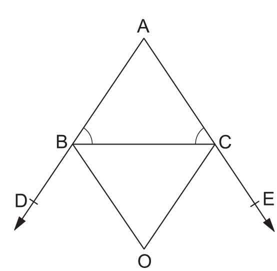
[sum of the angles of a triangle]

$$\Rightarrow \quad \left(90° - \frac{1}{2}\angle A\right) + \angle BOC = 180° \quad \text{[using (i)]}$$

$$\Rightarrow \quad \angle BOC = 180° - \left(90° - \frac{1}{2}\angle A\right) = \left(90° + \frac{1}{2}\angle A\right).$$

Hence, $\angle BOC = \left(90° + \frac{1}{2}\angle A\right)$.

EXAMPLE 6 *In a $\triangle ABC$, the sides AB and AC are produced to points D and E respectively. The bisectors of $\angle DBC$ and $\angle ECB$ intersect at a point O. Prove that $\angle BOC = \left(90° - \frac{1}{2}\angle A\right)$.*



SOLUTION Since ABD is a line, we have

$$\angle B + \angle CBD = 180° \quad \text{[linear pair]}$$

$$\Rightarrow \quad \frac{1}{2}\angle B + \frac{1}{2}\angle CBD = 90°$$

$$\Rightarrow \quad \frac{1}{2}\angle B + \angle CBO = 90°$$

$$\Rightarrow \quad \angle CBO = \left(90° - \frac{1}{2}\angle B\right). \quad \ldots \text{(i)}$$

Again, ACE is a straight line.

$\therefore \quad \angle C + \angle BCE = 180° \quad$ [linear pair]

$\Rightarrow \quad \frac{1}{2}\angle C + \frac{1}{2}\angle BCE = 90°$

$\Rightarrow \quad \frac{1}{2}\angle C + \angle BCO = 90°$

$\Rightarrow \quad \angle BCO = \left(90° - \frac{1}{2}\angle C\right)$. ... (ii)

We know that the sum of the angles of a triangle is 180°.

So, from $\triangle OBC$, we get

$\angle CBO + \angle BCO + \angle BOC = 180°$

$\Rightarrow \quad \left(90° - \frac{1}{2}\angle B\right) + \left(90° - \frac{1}{2}\angle C\right) + \angle BOC = 180°$

[using (i) and (ii)]

$\Rightarrow \quad 180° - \frac{1}{2}(\angle B + \angle C) + \angle BOC = 180°$

$\Rightarrow \quad \angle BOC = \frac{1}{2}(\angle B + \angle C)$

$\Rightarrow \quad \angle BOC = \frac{1}{2}(\angle A + \angle B + \angle C) - \frac{1}{2}\angle A$

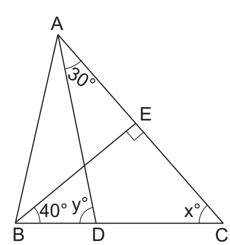
[adding and subtracting $\frac{1}{2}\angle A$]

$\Rightarrow \quad \angle BOC = \left(\frac{1}{2} \times 180°\right) - \frac{1}{2}\angle A \quad [\because \angle A + \angle B + \angle C = 180°]$

$\Rightarrow \quad \angle BOC = \left(90° - \frac{1}{2}\angle A\right)$.

Hence, $\angle BOC = \left(90° - \frac{1}{2}\angle A\right)$.

EXAMPLE 7 *In a $\triangle ABC$, $BE \perp AC$, $\angle EBC = 40°$ and $\angle CAD = 30°$. If $\angle ACD = x°$ and $\angle ADB = y°$, find the values of x and y.*



SOLUTION We know that the sum of the angles of a triangle is 180°.

In $\triangle BCE$, we have

$\angle CBE + \angle BEC + \angle ECB = 180°$

$\therefore \quad 40° + 90° + x° = 180° \Rightarrow x = 50.$

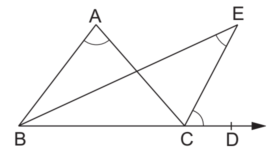
Now, in $\angle ACD$, the side CD has been produced to B.

$\therefore \quad$ ext. $\angle BDA = \angle DAC + \angle ACD$

$\Rightarrow \quad y° = 30° + x° \Rightarrow y = (30 + 50) = 80.$

Hence, $x = 50$ and $y = 80$.

EXAMPLE 8 *In the given figure, the side BC of $\triangle ABC$ has been produced to a point D. If the bisectors of $\angle ABC$ and $\angle ACD$ meet at point E then prove that $\angle BEC = \frac{1}{2}\angle BAC$.*



SOLUTION Side BC of $\triangle ABC$ has been produced to D.

$\therefore \quad \angle ACD = \angle BAC + \angle ABC$

$\Rightarrow \quad \frac{1}{2}\angle ACD = \frac{1}{2}\angle BAC + \frac{1}{2}\angle ABC$

$\Rightarrow \quad \angle ECD = \frac{1}{2}\angle BAC + \frac{1}{2}\angle ABC.$ ... (i)

Again, side BC of $\triangle EBC$ has been produced to D.

$\therefore \quad \angle ECD = \angle CBE + \angle BEC$

$\Rightarrow \quad \angle ECD = \frac{1}{2}\angle ABC + \angle BEC.$ ... (ii)

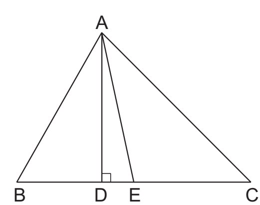
From (i) and (ii), we get

$$\frac{1}{2}\angle ABC + \angle BEC = \frac{1}{2}\angle BAC + \frac{1}{2}\angle ABC$$

[each equal to $\angle ECD$].

$\therefore \quad \angle BEC = \frac{1}{2}\angle BAC.$

EXAMPLE 9 *In a $\triangle ABC$, $\angle B > \angle C$. If $AD \perp BC$ and AE is the bisector of $\angle BAC$ then prove that $\angle DAE = \frac{1}{2}(\angle B - \angle C)$.*



SOLUTION Since AE is the bisector of $\angle BAC$, we have

$\angle BAE = \angle CAE.$ ... (i)

In $\triangle ABD$, we have $\angle ADB = 90°$.

$\therefore \quad \angle ABD + \angle BAD = 90°$

$\Rightarrow \quad \angle ABD = (90° - \angle BAD)$

$\Rightarrow \quad \angle B = (90° - \angle BAD).$ ... (ii)

In $\triangle ADC$, we have $\angle ADC = 90°$.

$\therefore \quad \angle CAD + \angle ACD = 90^\circ$

$\Rightarrow \quad \angle ACD = (90^\circ - \angle CAD)$

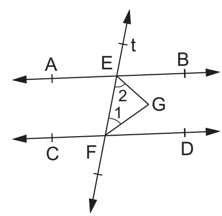
$\Rightarrow \quad \angle C = (90^\circ - \angle CAD)$. ... (iii)

On subtracting (iii) from (ii), we get

$$\begin{aligned}(\angle B - \angle C) &= (90^\circ - \angle BAD) - (90^\circ - \angle CAD)\\ &= \angle CAD - \angle BAD\\ &= (\angle CAE + \angle DAE) - (\angle BAE - \angle DAE)\\ &= 2\angle DAE \qquad \text{[using (i)]}.\end{aligned}$$

$\therefore \quad \angle DAE = \frac{1}{2}(\angle B - \angle C)$.

EXAMPLE 10 *If two parallel lines are cut by a transversal, prove that the bisectors of the interior angles on the same side of the transversal intersect each other at right angles.*



GIVEN $AB \parallel CD$ and t is a transversal, cutting them at E and F respectively. Also, EG and FG, the bisectors of $\angle BEF$ and $\angle DFE$ respectively, meet at G.

TO PROVE $\angle EGF = 90^\circ$.

PROOF $AB \parallel CD$ and t is a transversal.

$\therefore \quad \angle BEF + \angle DFE = 180^\circ$

[$\because$ co. int. $\angle$s are supplementary]

$\Rightarrow \quad \frac{1}{2}\angle BEF + \frac{1}{2}\angle DFE = 90^\circ \Rightarrow \angle 1 + \angle 2 = 90^\circ$. ... (i)

But, in $\triangle EGF$, we have

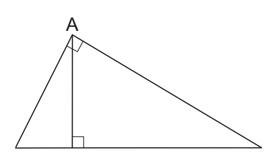
$\angle 1 + \angle 2 + \angle EGF = 180^\circ$ [sum of the $\angle$s of a $\triangle$]

$\Rightarrow \quad 90^\circ + \angle EGF = 180^\circ$ [using (i)]

$\Rightarrow \quad \angle EGF = 90^\circ$.

Hence, $\angle EGF = 90^\circ$.

EXAMPLE 11 *A $\triangle ABC$ is right angled at A and L is a point on BC such that $AL \perp BC$. Prove that $\angle BAL = \angle ACB$.*



SOLUTION We know that in a right-angled triangle, the sum of the two acute angles is 90°.

So, in right triangle $\triangle ALB$, we have

$\angle BAL + \angle ABL = 90^\circ \Rightarrow \angle BAL + \angle ABC = 90^\circ$. ... (i)

In right triangle BAC, we have

$$\angle ABC + \angle ACB = 90°. \quad \dots \text{(ii)}$$

From (i) and (ii), we get

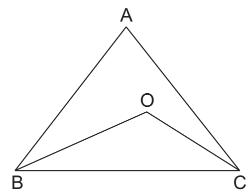
$$\angle BAL + \angle ABC = \angle ABC + \angle ACB.$$

Hence, $\angle BAL = \angle ACB$.

EXAMPLE 12 *Show that the bisectors of the base angles of a triangle can never enclose a right angle.*

GIVEN A $\triangle ABC$ in which BO and CO are the bisectors of the base angles $\angle B$ and $\angle C$ respectively.

TO PROVE $\angle BOC$ is not a right angle.



PROOF If possible, let $\angle BOC = 90°$.

Then, $\angle CBO + \angle BCO = 180°$

$\Rightarrow \quad \frac{1}{2}\angle B + \frac{1}{2}\angle C = 90°$

$\Rightarrow \quad \angle B + \angle C = 180°$

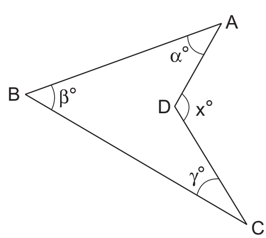
$\Rightarrow \quad \angle A = 0° \quad [\because \angle A + \angle B + \angle C = 180°]$.

This shows that the points A, B, C do not form a triangle, which is false.

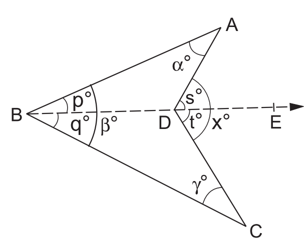
So, our assumption is wrong.

Hence, $\angle BOC$ is not a right angle.

EXAMPLE 13 *In the given figure, prove that* $x = \alpha + \beta + \gamma$.



SOLUTION Join B and D and produce BD to E.



Let $\angle ABD = p°$, $\angle CBD = q°$ and let $\angle ADE = s°$ and $\angle CDE = t°$.

Then, $p + q = \beta$ and $s + t = x$.

Now, side BD of $\triangle ABD$ has been produced to E.

$$\therefore \quad s = p + \alpha. \quad \dots \text{(i)}$$

Again, side BD of $\triangle CBD$ has been produced to E.

$\therefore \quad t = q + \gamma.$ … (ii)

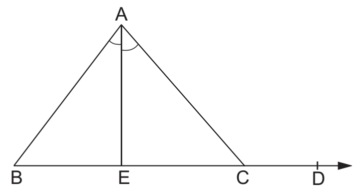
Adding the corresponding sides of (i) and (ii), we get

$$s + t = (p + q) + \alpha + \gamma$$

$\Rightarrow \quad x = \beta + \alpha + \gamma. \qquad [\because \; s + t = x$ and $p + q = \beta]$

Hence, $x = \alpha + \beta + \gamma$.

EXAMPLE 14 *The side BC of $\triangle ABC$ is produced to D. The bisector of $\angle A$ meets BC at E. Prove that $\angle ABC + \angle ACD = 2\angle AEC$.*



SOLUTION Side BE of $\triangle ABE$ has been produced to C.

$\therefore \quad \angle AEC = \angle ABE + \angle BAE$

$\Rightarrow \quad \angle AEC = \angle ABC + \frac{1}{2}\angle A$

$\Rightarrow \quad 2\angle AEC = 2\angle ABC + \angle A.$ … (i)

Again, side BC of $\triangle ABC$ has been produced to D.

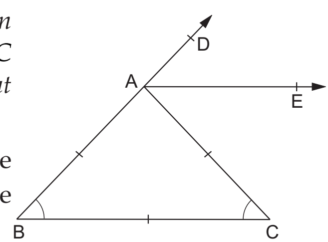
$\therefore \quad \angle ACD = \angle ABC + \angle A.$ … (ii)

On subtracting (ii) from (i), we get

$$2\angle AEC - \angle ACD = \angle ABC$$

$\therefore \quad \angle ABC + \angle ACD = 2\angle AEC.$

EXAMPLE 15 *In the given figure, $\triangle ABC$ is an isosceles triangle in which $AB = AC$ and AE bisects $\angle CAD$. Prove that $AE \parallel BC$.*



SOLUTION We know that the angles opposite to equal sides of a triangle are equal.

$\therefore \quad AB = AC \Rightarrow \angle B = \angle C.$

Now, side BA of $\triangle ABC$ has been produced to D.

$\therefore \quad \angle CAD = \angle B + \angle C \qquad$ [exterior angle = sum of int. opp. ∠s]

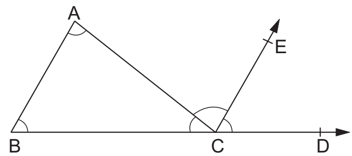
$\Rightarrow \quad 2\angle CAE = 2\angle C \qquad [\because \; \angle B = \angle C$ and $\angle CAD = 2\angle CAE]$

$\Rightarrow \quad \angle CAE = \angle C.$

But, these are alternate interior angles.

$\therefore \quad AE \parallel BC.$

EXAMPLE 16 *In the given figure, side BC of $\triangle ABC$ is produced to form ray BD and $CE \parallel BA$. Show that $\angle ACD = \angle A + \angle B$. Deduce that $\angle A + \angle B + \angle C = 180°$.*



SOLUTION $CE \parallel BA$ and AC is the transversal.

$\therefore \quad \angle ACE = \angle A \quad \ldots$ (i) [alternate interior $\angle$s].

Again, $CE \parallel BA$ and BD is the transversal.

$\therefore \quad \angle ECD = \angle B \quad \ldots$ (ii) [corresponding $\angle$s].

From (i) and (ii), we get

$\angle ACE + \angle ECD = \angle A + \angle B$

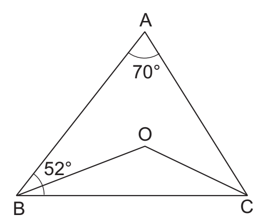
$\therefore \quad \angle ACD = \angle A + \angle B$

$\Rightarrow \quad \angle A + \angle B + \angle C = \angle ACD + \angle ACB$

$= \text{a straight angle} = 180°.$

Hence, $\angle ACD = \angle A + \angle B$ and $\angle A + \angle B + \angle C = 180°$.

EXAMPLE 17 *In $\triangle ABC$, it is given that $\angle A = 70°$, $\angle B = 52°$, BO and CO are the bisectors of $\angle B$ and $\angle C$ respectively. Find $\angle OCB$ and $\angle BOC$.*



SOLUTION We know that the sum of the angles of a triangle is 180°.

$\therefore \quad \angle A + \angle B + \angle C = 180°$

$\Rightarrow \quad 70° + 52° + \angle C = 180°$

$\Rightarrow \quad \angle C = (180° - 122°) = 58°.$

$\therefore \quad \angle OCB = \frac{1}{2}\angle C = \left(\frac{1}{2} \times 58°\right) = 29°$

and $\angle OBC = \frac{1}{2}\angle B = \left(\frac{1}{2} \times 52°\right) = 26°.$

In $\triangle BOC$, we have

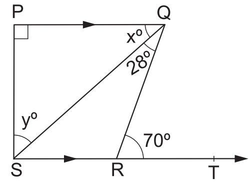
$\angle OBC + \angle OCB + \angle BOC = 180°$

$\Rightarrow \quad 26° + 29° + \angle BOC = 180°$

$\Rightarrow \quad \angle BOC = (180° - 55°) = 125°.$

Hence, $\angle OCB = 29°$ and $\angle BOC = 125°$.

EXAMPLE 18 *In the given figure, $PQ \perp PS$, $PQ \parallel SR$, $\angle SQR = 28^\circ$ and $\angle QRT = 70^\circ$. If $\angle PQS = x^\circ$ and $\angle PSQ = y^\circ$, find the values of x and y.*



SOLUTION $PQ \parallel SRT$ and QR is a transversal.

$\therefore$ $\angle PQR = \angle QRT$ [alternate interior angles]

$\Rightarrow$ $\angle PQS + \angle SQR = \angle QRT$

$\Rightarrow$ $x + 28 = 70$

$\Rightarrow$ $x = (70 - 28) = 42.$

In right $\triangle PQS$, we have

$\angle SPQ + \angle PQS + \angle QSP = 180^\circ$

[sum of the angles of a $\triangle$ is 180°]

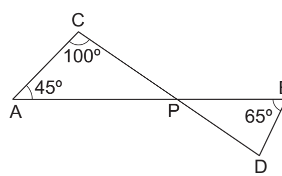
$\Rightarrow$ $90^\circ + x^\circ + y^\circ = 180^\circ$

$\Rightarrow$ $x^\circ + y^\circ = 90^\circ \Rightarrow 42 + y = 90$ [$\because$ $x = 42$]

$\Rightarrow$ $y = (90 - 42) = 48.$

Hence, $x = 42$ and $y = 48$.

EXAMPLE 19 *In the given figure, lines AB and CD intersect at a point P such that $\angle PAC = 45^\circ$, $\angle ACP = 100^\circ$ and $\angle PBD = 65^\circ$. Find $\angle CPA$, $\angle DPB$ and $\angle BDP$.*



SOLUTION We know that the sum of the angles of a triangle is 180°.

In $\triangle ACP$, we have

$\angle PAC + \angle ACP + \angle CPA = 180^\circ$

$\Rightarrow$ $45^\circ + 100^\circ + \angle CPA = 180^\circ$

$\Rightarrow$ $\angle CPA = (180^\circ - 145^\circ) = 35^\circ.$

$\therefore$ $\angle DPB = \angle CPA = 35^\circ$ [vertically opp. $\angle$s].

In $\triangle PBD$, we have

$\angle DPB + \angle PBD + \angle BDP = 180^\circ$

[sum of the angles of a triangle]

$\Rightarrow$ $35^\circ + 65^\circ + \angle BDP = 180^\circ$

$\Rightarrow$ $100^\circ + \angle BDP = 180^\circ$

$\Rightarrow$ $\angle BDP = (180^\circ - 100^\circ) = 80^\circ.$

Hence, $\angle CPA = 35^\circ$, $\angle DPB = 35^\circ$ and $\angle BDP = 80^\circ$.

EXERCISE 8

1. In $\triangle ABC$, if $\angle B = 76°$ and $\angle C = 48°$, find $\angle A$.

2. The angles of a triangle are in the ratio 2 : 3 : 4. Find the angles.

3. In $\triangle ABC$, if $3\angle A = 4\angle B = 6\angle C$, calculate $\angle A$, $\angle B$ and $\angle C$.

4. In $\triangle ABC$, if $\angle A + \angle B = 108°$ and $\angle B + \angle C = 130°$, find $\angle A$, $\angle B$ and $\angle C$.

5. In $\triangle ABC$, if $\angle A + \angle B = 125°$ and $\angle A + \angle C = 113°$, find $\angle A$, $\angle B$ and $\angle C$.

6. In $\triangle PQR$, if $\angle P - \angle Q = 42°$ and $\angle Q - \angle R = 21°$, find $\angle P$, $\angle Q$ and $\angle R$.

7. The sum of two angles of a triangle is 116° and their difference is 24°. Find the measure of each angle of the triangle.

8. Two angles of a triangle are equal and the third angle is greater than each one of them by 18°. Find the angles.

9. Of the three angles of a triangle, one is twice the smallest and another one is thrice the smallest. Find the angles.

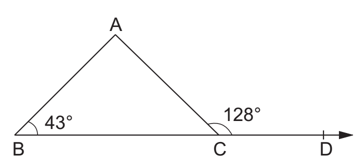
10. In a right-angled triangle, one of the acute angles measures 53°. Find the measure of each angle of the triangle.

11. If one angle of a triangle is equal to the sum of the other two, show that the triangle is right angled.

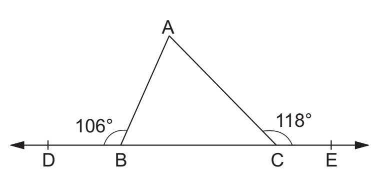
12. If each angle of a triangle is less than the sum of the other two, show that the triangle is acute angled.

13. If one angle of a triangle is greater than the sum of the other two, show that the triangle is obtuse angled.

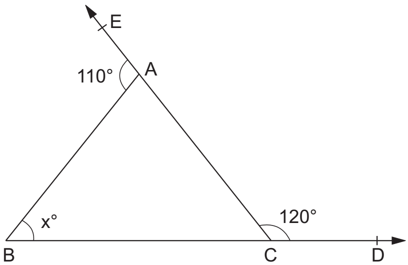
14. In the given figure, side BC of $\triangle ABC$ is produced to D. If $\angle ACD = 128°$ and $\angle ABC = 43°$, find $\angle BAC$ and $\angle ACB$.



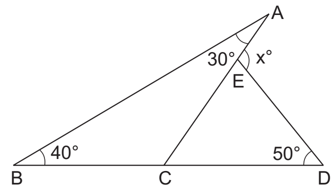
15. In the given figure, the side BC of $\triangle ABC$ has been produced on the left-hand side from B to D and on the right-hand side from C to E. If $\angle ABD = 106°$ and $\angle ACE = 118°$, find the measure of each angle of the triangle.



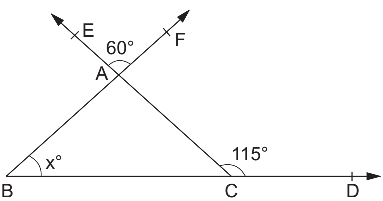
16. Calculate the value of x in each of the following figures.



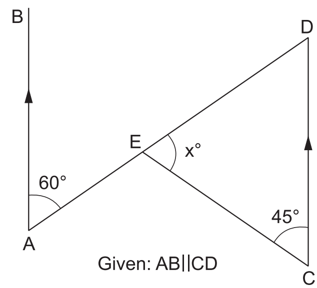
(i)



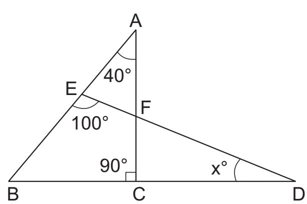
(ii)



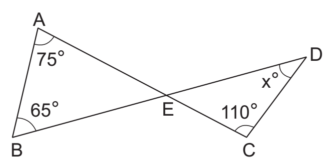
(iii)



(iv)

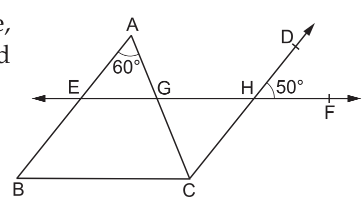


(v)

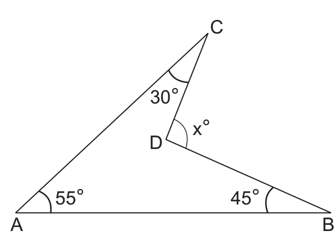


(vi)

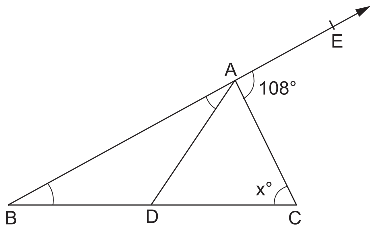
17. In the figure given alongside, $AB \parallel CD$, $EF \parallel BC$, $\angle BAC = 60°$ and $\angle DHF = 50°$. Find $\angle GCH$ and $\angle AGH$.



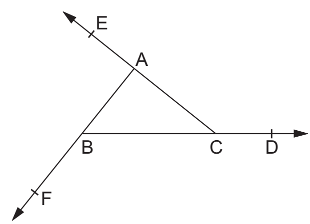
18. Calculate the value of x in the given figure.



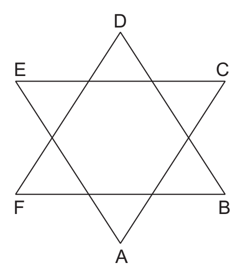
19. In the given figure, AD divides $\angle BAC$ in the ratio 1 : 3 and $AD = DB$. Determine the value of x.



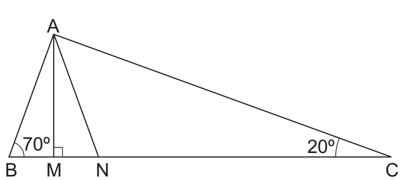
20. If the sides of a triangle are produced in order, prove that the sum of the exterior angles so formed is equal to four right angles.



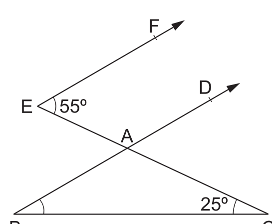
21. In the adjoining figure, show that $\angle A + \angle B + \angle C + \angle D + \angle E + \angle F = 360°$.



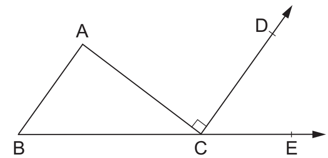
22. In the given figure, $AM \perp BC$ and AN is the bisector of $\angle A$. If $\angle ABC = 70°$ and $\angle ACB = 20°$, find $\angle MAN$.



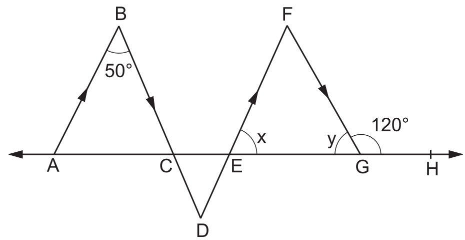
23. In the given figure, $BAD \parallel EF$, $\angle AEF = 55°$ and $\angle ACB = 25°$, find $\angle ABC$.



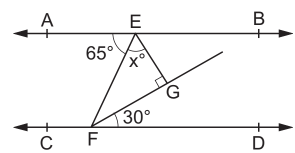
24. In a $\triangle ABC$, it is given that $\angle A : \angle B : \angle C = 3:2:1$ and $CD \perp AC$. Find $\angle ECD$.



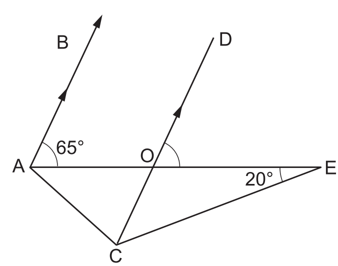
25. In the given figure, $AB \parallel DE$ and $BD \parallel FG$ such that $\angle ABC = 50°$ and $\angle FGH = 120°$. Find the values of x and y.



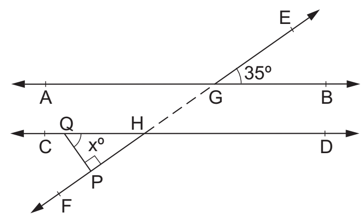
26. In the given figure, $AB \parallel CD$ and EF is a transversal. If $\angle AEF = 65°$, $\angle DFG = 30°$, $\angle EGF = 90°$ and $\angle GEF = x°$, find the value of x.



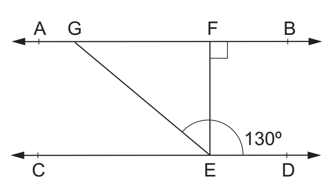
27. In the given figure, $AB \parallel CD$, $\angle BAE = 65°$ and $\angle OEC = 20°$. Find $\angle ECO$.



28. In the given figure, $AB \parallel CD$ and EF is a transversal, cutting them at G and H respectively. If $\angle EGB = 35°$ and $QP \perp EF$, find the measure of $\angle PQH$.



29. In the given figure, $AB \parallel CD$ and $EF \perp AB$. If EG is the transversal such that $\angle GED = 130°$, find $\angle EGF$.



ANSWERS (EXERCISE 8)

1. $\angle A = 56°$ **2.** $40°, 60°, 80°$ **3.** $\angle A = 80°, \angle B = 60°, \angle C = 40°$
4. $\angle A = 50°, \angle B = 58°, \angle C = 72°$ **5.** $\angle A = 58°, \angle B = 67°, \angle C = 55°$
6. $\angle P = 95°, \angle Q = 53°, \angle R = 32°$ **7.** $70°, 46°, 64°$
8. $54°, 54°, 72°$ **9.** $60°, 90°, 30°$ **10.** $53°, 37°, 90°$
14. $\angle BAC = 85°, \angle ACB = 52°$ **15.** $\angle A = 44°, \angle B = 74°, \angle C = 62°$
16. (i) $x = 50$ (ii) $x = 120$ (iii) $x = 55$ (iv) $x = 75$ (v) $x = 30$ (vi) $x = 30$
17. $\angle GCH = 60°, \angle AGH = 110°$ **18.** $x = 130$ **19.** $x = 90$
22. $\angle MAN = 25°$ **23.** $\angle ABC = 30°$ **24.** $\angle ECD = 60°$
25. $x = 70°, y = 60°$ **26.** $x = 55$ **27.** $\angle ECO = 45°$ **28.** $\angle PQH = 55°$
29. $\angle EGF = 50°$

HINTS TO SOME SELECTED QUESTIONS

6. $P - Q = 42 \Rightarrow P = (Q + 42), Q - R = 21 \Rightarrow R = (Q - 21)$.
$P + Q + R = 180° \Rightarrow (Q + 42) + Q + (Q - 21) = 180 \Rightarrow Q = 53$.

11. $A = B + C \Rightarrow A + A = A + B + C = 180° \Rightarrow 2A = 180° = A = 90°$.

12. $A < B + C \Rightarrow 2A < A + B + C = 180° \Rightarrow A < 90°$.
$B < A + C \Rightarrow 2B < A + B + C = 180° \Rightarrow B < 90°$, etc.

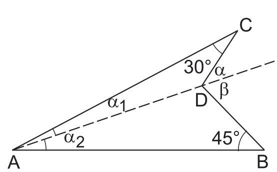
13. $A > B + C \Rightarrow 2A > A + B + C = 180° \Rightarrow A > 90°$.

14. $\angle BAC + \angle ABC = \text{ext.} \angle ACD \Rightarrow \angle BAC + 43° = 128° \Rightarrow \angle BAC = 85°$.
Also, $\angle BAC + \angle ABC + \angle ACB = 180°$. Find $\angle ACB$.

15. $\angle ABC = (180° - 106°) = 74°, \angle ACB = (180° - 118°) = 62°$.
$74° + 62° + \angle BAC = 180°$. Find $\angle BAC$.

17. $\angle ACD = \angle BAC = 60°$ [alternate int. ∠s].
$\angle GCH = \angle ACD = 60°, \angle CHG = 50°$ [vert. opp. ∠s].
In $\triangle GCH$, we have $60° + 50° + \angle HGC = 180°$.
$\therefore \quad \angle HGC = 70°$ and $\angle AGH = (180° - 70°) = 110°$.

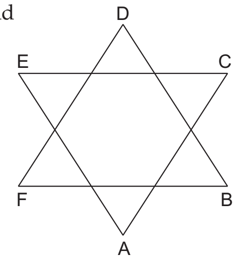
18. $30° + \alpha_1 = \alpha,\ 45° + \alpha_2 = \beta$.
Adding, $(30° + 45°) + (\alpha_1 + \alpha_2) = \alpha + \beta$
$\Rightarrow \quad 75° + 55° = x° \Rightarrow x = 130$.



19. $\angle BAC = (180° - 108°) = 72°$.
$\angle BAD : \angle CAD = 1 : 3$
$\therefore \quad \angle BAD = \left(\frac{1}{4} \times 72°\right) = 18°, \angle CAD = \left(\frac{3}{4} \times 72°\right) = 54°$.
$AD = DB \Rightarrow \angle DBA = \angle BAD = 18°$.
$\angle A + \angle B + \angle C = 180° \Rightarrow 72° + 18° + x° = 180° \Rightarrow x = 90$.

20. $\angle ACD = \angle A + \angle B, \angle BAE = \angle B + \angle C$ and $\angle CBF = \angle C + \angle A$.
$\therefore \quad \angle ACD + \angle BAE + \angle CBF = 2(\angle A + \angle B + \angle C) = (2 \times 180°) = 360°$.

21. Divide the whole figure into two triangles, namely $\triangle DFB$ and $\triangle EAC$.



$\therefore \quad \angle A + \angle B + \angle C + \angle D + \angle E + \angle F$
$= (\angle A + \angle C + \angle E) + (\angle B + \angle D + \angle F)$
$= (180° + 180°) = 360°.$

22. $\angle A + \angle B + \angle C = 180° \Rightarrow \angle A + 70° + 20° = 180° \Rightarrow \angle A = 90°.$
In $\triangle ABM$, $\angle ABM + \angle BMA + \angle BAM = 180°.$
$\therefore \quad 70° + 90° + \angle BAM = 180° \Rightarrow \angle BAM = 20°.$
$\angle BAN = \frac{1}{2}\angle A = 45° \Rightarrow \angle BAM + \angle MAN = 45°$
$\Rightarrow \angle MAN = (45° - 20°) = 25°.$

23. $\angle CAD = \angle AEF = 55°$ [corresponding $\angle$s].
$\angle BAC + \angle CAD = 180° \Rightarrow \angle BAC = 125°.$
$\angle ABC + \angle BAC + \angle ACB = 180° \Rightarrow \angle ABC = 30°.$

24. $\angle C = \left(180 \times \frac{1}{6}\right)^{\circ} = 30°.$
$\angle ACB + \angle ACD + \angle ECD = 180°$ [a straight angle]
$\Rightarrow \quad 30° + 90° + \angle ECD = 180° \Rightarrow \angle ECD = 60°.$

25. $\angle CDE = \angle ABC = 50°$ [alt. int. $\angle$s].
$BD \parallel FG$ and DF is the transversal.
$\therefore \quad \angle EFG = \angle CDE = 50°$. Also, $y = (180° - 120°) = 60°.$
$x + y + \angle EFG = 180° \Rightarrow x + 60° + 50° = 180° \Rightarrow x = 70°.$

26. $\angle EFD = \angle AEF = 65°$ [alternate $\angle$s].
$\therefore \quad \angle EFG = (65° - 30°) = 35°, \angle FGE = 90°.$
In $\triangle EFG$, we have $35° + 90° + x° = 180° \Rightarrow x = 55.$

27. $\angle DOE = \angle BAE = 65°$ [corresponding $\angle$s].
Now, COD is a straight line.
$\therefore \quad \angle COE + \angle DOE = 180° \Rightarrow \angle COE + 65° = 180°$
$\Rightarrow \angle COE = 115°.$
In $\triangle OCE$, we have
$115° + 20° + \angle OCE = 180° \Rightarrow \angle OCE = 45°.$

28. $\angle GHD = \angle EGB = 35° \Rightarrow \angle PHQ = \angle GHD = 35°$ (vert. opp. $\angle$s).

MULTIPLE-CHOICE QUESTIONS (MCQ)

Choose the correct answer in each of the following questions:

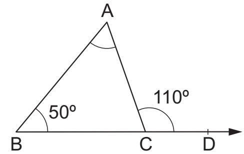
1. In a $\triangle ABC$, if $3\angle A = 4\angle B = 6\angle C$ then $A : B : C = ?$

(a) 3 : 4 : 6 (b) 4 : 3 : 2 (c) 2 : 3 : 4 (d) 6 : 4 : 3

2. In a $\triangle ABC$, if $\angle A - \angle B = 42^\circ$ and $\angle B - \angle C = 21^\circ$ then $\angle B = ?$

(a) 32° (b) 63° (c) 53° (d) 95°

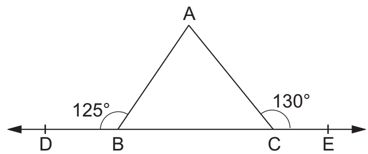
3. In a $\triangle ABC$, side BC is produced to D. If $\angle ABC = 50^\circ$ and $\angle ACD = 110^\circ$ then $\angle A = ?$



(a) 160° (b) 60°

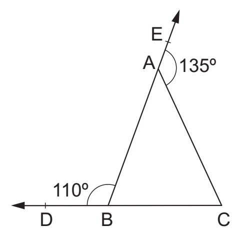
(c) 80° (d) 30°

4. Side BC of $\triangle ABC$ has been produced to D on left and to E on right-hand side of BC such that $\angle ABD = 125^\circ$ and $\angle ACE = 130^\circ$. Then, $\angle A = ?$



(a) 50° (b) 55° (c) 65° (d) 75°

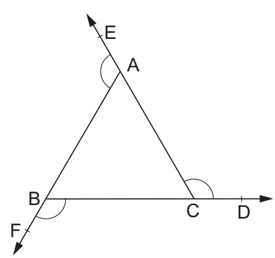
5. In the given figure, the sides CB and BA of $\triangle ABC$ have been produced to D and E respectively such that $\angle ABD = 110^\circ$ and $\angle CAE = 135^\circ$. Then, $\angle ACB = ?$



(a) 65° (b) 45°

(c) 55° (d) 35°

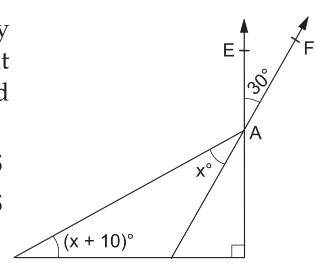
6. The sides BC, CA and AB of $\triangle ABC$ have been produced to D, E and F respectively. $\angle BAE + \angle CBF + \angle ACD = ?$



(a) 240° (b) 300°

(c) 320° (d) 360°

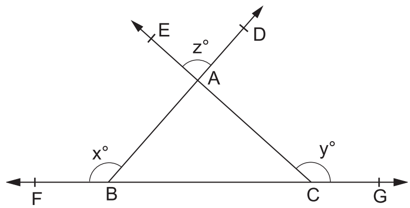
7. In the given figure, $EAD \perp BCD$. Ray FAC cuts ray EAD at a point A such that $\angle EAF = 30^\circ$. Also, in $\triangle BAC$, $\angle BAC = x^\circ$ and $\angle ABC = (x+10)^\circ$. Then, the value of x is



(a) 20 (b) 25

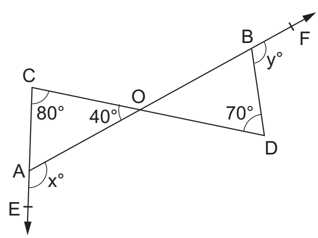
(c) 30 (d) 35

8. In the given figure, two rays BD and CE intersect at a point A. The side BC of $\triangle ABC$ have been produced on both sides to points F and G respectively. If $\angle ABF = x°$, $\angle ACG = y°$ and $\angle DAE = z°$ then $z = ?$



(a) $x + y - 180$ (b) $x + y + 180$ (c) $180 - (x + y)$ (d) $x + y + 360°$

9. In the given figure, lines AB and CD intersect at a point O. The sides CA and OB have been produced to E and F respectively such that $\angle OAE = x°$ and $\angle DBF = y°$.

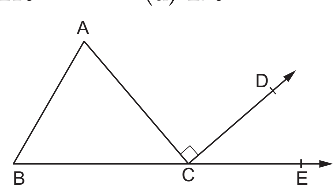


If $\angle OCA = 80°$, $\angle COA = 40°$ and $\angle BDO = 70°$ then $x° + y° = ?$

(a) 190° (b) 230° (c) 210° (d) 270°

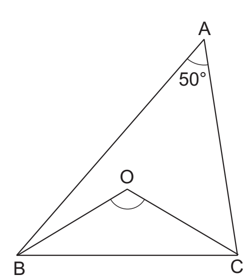
10. In a $\triangle ABC$, it is given that $\angle A : \angle B : \angle C = 3 : 2 : 1$ and $\angle ACD = 90°$. If BC is produced to E then $\angle ECD = ?$

(a) 60°
(b) 50°
(c) 40°
(d) 25°

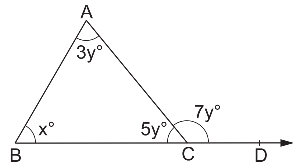


11. In the given figure, BO and CO are the bisectors of $\angle B$ and $\angle C$ respectively. If $\angle A = 50°$ then $\angle BOC = ?$

(a) 130° (b) 100°
(c) 115° (d) 120°



12. In the given figure, side BC of $\triangle ABC$ has been produced to a point D. If $\angle A = 3y^\circ$, $\angle B = x^\circ$, $\angle C = 5y^\circ$ and $\angle CBD = 7y^\circ$. Then, the value of x is



(a) 60 (b) 50

(c) 45 (d) 35

ANSWERS (MCQ)

1. (b) **2.** (c) **3.** (b) **4.** (d) **5.** (a) **6.** (d) **7.** (b) **8.** (a)
9. (b) **10.** (a) **11.** (c) **12.** (a)

HINTS TO SOME SELECTED QUESTIONS

1. Let $3A = 4B = 6C = k$. Then, $A = \frac{k}{3}, B = \frac{k}{4}, C = \frac{k}{6}$.

$\therefore \quad A:B:C = \frac{k}{3}:\frac{k}{4}:\frac{k}{6} = 4:3:2.$

2. $\angle A = \angle B + 42^\circ$ and $\angle C = (\angle B - 21^\circ)$.

$A + B + C = 180^\circ \Rightarrow (B+42) + B + (B-21) = 180$

$\Rightarrow 3B = 159 \Rightarrow B = 53^\circ$.

3. Ext. $\angle ACD = \angle BAC + \angle ABC \Rightarrow x + 50 = 110 \Rightarrow x = 60$.

4. $\angle ABC = (180^\circ - 125^\circ) = 55^\circ$ and $\angle ACB = (180^\circ - 130^\circ) = 50^\circ$.

$\angle A + \angle B + \angle C = 180^\circ \Rightarrow \angle A + 55^\circ + 50^\circ = 180^\circ$

$\Rightarrow \angle A = (180^\circ - 105^\circ) = 75^\circ$.

5. $\angle BAC = (180^\circ - 135^\circ) = 45^\circ$ and $\angle ABD = (180^\circ - 110^\circ) = 70^\circ$.

$\angle A + \angle B + \angle C = 180^\circ \Rightarrow 45^\circ + 70^\circ + \angle ACB = 180^\circ \Rightarrow \angle ACB = 65^\circ$.

7. $\angle CAD = \angle EAF = 30^\circ$ [vert. opp. $\angle$s].

$\angle ACD = 180^\circ - (30^\circ + 90^\circ) = 60^\circ \Rightarrow x + (x+10) = 60 \Rightarrow x = 25$.

8. $\angle BAC = \angle EAD = z^\circ, \angle ABC = (180^\circ - x^\circ), \angle BCA = (180^\circ - y^\circ)$.

$\angle BAC + \angle ABC + \angle BCA = 180^\circ$

$\Rightarrow \quad z + (180 - x) + (180 - y) = 180 \Rightarrow z = (x + y) - 180$.

9. Ext. $\angle OAE = \angle OCA + \angle COA \Rightarrow x^\circ = (80^\circ + 40^\circ) = 120^\circ$.

$\angle BOD = \angle COA = 40^\circ$ (vert. opp. $\angle$s)

Ext. $\angle DBF = \angle BOD + \angle BDO = (40^\circ + 70^\circ) = 110^\circ \Rightarrow y^\circ = 110^\circ$.

$\therefore \quad x^\circ + y^\circ = (120^\circ + 110^\circ) = 230^\circ$.

10. $\angle A + \angle B + \angle C = 180^\circ \Rightarrow 3x + 2x + x = 180$

$\Rightarrow 6x = 180 \Rightarrow x = 30$.

$\therefore \quad \angle A = 90^\circ, \angle B = 60^\circ$ and $\angle C = 30^\circ$.

$\angle ACE = \angle A + \angle B \Rightarrow 90^\circ + \angle ECD = 90^\circ + 60^\circ \Rightarrow \angle ECD = 60^\circ$.

11. $\angle B + \angle C = (180° - \angle A) = (180° - 150°) = 130°.$

$\frac{1}{2}\angle B + \frac{1}{2}\angle C = \left(\frac{1}{2} \times 130°\right) = 65°.$

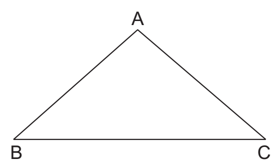
$\frac{1}{2}\angle B + \frac{1}{2}\angle C + \angle BOC = 180° \Rightarrow 65° + \angle BOC = 180° \Rightarrow \angle BOC = 115°.$

12. $5y° + 7y° = 180° \Rightarrow 12y° = 180° \Rightarrow y = 15.$

$3y + x + 5y = 180 \Rightarrow 8y + x = 180 \Rightarrow 120 + x = 180 \Rightarrow x = 60.$

SUMMARY OF FACTS AND FORMULAE

I. TRIANGLE *A plane figure bounded by three line segments is called a triangle, denoted by* $\triangle$.



II. A $\triangle ABC$ has

(i) 3 vertices, namely A, B and C;

(ii) 3 sides, namely AB, BC and CA;

(iii) 3 angles, namely $\angle A$, $\angle B$ and $\angle C$.

III. A triangle has 6 parts, namely 3 sides and 3 angles.

IV. TYPES OF TRIANGLES ON THE BASIS OF SIDES

(i) **EQUILATERAL TRIANGLE** *A triangle having all sides equal is called an equilateral triangle.*

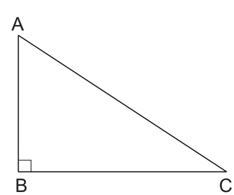
(ii) **ISOSCELES TRIANGLE** *A triangle having two sides equal is called an isosceles triangle.*

(iii) **SCALENE TRIANGLE** *A triangle in which all the sides are of unequal lengths is called a scalene triangle.*

V. *The sum of the lengths of three sides of a triangle is called its perimeter.*

VI. *The sum of the angles of a triangle is always* 180°.

VII. TYPES OF TRIANGLES ON THE BASIS OF ANGLES



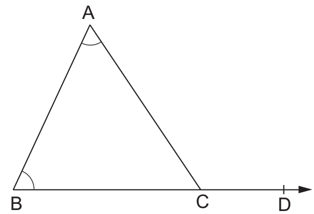
(i) **RIGHT-ANGLED TRIANGLE** *A triangle in which one of the angles measures* 90° *is called a right-angled triangle.*

The side opposite to the right angle is called its *hypotenuse*.

(ii) **ACUTE-ANGLED TRIANGLE** *A triangle in which every angle is less than* 90° *is called an acute-angled triangle.*

(iii) **OBTUSE-ANGLED TRIANGLE** *A triangle in which one angle measures more than* 90° *but less than* 180° *is called an obtuse-angled triangle.*

VIII. *If one side of a triangle is produced then the exterior angle so formed is equal to the sum of the two interior opposite angles.*



In the given figure, the side BC of $\triangle ABC$ has been produced to D.

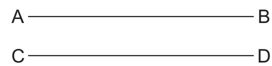
$\therefore$ ext. $\angle ACD = \angle A + \angle B$.

9 Congruence of Triangles and Inequalities in a Triangle

CONGRUENT FIGURES *Two geometrical figures having exactly the same shape and size are known as congruent figures.*

In such figures, one can be superposed on the other to cover it exactly.

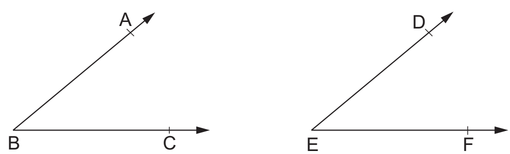
For congruence, we use the symbol '$\cong$'.



Two line segments are congruent only when their lengths are equal.

Thus, $\overline{AB} \cong \overline{CD}$ if $l(AB) = l(CD)$.

Two angles are congruent only when their measures are equal.



$\angle ABC \cong \angle DEF$, if $m(\angle ABC) = m(\angle DEF)$.

CONGRUENT TRIANGLES *Two triangles are congruent if and only if one of them can be made to superpose on the other so as to cover it exactly.*

Thus, congruent triangles are exactly identical.

EXAMPLE 1 If $\triangle ABC \cong \triangle DEF$ then we have

$$\angle A = \angle D,\ \angle B = \angle E,\ \angle C = \angle F;$$

and $AB = DE$, $BC = EF$, $AC = DF$.

EXAMPLE 2 If $\triangle ABC \cong \triangle EDF$ then we have

$$\angle A = \angle E,\ \angle B = \angle D,\ \angle C = \angle F;$$

and $AB = ED$, $BC = DF$, $AC = EF$.

CONGRUENCE RELATION IN THE SET OF ALL TRIANGLES

The following results are quite obvious.

(i) Every triangle is congruent to itself, i.e., $\triangle ABC \cong \triangle ABC$.

(ii) If $\triangle ABC \cong \triangle DEF$ then $\triangle DEF \cong \triangle ABC$.

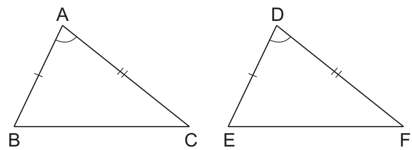
(iii) If $\triangle ABC \cong \triangle DEF$, and $\triangle DEF \cong \triangle PQR$ then $\triangle ABC \cong \triangle PQR$.

NOTE We shall use the abbreviation 'c.p.c.t.' for 'corresponding parts of congruent triangles'.

SAS-CRITERIA FOR CONGRUENCE OF TRIANGLES

THEOREM 1 *If two triangles have two sides and the included angle of the one equal to the corresponding sides and the included angle of the other then the triangles are congruent.*

GIVEN $\triangle ABC$ and $\triangle DEF$ in which $AB = DE$, $AC = DF$ and $\angle A = \angle D$.



TO PROVE $\triangle ABC \cong \triangle DEF$.

PROOF Place $\triangle ABC$ over $\triangle DEF$ such that A falls on D and AB falls along DE.

Since $AB = DE$, so B falls on E.

Since $\angle A = \angle D$, so AC will fall along DF.

But, $AC = DF$.

$\therefore$ C will fall on F.

Thus, AC will coincide with DF.

And, therefore, BC will coincide with EF.

$\therefore$ $\triangle ABC$ coincides with $\triangle DEF$.

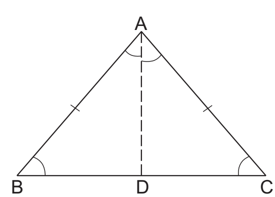
Hence, $\triangle ABC \cong \triangle DEF$.

THEOREM 2 *The angles opposite to two equal sides of a triangle are equal.*

GIVEN A $\triangle ABC$ in which $AB = AC$.

TO PROVE $\angle B = \angle C$.

CONSTRUCTION Draw AD, the bisector of $\angle A$, to meet BC in D.



PROOF In $\triangle ABD$ and $\triangle ACD$, we have

$AB = AC$ (given),

$AD = AD$ (common), $\angle BAD = \angle CAD$ (by construction).

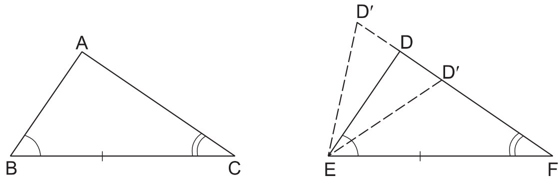
$\therefore$ $\triangle ABD \cong \triangle ACD$ (SAS-criteria).

Hence, $\angle B = \angle C$ (c.p.c.t.).

ASA-CRITERIA FOR CONGRUENCE OF TRIANGLES

THEOREM 3 *If two angles and the included side of one triangle are equal to the corresponding two angles and the included side of the other triangle then the two triangles are congruent.*

GIVEN $\triangle ABC$ and $\triangle DEF$ in which $\angle ABC = \angle DEF$, $\angle ACB = \angle DFE$ and $BC = EF$.



TO PROVE $\triangle ABC \cong \triangle DEF$.

PROOF **Case I**

Let $AC = DF$.

In this case, $AC = DF$, $BC = EF$ and $\angle C = \angle F$.

$\therefore \quad \triangle ABC \cong \triangle DEF$ (SAS-criteria).

Case II

If possible, let $AC \neq DF$.

Then, construct $D'F = AC$. Join $D'E$.

Now, in $\triangle ABC$ and $\triangle D'EF$, we have $AC = D'F$, $BC = EF$ and $\angle C = \angle F$.

$\therefore \quad \triangle ABC \cong \triangle D'EF$ (SAS-criteria).

$\therefore \quad \angle ABC = \angle D'EF$ (c.p.c.t.).

But, $\angle ABC = \angle DEF$ (given)

$\therefore \quad \angle D'EF = \angle DEF$.

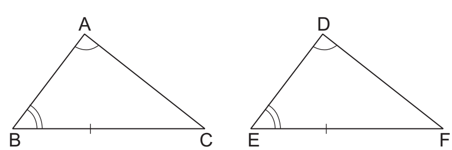
This is possible only when D and D' coincide.

$\therefore \quad \triangle ABC \cong \triangle DEF$.

AAS-CRITERIA FOR CONGRUENCE OF TRIANGLES

COROLLARY *If two angles and any side of a triangle are equal to the corresponding angles and side of another triangle then the two triangles are congruent.*

GIVEN $\triangle ABC$ and $\triangle DEF$ in which $\angle A = \angle D$, $\angle B = \angle E$ and $BC = EF$.



TO PROVE $\triangle ABC \cong \triangle DEF$.

PROOF We know that the sum of the angles of a triangle is 180°.

$\therefore \quad \angle A + \angle B + \angle C = \angle D + \angle E + \angle F = 180°$.

But, $\angle A = \angle D$ and $\angle B = \angle E$.

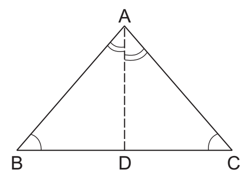
$\therefore \quad \angle C = \angle F$.

Now, in $\triangle ABC$ and $\triangle DEF$, we have

$\angle B = \angle E$, $\angle C = \angle F$ and $BC = EF$.

$\therefore$ $\triangle ABC \cong \triangle DEF$ (ASA-criteria).

THEOREM 4 (Converse of Theorem 2) *If two angles of a triangle are equal then the sides opposite to them are also equal.*



GIVEN A $\triangle ABC$ in which $\angle B = \angle C$.

TO PROVE $AB = AC$.

CONSTRUCTION Draw AD, the bisector of $\angle A$ to meet BC at D.

PROOF In $\triangle ABD$ and $\triangle ACD$, we have

$\angle BAD = \angle CAD$ (by construction),

$AD = AD$ (common),

$\angle B = \angle C$ (given).

$\therefore$ $\triangle ABD \cong \triangle ACD$ (AAS-criteria).

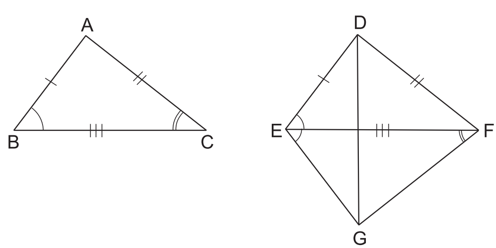
Hence, $AB = AC$ (c.p.c.t.).

SSS-CRITERIA FOR CONGRUENCE OF TRIANGLES

THEOREM 5 *If the three sides of one triangle are equal to the corresponding three sides of another triangle then the two triangles are congruent.*

GIVEN $\triangle ABC$ and $\triangle DEF$ in which $AB = DE$, $AC = DF$ and $BC = EF$.

TO PROVE $\triangle ABC \cong \triangle DEF$.



CONSTRUCTION Suppose BC is the longest side of $\triangle ABC$. Draw EG and FG such that $\angle FEG = \angle CBA$ and $\angle EFG = \angle BCA$. Join DG.

PROOF In $\triangle$s ABC and GEF, we have

$BC = EF$ (given), $\angle CBA = \angle FEG$ (by construction)

and $\angle BCA = \angle EFG$ (by construction).

$\therefore$ $\triangle ABC \cong \triangle GEF$ (ASA-criteria).

$\therefore$ $\angle A = \angle EGF$, $AB = GE$ and $AC = GF$ (c.p.c.t.)

$\Rightarrow$ $\angle A = \angle EGF$, $DE = GE$ and $DF = GF$

[$\because$ $AB = DE$ and $AC = DF$].

Now, in $\triangle EGD$, $DE = GE \Rightarrow \angle EGD = \angle EDG$.

And, in $\triangle FGD$, $DF = GF \Rightarrow \angle FGD = \angle FDG$.

$\therefore \quad \angle EGD + \angle FGD = \angle EDG + \angle FDG$

$\Rightarrow \quad \angle EGF = \angle EDF \Rightarrow \angle A = \angle EDF \quad [\because \angle EGF = \angle A]$

$\Rightarrow \quad \angle A = \angle D$.

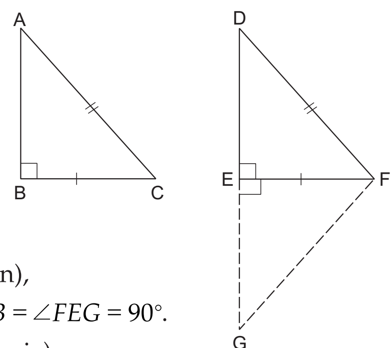
Now, in $\triangle$s ABC and DEF, we have

$AB = DE$, $AC = DF$ and $\angle A = \angle D$.

$\therefore \quad \triangle ABC \cong \triangle DEF$ (SAS-criteria).

RHS-CRITERIA FOR CONGRUENCE OF RIGHT TRIANGLES

THEOREM 6 *Two right-angled triangles are congruent if one side and the hypotenuse of the one are respectively equal to the corresponding side and the hypotenuse of the other.*



GIVEN Two right-angled triangles $\triangle ABC$ and $\triangle DEF$ in which $\angle B = \angle E = 90°$, $BC = EF$ and $AC = DF$.

TO PROVE $\triangle ABC \cong \triangle DEF$.

CONSTRUCTION Produce DE to G such that $GE = AB$. Join GF.

PROOF In $\triangle$s ABC and GEF, we have

$AB = GE$ (by construction),

$BC = EF$ (given), $\angle B = \angle FEG = 90°$.

$\therefore \quad \triangle ABC \cong \triangle GEF$ (SAS-criteria)

$\therefore \quad \angle A = \angle G$ and $AC = GF$ (c.p.c.t.).

Now, $AC = GF$ and $AC = DF \Rightarrow GF = DF \Rightarrow \angle G = \angle D$

$\Rightarrow \angle A = \angle D \quad [\because \angle G = \angle A]$.

Now, $\angle A = \angle D$, $\angle B = \angle E \Rightarrow$ 3rd $\angle C$ = 3rd $\angle F$.

Thus, in $\triangle$s ABC and DEF, we have

$BC = EF$, $AC = DF$ and $\angle C = \angle F$.

$\therefore \quad \triangle ABC \cong \triangle DEF$. (SAS-criteria).

SUMMARY OF THE RESULTS

(i) (SAS-CRITERIA) Two $\triangle$s are congruent if two sides and the included angle are correspondingly equal.

(ii) (AAS-CRITERIA) Two $\triangle$s are congruent if two angles and one side are correspondingly equal.

(iii) (SSS-CRITERIA) Two $\triangle s$ are congruent if three sides are correspondingly equal.

(iv) (RHS-CRITERIA) Two right $\triangle s$ are congruent if hypotenuse and one side are correspondingly equal.

EQUIANGULAR TRIANGLES *If two triangles have three angles of the one correspondingly equal to three angles of the other, they are said to be equiangular.*

Clearly, two congruent triangles are surely equiangular but equiangular triangles need not be congruent.

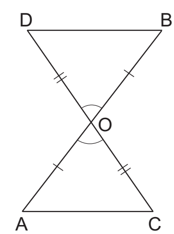
DISTANCE BETWEEN A LINE AND A POINT *The distance between a line and a point not on it, is the length of perpendicular from the point to the given line.*

The distance between a line and a point lying on it, is zero.

SOLVED EXAMPLES

EXAMPLE 1 *In the given figure, $OA = OB$ and $OC = OD$. Show that:*

(i) $\triangle AOC \cong \triangle BOD$ (ii) $AC \parallel BD$.



SOLUTION (i) In $\triangle OAC$ and $\triangle OBD$, we have

$OA = OB$ (given),

$OC = OD$ (given)

and $\angle AOC = \angle BOD$ (vertically opposite $\angle s$).

$\therefore$ $\triangle OAC \cong \triangle OBD$ (by SAS-criteria).

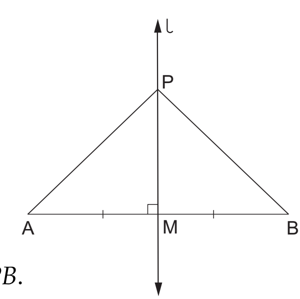
(ii) We have, $\triangle OAC \cong \triangle OBD$ (proved above).

$\therefore$ $\angle OAC = \angle OBD$ (c.p.c.t.).

But, these are alternate angles.

$\therefore$ $AC \parallel BD$.

EXAMPLE 2 *AB is a line segment and line l is its perpendicular bisector. If a point P lies on l, show that P is equidistant from A and B.*



SOLUTION Let M be the midpoint of AB. Then, $l \perp AB$ and passes through M. Let P be a given point on l.

Then, we have to show that $PA = PB$.

In $\triangle PMA$ and $\triangle PMB$, we have

$MA = MB$ [$\because$ M is the midpoint of AB],

$\angle PMA = \angle PMB = 90°$ (given)

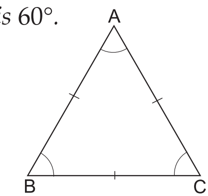
and $PM = PM$ (common).

$\therefore$ $\triangle PMA \cong \triangle PMB$ (by SAS-criteria).

Hence, $PA = PB$ (c.p.c.t.).

This shows that P is equidistant from A and B.

EXAMPLE 3 *Show that each angle of an equilateral triangle is* 60°.



SOLUTION Let $\triangle ABC$ be an equilateral triangle.

Then, $BC = CA = AB$

$\Rightarrow$ $BC = CA$ and $CA = AB$

$\Rightarrow$ $\angle A = \angle B$ and $\angle B = \angle C$

[$\because$ angles opposite to equal sides are equal]

$\Rightarrow$ $\angle A = \angle B = \angle C = x°$ (say).

But, we know that the sum of all angles of a triangle is 180°.

$\therefore$ $\angle A + \angle B + \angle C = 180°$

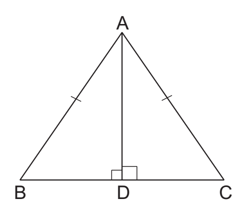
$\Rightarrow$ $x° + x° + x° = 180°$

$\Rightarrow$ $3x° = 180° \Rightarrow x = 60$.

$\therefore$ $\angle A = \angle B = \angle C = 60°$.

Hence, each angle of an equilateral triangle is 60°.

EXAMPLE 4 *In an isosceles triangle, prove that the altitude from the vertex bisects the base.*



GIVEN A $\triangle ABC$ in which $AB = AC$ and $AD \perp BC$.

TO PROVE $BD = DC$.

PROOF In right-angled $\triangle$s ADB and ADC, we have

hyp. AB = hyp. AC (given)

and $AD = AD$ (common).

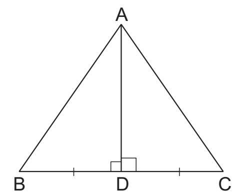
$\therefore$ $\triangle ADB \cong \triangle ADC$ [RHS-criteria].

Hence, $BD = DC$ (c.p.c.t.).

EXAMPLE 5 *If the altitude from one vertex of a triangle bisects the opposite side, prove that the triangle is isosceles.*

GIVEN A $\triangle ABC$ in which $AD \perp BC$ and $BD = DC$.

TO PROVE $AB = AC$.



PROOF In $\triangle ADB$ and $\triangle ADC$, we have

$BD = DC$ (given),

$AD = AD$ (common)

and $\angle ADB = \angle ADC = 90°$

$\therefore$ $\triangle ADB \cong \triangle ADC$ (SAS-criteria).

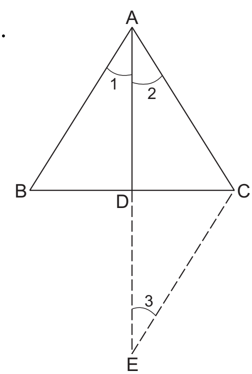
Hence, $AB = AC$ (c.p.c.t.).

<u>EXAMPLE 6</u> *If the bisector of the vertical angle of a triangle bisects the base, prove that the triangle is isosceles.*

GIVEN A $\triangle ABC$ in which AD is the bisector of $\angle A$, which meets BC in D such that $BD = DC$.

TO PROVE $AB = AC$.

CONSTRUCTION Produce AD to E such that $AD = DE$. Join EC.



PROOF In $\triangle ABD$ and $\triangle ECD$, we have

$BD = DC$ (given),

$AD = DE$ (by construction),

$\angle ADB = \angle EDC$ (vert. opp. $\angle$s).

$\therefore$ $\triangle ABD \cong \triangle ECD$ (SAS-criteria).

$\therefore$ $AB = EC$ and $\angle 1 = \angle 3$ (c.p.c.t.).

Also, $\angle 1 = \angle 2$ [$\because$ AD bisects $\angle A$]

$\therefore$ $\angle 2 = \angle 3$.

But, we know that the sides opposite to equal angles of a triangle are equal.

$\therefore$ $EC = AC$.

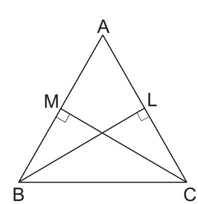
So, $AB = AC$ [$\because$ $EC = AB$].

Hence, $\triangle ABC$ is isosceles.

<u>EXAMPLE 7</u> *Prove that the perpendiculars drawn from the vertices of equal angles of an isosceles triangle to the opposite sides are equal.*

GIVEN A $\triangle ABC$ in which $\angle BCA = \angle CBA$, $BL \perp AC$ and $CM \perp AB$.

TO PROVE $BL = CM$.



PROOF In $\triangle BCL$ and $\triangle CBM$, we have

$\angle BCL = \angle CBM$ [$\because$ $\angle BCA = \angle CBA$],

$\angle BLC = \angle CMB = 90°$

and $BC = CB$ (common).

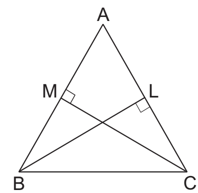
$\therefore$ $\triangle BCL \cong \triangle CBM$ (AAS-criteria).

Hence, $BL = CM$ (c.p.c.t.).

EXAMPLE 8 *If the altitudes from two vertices of a triangle to the opposite sides are equal, prove that the triangle is isosceles.*

GIVEN A $\triangle ABC$ in which $BL \perp AC$ and $CM \perp AB$ such that $BL = CM$.

TO PROVE $AB = AC$.



PROOF In $\triangle ABL$ and $\triangle ACM$, we have

$\angle ALB = \angle AMC$ [each equal to 90°],

$\angle BAL = \angle CAM$ [common]

and $BL = CM$ (given).

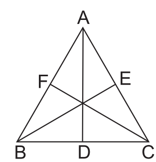
$\therefore$ $\triangle ABL \cong \triangle ACM$ (AAS-criteria).

$\therefore$ $AB = AC$ (c.p.c.t.).

Hence, $\triangle ABC$ is isosceles.

EXAMPLE 9 *Prove that the medians of an equilateral triangle are equal.*

GIVEN A $\triangle ABC$ in which $AB = BC = AC$ and AD, BE and CF are its medians.

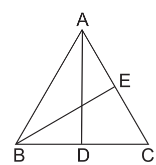


TO PROVE $AD = BE = CF$.

PROOF In $\triangle ADC$ and $\triangle BEA$, we have

$AC = BA$ (given),

$DC = EA$ $\left[\because\ BC = AC \Rightarrow \frac{1}{2}BC = \frac{1}{2}AC\right]$



and $\angle ACD = \angle BAE$ [each equal to 60°].

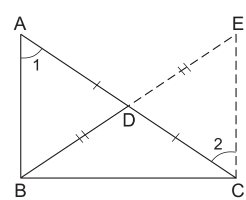
$\therefore$ $\triangle ADC \cong \triangle BEA$ (SAS-criteria).

$\therefore$ $AD = BE$ (c.p.c.t.).

Similarly, $BE = CF$.

Hence, $AD = BE = CF$.

EXAMPLE 10 *If D is the midpoint of the hypotenuse AC of a right-angled $\triangle ABC$, prove that $BD = \frac{1}{2}AC$.*



GIVEN A $\triangle ABC$ in which $\angle B = 90°$ and D is the midpoint of AC.

TO PROVE $BD = \frac{1}{2}AC$.

CONSTRUCTION Produce BD to E such that $BD = DE$. Join EC.

PROOF In $\triangle ADB$ and $\triangle CDE$, we have

$AD = CD$ (given),

$BD = ED$ (by construction)

and $\angle ADB = \angle CDE$ (vert. opp. $\angle$s).

$\therefore$ $\triangle ADB \cong \triangle CDE$ (SAS-criteria).

$\therefore$ $AB = EC$ and $\angle 1 = \angle 2$ (c.p.c.t.).

But, $\angle 1$ and $\angle 2$ are alternate interior angles.

$\therefore$ $CE \parallel BA$.

Now, $CE \parallel BA$ and BC is the transversal.

$\therefore$ $\angle ABC + \angle BCE = 180°$ [co. int. $\angle$s]

$\Rightarrow$ $90° + \angle BCE = 180°$ [$\because$ $\angle ABC = 90°$]

$\Rightarrow$ $\angle BCE = 90°$.

Now, in $\triangle ABC$ and $\triangle ECB$, we have

$BC = CB$ (common),

$AB = EC$ (proved)

and $\angle CBA = \angle BCE$ (each equal to 90°).

$\therefore$ $\triangle ABC \cong \triangle ECB$ (SAS-criteria).

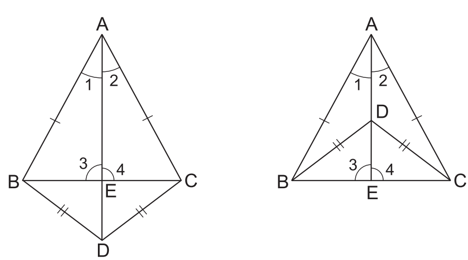
$\therefore$ $AC = EB \Rightarrow \frac{1}{2}EB = \frac{1}{2}AC \Rightarrow BD = \frac{1}{2}AC$.

Hence, $BD = \frac{1}{2}AC$.

EXAMPLE 11 *If two isosceles triangles have a common base, prove that the line segment joining their vertices bisects the common base at right angles.*

GIVEN Two $\triangle$s ABC and DBC with the same base BC, in which $AB = AC$ and $DB = DC$. Also, AD (or AD produced) meets BC in E.

TO PROVE $BE = CE$ and $\angle AEB = \angle AEC = 90°$.



PROOF In $\triangle ABD$ and $\triangle ACD$, we have

$AB = AC$ (given),

$DB = DC$ (given)

and $AD = AD$ (common).

$\therefore$ $\triangle ABD \cong \triangle ACD$ (SSS-criteria).

$\therefore$ $\angle 1 = \angle 2$ (c.p.c.t.).

Now, in $\triangle ABE$ and $\triangle ACE$, we have

$AB = AC$ (given),

$AE = AE$ (common)

and $\angle 1 = \angle 2$ (proved).

$\therefore$ $\triangle ABE \cong \triangle ACE$ (SAS-criteria).

$\therefore$ $BE = CE$ and $\angle 3 = \angle 4$ (c.p.c.t.).

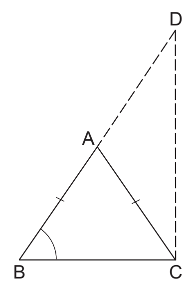
But, $\angle 3 + \angle 4 = 180°$ (linear pair).

$\therefore$ $2\angle 4 = 180° \Rightarrow \angle 4 = 90°$ [$\because$ $\angle 3 = \angle 4$].

Thus, $\angle 3 = \angle 4 = 90°$.

Hence, $BE = CE$ and $\angle AEB = \angle AEC = 90°$.

EXAMPLE 12 *$\triangle ABC$ is an isosceles triangle with $AB = AC$. Side BA is produced to D such that $AB = AD$. Prove that $\angle BCD$ is a right angle.*



GIVEN A $\triangle ABC$ in which $AB = AC$ and side BA is produced to D such that $AB = AD$.

TO PROVE $\angle BCD = 90°$.

CONSTRUCTION Produce BA to D such that $BA = AD$. Join DC.

PROOF $AB = AC$ and $AB = AD \Rightarrow AD = AC$.

Now, $AB = AC \Rightarrow \angle ABC = \angle ACB$,

$AD = AC \Rightarrow \angle ADC = \angle ACD$.

$\therefore$ $\angle ABC + \angle ADC = \angle ACB + \angle ACD$

$\Rightarrow$ $\angle ABC + \angle ADC = \angle BCD$ [$\because$ $\angle ACB + \angle ACD = \angle BCD$]

$\Rightarrow$ $\angle ABC + \angle ADC + \angle BCD = 2\angle BCD$

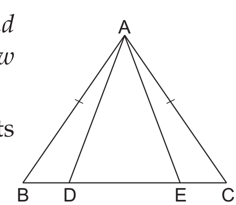
[adding $\angle BCD$ on both sides]

$\Rightarrow$ $2\angle BCD = 180°$ [$\because$ sum of the $\angle$s of $\triangle BCD$ is 180°]

$\Rightarrow$ $\angle BCD = 90°$.

Hence, $\angle BCD$ is a right angle.

EXAMPLE 13 *In an isosceles $\triangle ABC$ with $AB = AC$, D and E are points on BC such that $BE = CD$. Show that $AD = AE$.*



GIVEN A $\triangle ABC$ in which $AB = AC$. D and E are points on BC such that $BE = CD$.

TO PROVE $AD = AE$.

PROOF $BE = CD \Rightarrow BE - DE = CD - DE$

$\Rightarrow BD = CE$. ... (i)

Now, in $\triangle ABD$ and $\triangle ACE$, we have

$AB = AC$ (given),

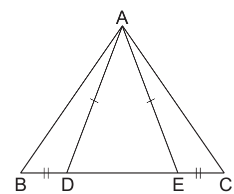
$\angle B = \angle C$ [angles opposite to equal sides]

and $BD = CE$ [proved above in (i)].

$\therefore$ $\triangle ABD \cong \triangle ACE$ (SAS-criteria).

Hence, $AD = AE$ (c.p.c.t.).

EXAMPLE 14 *In the given figure, D and E are points on the side BC of a $\triangle ABC$ such that $BD = CE$ and $AD = AE$. Show that $\triangle ABD \cong \triangle ACE$.*



GIVEN A $\triangle ABC$ in which D and E are points on BC such that $BD = CE$ and $AD = AE$.

TO PROVE $\triangle ABD \cong \triangle ACE$.

PROOF We know that the angles opposite to equal sides of a triangle are equal.

So, in $\triangle ADE$, we have

$AD = AE \Rightarrow \angle ADE = \angle AED$

$\Rightarrow 180^\circ - \angle ADE = 180^\circ - \angle AED$

$\Rightarrow \angle ADB = \angle AEC$. ... (i)

Now, in $\triangle ABD$ and $\triangle ACE$, we have

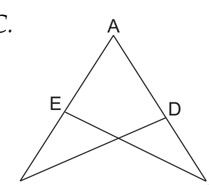
$AD = AE$ (given)

$\angle ADB = \angle AEC$ [proved in (i)]

$BD = CE$ (given)

$\therefore$ $\triangle ABD \cong \triangle ACE$. (SAS-criteria).

EXAMPLE 15 *In the given figure, $AB = AC$ and $\angle B = \angle C$. Prove that $\triangle ABD \cong \triangle ACE$.*



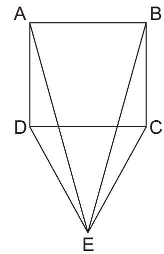
SOLUTION In $\triangle ABD$ and $\triangle ACE$, we have

$AB = AC$ (given)

$\angle ABD = \angle ACE$ [$\because$ $\angle B = \angle C$ (given)]

$\angle BAD = \angle CAE$ [common]

$\therefore$ $\triangle ABD \cong \triangle ACE$ [ASA-criteria].



EXAMPLE 16 *In the given figure, $\triangle CDE$ is an equilateral triangle on a side CD of a square ABCD. Show that $\triangle ADE \cong \triangle BCE$.*

GIVEN A square $ABCD$ and an equilateral $\triangle CDE$ on side CD.

TO PROVE $\triangle ADE \cong \triangle BCE$.

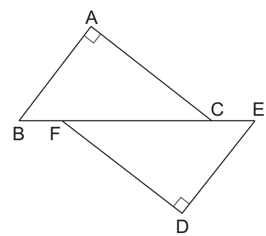
PROOF In $\triangle ADE$ and $\triangle BCE$, we have

$AD = BC$ (sides of the same square)

$DE = CE$ (sides of an equilateral triangle)

$\angle ADE = \angle BCE$ [each equal to $(90° + 60°) = 150°$]

$\therefore$ $\triangle ADE \cong \triangle BCE$ [by SAS-criteria].



EXAMPLE 17 *In the given figure, $BA \perp AC$ and $DE \perp DF$ such that $AB = DE$ and $BF = EC$. Show that $\triangle ABC \cong \triangle DEF$.*

GIVEN $BA \perp AC$ and $DE \perp DF$ such that $AB = DE$ and $BF = EC$.

TO PROVE $\triangle ABC \cong \triangle DEF$.

PROOF $EC = BF \Rightarrow BE - EC = BE - BF$

$\Rightarrow BC = EF.$ ... (i)

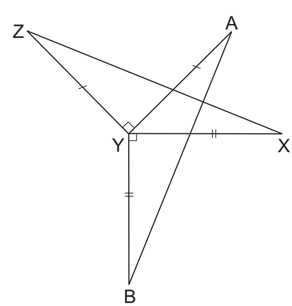
In $\triangle ABC$ and $\triangle DEF$, we have

$AB = DE$ (given)

$\angle BAC = \angle EDF = 90°$

hyp. BC = hyp. EF

$\therefore$ $\triangle ABC \cong \triangle DEF$ [by RHS-criteria].



EXAMPLE 18 *In the given figure, $AY \perp ZY$ and $BY \perp XY$ such that $AY = ZY$ and $BY = XY$. Prove that $AB = ZX$.*

GIVEN $\angle BYX = 90°$, $\angle AYZ = 90°$, $AY = ZY$ and $BY = XY$.

TO PROVE $AB = ZX$.

PROOF $\angle BYX = \angle AYZ$ [each equal to 90°]

$\Rightarrow$ $\angle BYX + \angle AYX = \angle AYZ + \angle AYX$

$\Rightarrow$ $\angle AYB = \angle ZYX.$ ... (i)

Now, in $\triangle AYB$ and $\triangle ZYX$, we have

$AY = ZY$ (given)

$BY = XY$ (given)

$\angle AYB = \angle ZYX$ [proved in (i)]

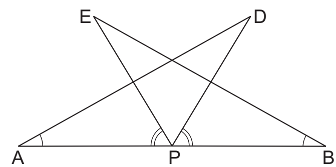
$\therefore \quad \triangle AYB \cong \triangle ZYX$ (SAS-criteria).

Hence, $AB = ZX$ [c.p.c.t.].

EXAMPLE 19 *AB is a line segment and P is its midpoint. D and E are points on the same side of AD such that* $\angle BAD = \angle ABE$ *and* $\angle EPA = \angle DPB$.

Show that (i) $\triangle DAP \cong \triangle EBP$,

(ii) $AD = BE$.



SOLUTION $\angle EPA = \angle DPB$

$\Rightarrow \quad \angle EPA + \angle EPD = \angle DPB + \angle EPD$

$\Rightarrow \quad \angle APD = \angle BPE.$ ... (A)

(i) In $\triangle DAP$ and $\triangle EBP$, we have

$AP = BP$ [$\because$ P is the midpoint of AB]

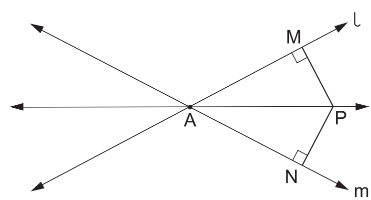
$\angle APD = \angle BPE$ [proved in (A)]

$\angle PAD = \angle PBE$ [$\because$ $\angle BAD = \angle ABE$]

$\therefore \quad \triangle DAP \cong \triangle EBP$ (by SAS-criteria).

(ii) $\triangle DAP \cong \triangle EBP \Rightarrow AD = BE$ (c.p.c.t.).

EXAMPLE 20 *P is a point equidistant from two lines l and m intersecting at a point A, as shown in the given figure. Show that the line AP bisects the angle between them.*



GIVEN Two lines l and m intersecting at a point A and P is a point such that $PM = PN$, where $PM \perp l$ and $PN \perp m$.

TO PROVE $\angle PAM = \angle PAN$.

PROOF In $\triangle PAM$ and $\triangle PAN$, we have

$\angle PMA = \angle PNA = 90°$

$PM = PN$ (given)

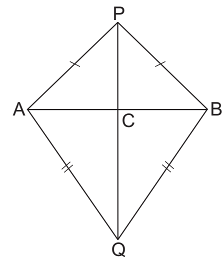
$AP = AP$ (common)

$\therefore \quad \triangle PAM \cong \triangle PAN$ (by RHS-criteria).

$\therefore \quad \angle PAM = \angle PAN$ (c.p.c.t.).

Hence, the line AP bisects the angle between l and m.

EXAMPLE 21 *In the given figure, AB is a line segment. P and Q are points on opposite sides of AB such that each of them is equidistant from the points A and B. Show that the line PQ is the perpendicular bisector of AB.*



GIVEN A line segment AB and two points P and Q such that $PA = PB$ and $QA = QB$.

TO PROVE Let AB and PQ intersect at C. Then, we have to prove that $AC = BC$ and $\angle ACP = 90°$.

PROOF In $\triangle PAQ$ and $\triangle PBQ$, we have

$PA = PB$ (given)

$QA = QB$ (given)

$PQ = PQ$ (common)

$\therefore$ $\triangle PAQ \cong \triangle PBQ$ (by SSS-criteria).

$\therefore$ $\angle APQ = \angle BPQ$ ... (i) (c.p.c.t.).

Now, in $\triangle PAC$ and $\triangle PBC$, we have

$PA = PB$ (given)

$\angle APC = \angle BPC$ [$\because$ $\angle APQ = \angle BPQ$ in (i)]

$PC = PC$ (common)

$\therefore$ $\triangle PAC \cong \triangle PBC$ (by SAS-criteria).

$\therefore$ $AC = BC$ ... (ii) (c.p.c.t.).

And, $\angle ACP = \angle BCP$ ... (iii) (c.p.c.t.).

But, $\angle ACP + \angle BCP = 180°$ (linear pair).

$\therefore$ $2\angle ACP = 180°$ [using (iii)].

So, $\angle ACP = 90°$.

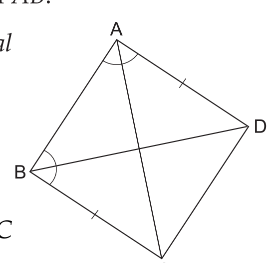
Hence, PQ is the perpendicular bisector of AB.

EXAMPLE 22 *In the given figure, ABCD is a quadrilateral in which $AD = BC$ and $\angle DAB = \angle CBA$.*

Prove that (i) $\triangle ABD \cong \triangle BAC$,

(ii) $BD = AC$,

(iii) $\angle ABD = \angle BAC$.



GIVEN A quadrilateral $ABCD$ in which $AD = BC$ and $\angle DAB = \angle CBA$.

TO PROVE (i) $\triangle ABD \cong \triangle BAC$, (ii) $BD = AC$, (iii) $\angle ABD = \angle BAC$.

PROOF (i) In $\triangle ABD$ and $\triangle BAC$, we have

$AD = BC$ (given)

$\angle DAB = \angle CBA$ (given)

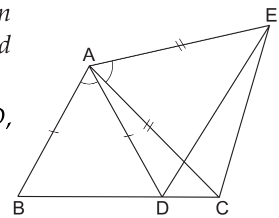
$AB = BA$ (common)

$\therefore$ $\triangle ABD \cong \triangle BAC$ (SAS-criteria).

(ii) As shown above, $\triangle ABD \cong \triangle BAC$.

$\therefore$ $BD = AC$ (c.p.c.t.).

(iii) And, $\angle ABD = \angle BAC$ (c.p.c.t.).



EXAMPLE 23 *In the given figure, $AB = AD$ in $\triangle ABD$ and $AC = AE$ in $\triangle ACE$ and $\angle BAD = \angle EAC$. Show that $BC = DE$.*

GIVEN $\triangle ABD$ and $\triangle ACE$ in which $AB = AD$, $AC = AE$ and $\angle BAD = \angle EAC$.

TO PROVE $BC = DE$.

PROOF In $\triangle ABC$ and $\triangle ADE$, we have

$AB = AD$ (given)

$\angle BAC = \angle BAD + \angle CAD$

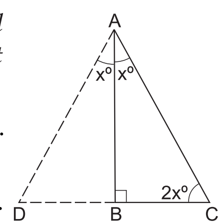
$= \angle EAC + \angle CAD$ [$\because$ $\angle BAD = \angle EAC$]

$= \angle DAE$

$AC = AE$ (given)

$\therefore$ $\triangle ABC \cong \triangle ADE$ (SAS-criteria).

Hence, $BC = DE$ (c.p.c.t.).



EXAMPLE 24 *In the given figure, $\triangle ABC$ is right angled at B such that $\angle BCA = 2\angle BAC$. Show that hypotenuse $AC = 2BC$.*

GIVEN A $\triangle ABC$ in which $\angle B = 90^\circ$ and $\angle BCA = 2\angle BAC$.

TO PROVE $AC = 2BC$.

CONSTRUCTION Produce CB to D such that $BD = BC$. Join AD.

PROOF Let $\angle BAC = x^\circ$. Then, $\angle BCA = 2x^\circ$.

In $\triangle ABC$ and $\triangle ABD$, we have

$BC = BD$ (by construction)

$AB = AB$ (common)

$\angle ABC = \angle ABD = 90^\circ$

$\therefore$ $\triangle ABC \cong \triangle ABD$ (SAS-criteria)

$\therefore$ $\angle CAB = \angle DAB = x^\circ$ … (i) (c.p.c.t.)

and $AC = AD$ … (ii) (c.p.c.t.).

In $\triangle CAD$, we have

$\angle CAD = \angle CAB + \angle DAB = (x^\circ + x^\circ) = 2x^\circ$.

$\angle ACD = \angle ACB = 2x^\circ$.

But, we know that the sides opposite to equal angles are equal.

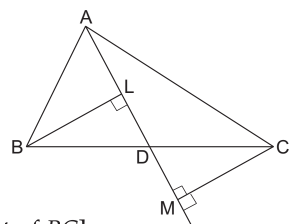
$\therefore \quad \angle ACD = \angle CAD \;\Rightarrow\; AD = CD.$ … (iii)

From (ii) and (iii), we get

$$AC = CD \;\Rightarrow\; AC = 2BC \quad [\because\; CD = BC + BD = 2BC].$$

Hence, $AC = 2BC$.

EXAMPLE 25 *In the adjoining figure, AD is a median of $\triangle ABC$. If BL and CM are drawn perpendiculars on AD and AD produced, prove that $BL = CM$.*



SOLUTION In right triangles $\triangle BLD$ and $\triangle CMD$, we have

$BD = CD$ [$\because$ D is the midpoint of BC]

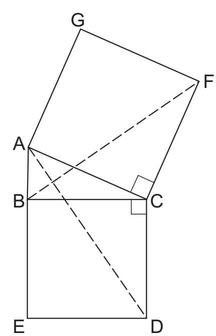
$\angle BLD = \angle CMD$ [each equal to 90°]

$\angle BDL = \angle CDM$ [vert. opp. $\angle$s]

$\therefore \quad \triangle BLD \cong \triangle CMD$ [AAS-criteria].

Hence, $BL = CM$ [c.p.c.t.].

EXAMPLE 26 *In the given figure, ABC is a triangle, right angled at B. If $BCDE$ is a square on side BC and $ACFG$ is a square on AC, prove that $AD = FB$.*



SOLUTION In $\triangle ACD$ and $\triangle FCB$, we have

$\angle ACD = 90^\circ + \angle BCA$

and $\angle FCB = 90^\circ + \angle BCA$

$\Rightarrow \quad \angle ACD = \angle FCB$

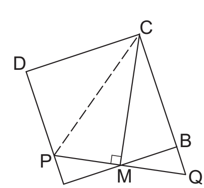
$CA = CF$ (sides of the same square)

$CD = CB$ (sides of the same square)

$\therefore \quad \triangle ACD \cong \triangle FCB$ (SAS-criteria).

Hence, $AD = FB$ (c.p.c.t.).

EXAMPLE 27 *In the given figure, $ABCD$ is a square, M is the midpoint of AB and $PQ \perp CM$ meets AD at P and CB produced at Q. Prove that (i) $PA = QB$ and (ii) $CP = AB + PA$.*



GIVEN A square $ABCD$ in which M is the midpoint of AB. $PQ \perp CM$ meets AD at P and CB produced at Q.

TO PROVE (i) $PA = QB$ and (ii) $CP = AB + PA$.

PROOF (i) In $\triangle PAM$ and $\triangle QBM$, we have

$AM = BM$ $\quad$ ($\because$ M is the midpoint of AB)

$\angle PAM = \angle QBM$ $\quad$ (each equal to 90°)

$\angle AMP = \angle BMQ$ $\quad$ (vertically opp. $\angle$s)

$\therefore$ $\triangle PAM \cong \triangle QBM$ $\quad$ (AAS-criteria).

Hence, $PA = QB$ $\quad$ (c.p.c.t.).

(ii) Join PC.

Now, in $\triangle CMP$ and $\triangle CMQ$, we have

$PM = QM$ $\quad$ ($\because$ $\triangle PAM \cong \triangle QBM$)

$\angle CMP = \angle CMQ$ $\quad$ (each equal to 90°)

$CM = CM$ $\quad$ (common)

$\therefore$ $\triangle CMP \cong \triangle CMQ$ $\quad$ (SAS-criteria).

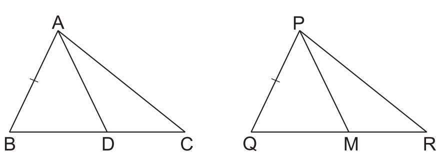
$\therefore$ $CP = CQ$ $\quad$ (c.p.c.t.).

$\Rightarrow$ $CP = CB + QB = AB + PA$ $\quad$ [$\because$ $CB = AB$ and $QB = PA$].

Hence, $CP = AB + PA$.

EXAMPLE 28 *In the given figure, the two sides AB and BC, and the median AD of $\triangle ABC$ are correspondingly equal to the two sides PQ and QR, and the median PM of $\triangle PQR$. Prove that $\triangle ABC \cong \triangle PQR$.*

GIVEN $\triangle ABC$ and $\triangle PQR$ in which $AB = PQ$, $BC = QR$ and median AD of $\triangle ABC$ = median PM of $\triangle PQR$.



TO PROVE $\triangle ABC \cong \triangle PQR$.

PROOF In $\triangle ABD$ and $\triangle PQM$, we have

$AB = PQ$ $\quad$ (given)

$BD = QM$ $\quad \left[\because\ BC = QR \Rightarrow \frac{1}{2}BC = \frac{1}{2}QR \Rightarrow BD = QM\right]$

med. AD = med. PM $\quad$ (given)

$\therefore$ $\triangle ABD \cong \triangle PQR$ $\quad$ (SSS-criteria).

$\therefore$ $\angle ABD = \angle PQM$ $\quad$ (c.p.c.t.)

$\Rightarrow$ $\angle ABC = \angle PQR$.

Now, in $\triangle ABC$ and $\triangle PQR$, we have

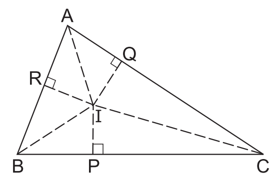
$AB = PQ$ $\quad$ (given)

$\angle ABC = \angle PQR$ (proved above)

$BC = QR$ (given).

$\therefore$ $\triangle ABC \cong \triangle PQR$ (SAS-criteria).

EXAMPLE 29 *In the given figure, the bisectors of $\angle B$ and $\angle C$ of $\triangle ABC$ meet at I. If $IP \perp BC$, $IQ \perp CA$ and $IR \perp AB$, prove that (i) $IP = IQ = IR$, (ii) IA bisects $\angle A$.*



GIVEN A $\triangle ABC$ in which BI and CI are the bisectors of $\angle B$ and $\angle C$ respectively. $IP \perp BC$, $IQ \perp CA$ and $IR \perp AB$.

TO PROVE (i) $IP = IQ = IR$ and (ii) IA bisects $\angle A$.

PROOF (i) In $\triangle IPC$ and $\triangle IQC$, we have

$\angle IPC = \angle IQC = 90°$ (given)

$\angle ICP = \angle ICQ$ [$\because$ CI is the bisector of $\angle C$]

$CI = CI$ (common)

$\therefore$ $\triangle IPC \cong \triangle IQC$ (AAS-criteria).

$\therefore$ $IP = IQ$ (c.p.c.t.).

Similarly, $IQ = IR$.

Hence, $IP = IQ = IR$.

(ii) In $\triangle IQA$ and $\triangle IRA$, we have

$IQ = IR$ [proved in (i)]

$\angle IQA = \angle IRA$ [each = 90°]

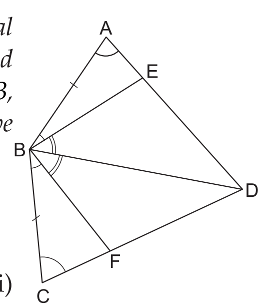
hyp. IA = hyp. IA (common)

$\therefore$ $\triangle IQA \cong \triangle IRA$ (RHS-criteria).

$\therefore$ $\angle IAQ = \angle IAR$ (c.p.c.t.).

Hence, IA bisects $\angle A$.

EXAMPLE 30 *In the given figure, ABCD is a quadrilateral and E and F are points on AD and CD respectively such that $AB = CB$, $\angle ABE = \angle CBF$ and $\angle EBD = \angle FBD$. Prove that $BE = BF$.*



SOLUTION $\angle ABE = \angle CBF$ and $\angle EBD = \angle FBD$

$\Rightarrow$ $\angle ABE + \angle EBD = \angle CBF + \angle FBD$

$\Rightarrow$ $\angle ABD = \angle CBD$. … (i)

Now, in $\triangle ABD$ and $\triangle CBD$, we have

$AB = CB$ (given)

$\angle ABD = \angle CBD$ [from (i)]

$BD = BD$ (common)

$\therefore$ $\triangle ABD \cong \triangle CBD$ [SAS-criteria].

$\therefore$ $\angle BAD = \angle BCD$ [c.p.c.t.]

$\Rightarrow$ $\angle BAE = \angle BCF$. … (ii)

Now, in $\triangle ABE$ and $\triangle CBF$, we have

$AB = CB$ (given)

$\angle ABE = \angle CBF$ (given)

$\angle BAE = \angle BCF$ [proved in (ii)]

$\therefore$ $\triangle ABE \cong \triangle CBF$ [AAS-criteria].

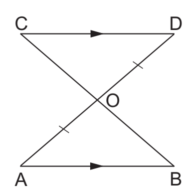
Hence, $BE = BF$.

EXERCISE 9A

1. In the given figure, $AB \parallel CD$ and O is the midpoint of AD.

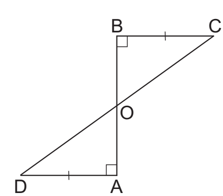
Show that (i) $\triangle AOB \cong \triangle DOC$.

(ii) O is the midpoint of BC.



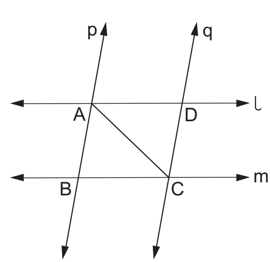
2. In the given figure, AD and BC are equal perpendiculars to a line segment AB.

Show that CD bisects AB.



3. In the given figure, two parallel lines l and m are intersected by two parallel lines p and q.

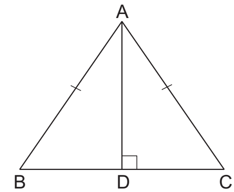
Show that $\triangle ABC \cong \triangle CDA$.



4. AD is an altitude of an isosceles $\triangle ABC$ in which $AB = AC$.

Show that (i) AD bisects BC,

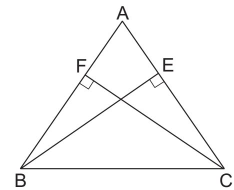
(ii) AD bisects $\angle A$.



5. In the given figure, BE and CF are two equal altitudes of $\triangle ABC$.

Show that (i) $\triangle ABE \cong \triangle ACF$,

(ii) $AB = AC$.



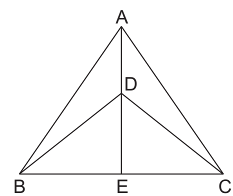
6. $\triangle ABC$ and $\triangle DBC$ are two isosceles triangles on the same base BC and vertices A and D are on the same side of BC. If AD is extended to intersect BC at E, show that

(i) $\triangle ABD \cong \triangle ACD$

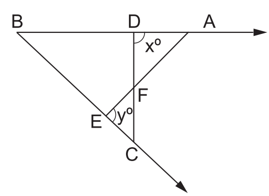
(ii) $\triangle ABE \cong \triangle ACE$

(iii) AE bisects $\angle A$ as well as $\angle D$

(iv) AE is the perpendicular bisector of BC.



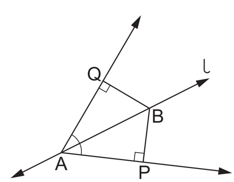
7. In the given figure, if $x = y$ and $AB = CB$ then prove that $AE = CD$.



8. In the given figure, line l is the bisector of an angle $\angle A$ and B is any point on l. If BP and BQ are perpendiculars from B to the arms of $\angle A$, show that

(i) $\triangle APB \cong \triangle AQB$

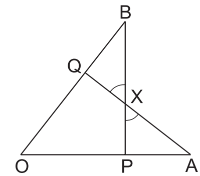
(ii) $BP = BQ$, i.e., B is equidistant from the arms of $\angle A$.



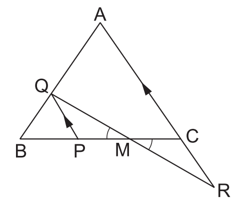
9. $ABCD$ is a quadrilateral such that diagonal AC bisects the angles $\angle A$ and $\angle C$. Prove that $AB = AD$ and $CB = CD$.

10. $\triangle ABC$ is a right triangle right angled at A such that $AB = AC$ and bisector of $\angle C$ intersects the side AB at D. Prove that $AC + AD = BC$.

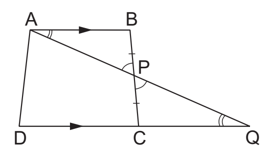
11. In the given figure, $OA = OB$ and $OP = OQ$. Prove that (i) $PX = QX$, (ii) $AX = BX$.



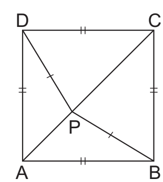
12. In the given figure, ABC is an equilateral triangle; $PQ \parallel AC$ and AC is produced to R such that $CR = BP$. Prove that QR bisects PC.



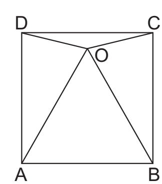
13. In the given figure, $ABCD$ is a quadrilateral in which $AB \parallel DC$ and P is the midpoint of BC. On producing, AP and DC meet at Q. Prove that (i) $AB = CQ$, (ii) $DQ = DC + AB$.



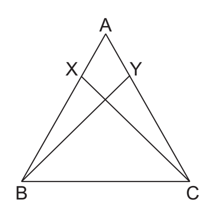
14. In the given figure, $ABCD$ is a square and P is a point inside it such that $PB = PD$. Prove that CPA is a straight line.



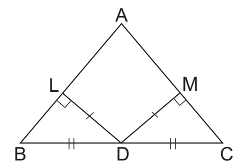
15. In the given figure, O is a point in the interior of square $ABCD$ such that $\triangle OAB$ is an equilateral triangle. Show that $\triangle OCD$ is an isosceles triangle.



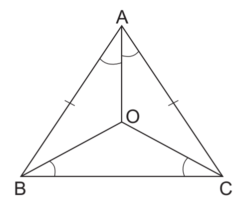
16. In the adjoining figure, X and Y are respectively two points on equal sides AB and AC of $\triangle ABC$ such that $AX = AY$. Prove that $CX = BY$.



17. In $\triangle ABC$, D is the midpoint of BC. If $DL \perp AB$ and $DM \perp AC$ such that $DL = DM$, prove that $AB = AC$.



18. In $\triangle ABC$, $AB = AC$ and the bisectors of $\angle B$ and $\angle C$ meet at a point O. Prove that $BO = CO$ and the ray AO is the bisector of $\angle A$.



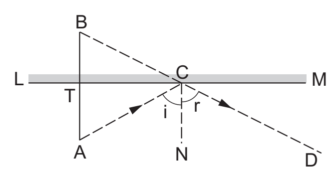
19. The line segments joining the midpoints M and N of parallel sides AB and DC respectively of a trapezium $ABCD$ is perpendicular to both the sides AB and DC. Prove that $AD = BC$.

20. The bisectors of $\angle B$ and $\angle C$ of an isosceles triangle with $AB = AC$ intersect each other at a point O. BO is produced to meet AC at a point M. Prove that $\angle MOC = \angle ABC$.

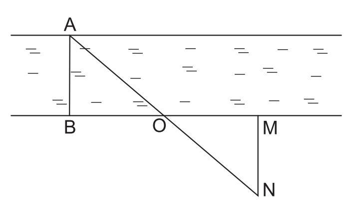
21. The bisectors of $\angle B$ and $\angle C$ of an isosceles $\triangle ABC$ with $AB = AC$ intersect each other at a point O. Show that the exterior angle adjacent to $\angle ABC$ is equal to $\angle BOC$.

22. P is a point on the bisector of $\angle ABC$. If the line through P, parallel to BA meets BC at Q, prove that $\triangle BPQ$ is an isosceles triangle.

23. The image of an object placed at a point A before a plane mirror LM is seen at the point B by an observer at D, as shown in the figure. Prove that the image is as far behind the mirror as the object is in front of the mirror.



24. In the adjoining figure, explain how one can find the breadth of the river without crossing it.



25. In a $\triangle ABC$, D is the midpoint of side AC such that $BD = \frac{1}{2}AC$. Show that $\angle ABC$ is a right angle.

26. "If two sides and an angle of one triangle are equal to two sides and an angle of another triangle then the two triangles must be congruent."
Is the statement true? Why?

27. "If two angles and a side of one triangle are equal to two angles and a side of another triangle then the two triangles must be congruent."
Is the statement true? Why?

HINTS TO SOME SELECTED QUESTIONS

5. $\triangle ABE \cong \triangle ACF$ as $BE = CF$, $\angle BAE = \angle CAF = \angle A$, $\angle AEB = \angle AFC = 90°$.

6. (i) $\triangle ABD \cong \triangle ACD$ as $AB = AC$, $BD = CD$, $AD = AD$.
$\therefore \quad \angle BAD = \angle CAD \Rightarrow \angle BAE = \angle CAE$.

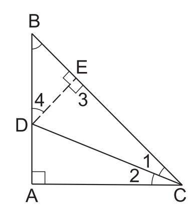
(ii) $\triangle ABE \cong \triangle ACE$ as $AB = AC$, $AE = AE$, $\angle BAE = \angle CAE$.
Similarly, $\triangle BDE \cong \triangle CDE$.

(iii) $\triangle ABE \cong \triangle ACE \Rightarrow \angle BAE = \angle CAE$ and $\angle BDE = \angle CDE$.

(iv) $BE = CE$ and $\angle AEB = \angle AEC = 90°$ as $\angle AEB + \angle AEC = 180°$.

7. $\triangle ABE \cong \triangle CBD$ as $AB = CB$, $\angle ABE = \angle CBD$.
And, $x = y \Rightarrow 180° - x = 180° - y \Rightarrow \angle CDB = \angle AEB$.

10. GIVEN A $\triangle ABC$ in which $AB = AC$, $\angle A = 90°$ and CD bisects $\angle C$.



TO PROVE $AC + AD = BC$.
CONSTRUCTION Draw $DE \perp BC$.
PROOF $\triangle DAC \cong \triangle DEC$ as $DC = DC$, $\angle 1 = \angle 2$, $\angle A = \angle 3 = 90°$.
$\therefore \quad DA = DE$ and $AC = EC$.
Now, $AB = AC \Rightarrow \angle B = \angle C$.
$$\angle A + \angle B + \angle C = 180° \Rightarrow 90° + \angle B + \angle B = 180° \Rightarrow \angle B = 45°.$$
In $\triangle BED$, $\angle 4 + \angle B = 90° \Rightarrow \angle 4 + 45° = 90° \Rightarrow \angle 4 = 45°$.
In $\triangle BDE$, $\angle 4 = \angle B \Rightarrow DE = BE \Rightarrow DA = DE = BE$.
Now, $BC = BE + EC = AD + AC$.

11. $\triangle OQA \cong \triangle OPB$ as $OQ = OP$, $OA = OB$, $\angle AOQ = \angle BOP$.
$\therefore \quad \angle A = \angle B$.
$\triangle AXP \cong \triangle BXQ$ as $\angle A = \angle B$, $\angle AXP = \angle BXQ$, $(OA - OP) = (OB - OQ)$.

12. Let QR intersect PC at M.
$\therefore \quad \angle BPQ = \angle BCA = 60°$ and $\angle B = 60°$.
$\therefore \quad PQ = BP = CR$.
Now, $\triangle PMQ \cong \triangle CMR$ and hence, $PM = MC$.

13. Show that $\triangle ABP \cong \triangle QCP$. So, $AB = CQ$, $DQ = DC + CQ = DC + AB$.

14. $\triangle PAD \cong \triangle PAB \Rightarrow \angle APD = \angle APB$.
$\triangle CPD \cong \triangle CPB \Rightarrow \angle CPD = \angle CPB$.
$\therefore \quad \angle APD + \angle CPD = \angle APB + \angle CPB$.
But, $\angle APD + \angle CPD + \angle APB + \angle CPB = 360°$ [$\angle$s around a point].
$\therefore \quad \angle APD + \angle CPD = 180°$, i.e., CPA is a straight line.

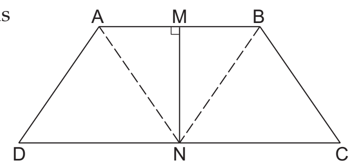
15. $\triangle OBC \cong \triangle OAD$ as $OB = OA$, $BC = AD$, $\angle OBC = \angle OAD = (90° - 60°) = 30°$.
Hence, $OC = OD$ (c.p.c.t.).

16. $\triangle AXC \cong \triangle AYB$ as $AX = AY$, $AC = AB$, $\angle A = \angle A$.

17. $\triangle BDL \cong \triangle CDM$ as $BD = CD$, $DL = DM$, $\angle BLD = \angle CMD = 90°$.
$\therefore$ $\angle B = \angle C$ and hence $AB = AC$.

18. $AB = AC \Rightarrow \angle B = \angle C \Rightarrow \frac{1}{2}\angle B = \frac{1}{2}\angle C \Rightarrow \angle OBC = \angle OCB \Rightarrow BO = CO$.
Now, $\triangle AOB \cong \triangle AOC$ as $AB = AC$, $BO = CO$, $AO = AO$.

19. Join NA and NB. Then, $\triangle AMN \cong \triangle BMN$ as $AM = BM$, $MN = MN$, $\angle AMN = \angle BMN = 90°$.

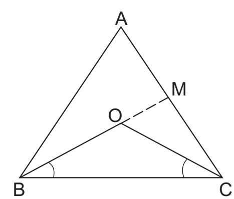


$\therefore$ $AN = BN$ and $\angle ANM = \angle BNM$.
So, $90° - \angle ANM = 90° - \angle BNM$
$\Rightarrow$ $\angle AND = \angle BNC$.
$\therefore$ $\triangle ADN \cong \triangle BCN$,
as $AN = BN$, $\angle AND = \angle BNC$ and $DN = CN$.
$\therefore$ $AD = BC$.

20. $AB = AC \Rightarrow \angle ACB = \angle ABC$

$$\Rightarrow \frac{1}{2}\angle ACB = \frac{1}{2}\angle ABC \Rightarrow \angle OCB = \angle OBC$$

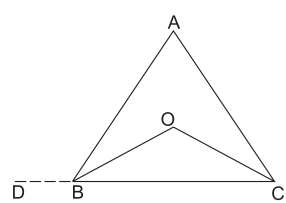
$\angle MOC = \angle OBC + \angle OCB = 2\angle OBC = \angle B = \angle ABC$.



21. $AB = AC \Rightarrow \angle ACB = \angle ABC$

$$\Rightarrow \frac{1}{2}\angle ACB = \frac{1}{2}\angle ABC$$

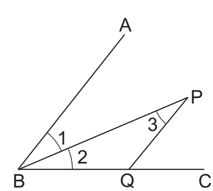
$$\Rightarrow \angle OCB = \angle OBC.$$



In $\triangle BOC$, we have

$\angle OBC + \angle OCB + \angle BOC = 180°$
$\Rightarrow$ $2\angle OBC + \angle BOC = 180°$
$\Rightarrow$ $\angle ABC + \angle BOC = 180°$
$\Rightarrow$ $180° - \angle DBA + \angle BOC = 180° \Rightarrow \angle DBA = \angle BOC$.

22. $BA \parallel PQ$ and BP is the transversal.
$\therefore$ $\angle 1 = \angle 3 \Rightarrow \angle 2 = \angle 3$ [$\because$ $\angle 1 = \angle 2$ (given)].
$\therefore$ $PQ = BQ$.



23. GIVEN A is an object in front of mirror LM and B is its image. Let AB cut LM at T.
TO PROVE $AT = BT$.
PROOF We know that angle of incidence is equal to the angle of reflection.
$\therefore$ $\angle i = \angle r \Rightarrow \angle ACN = \angle DCN$.
In $\triangle ACT$ and $\triangle BCT$, we have

$\angle ATC = \angle BTC = 90°$, $CT = CT$,

$\angle CAT = \angle ACN = i$ [alt. int. $\angle$s]

$\angle CBT = r$ [corr. $\angle$s]

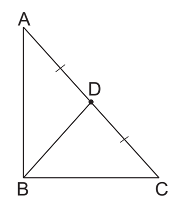
$\therefore$ $\angle CAT = \angle CBT$ [$\because$ $i = r$]

$\therefore$ $\triangle ACT \cong \triangle BCT$.

Hence, $AT = BT$.

24. Let the breadth of the river be AB.
    Mark a point M on the bank of the river at which B is situated.
    Let O be the midpoint of BM.
    From M move along $MN \perp BM$ to a point N such that A, O, N are in the same straight line.
    Then, $\triangle OBA \cong \triangle OMN$ and MN is the required breadth of the river.

25. GIVEN A $\triangle ABC$ in which $AD = CD$ and $BD = \frac{1}{2}AC$.



TO PROVE $\angle ABC = 90°$.

PROOF $BD = \frac{1}{2}AC$ and $AD = CD = \frac{1}{2}AC$

$\Rightarrow$ $AD = CD = BD$

$\Rightarrow$ $AD = BD$ and $BD = CD$

$\Rightarrow$ $\angle BAD = \angle ABD$ and $\angle BCD = \angle CBD$

$\Rightarrow$ $\angle BAC = \angle ABD$ and $\angle BCA = \angle CBD$.

In $\triangle ABC$, we have

$\angle ABC + \angle BAC + \angle BCA = 180°$

$\Rightarrow$ $\angle ABC + \angle ABD + \angle CBD = 180°$

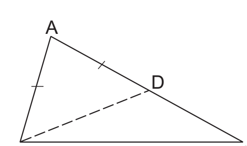
$\Rightarrow$ $2\angle ABC = 180°$ [$\because$ $\angle ABD + \angle CBD = \angle ABC$]

$\Rightarrow$ $\angle ABC = 90°$.

26. The given statement is not true.
    It must be two sides and the included angle.
27. The given statement is not true.
    The sides must be corresponding sides.

INEQUALITIES IN A TRIANGLE

THEOREM 1 *If two sides of a triangle are unequal, prove that the angle opposite to the longer side is greater.*



GIVEN A $\triangle ABC$ in which $AC > AB$.

TO PROVE $\angle ABC > \angle BCA$.

CONSTRUCTION Mark a point D on AC such that $AD = AB$. Join BD.

PROOF We know that in a triangle, the angles opposite to equal sides are equal.

So, in $\triangle ABD$, we have

$$AB = AD = \angle BDA = \angle ABD. \quad \ldots \text{(i)}$$

Now, in $\triangle BCD$, side CD has been produced to A, forming exterior angle $\angle BDA$.

$\therefore \quad \angle BDA > \angle BCD$ [exterior angle is greater than int. opp. angle]

$\Rightarrow \quad \angle BDA = \angle BCA$ [$\because \angle BCD = \angle BCA$]

$\Rightarrow \quad \angle ABD > \angle BCA$ [using (i)]

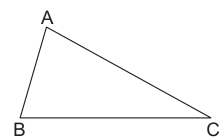
$\Rightarrow \quad \angle ABC > \angle ABD > \angle BCA$ [$\because \angle ABC > \angle ABD$]

$\Rightarrow \quad \angle ABC > \angle BCA$.

Hence, $\angle ABC > \angle BCA$.

THEOREM 2 *In any triangle, prove that the side opposite to the greater angle is longer.*

GIVEN A $\triangle ABC$ in which $\angle ABC > \angle ACB$.



TO PROVE $AC > AB$.

PROOF We have the following possibilities only.

(i) $AC = AB$ (ii) $AC < AB$ (iii) $AC > AB$

Out of these possibilities, exactly one must be true.

CASE I

If possible, let $AC = AB$.

We know that the angles opposite to equal sides of a triangle are equal.

$\therefore \quad AC = AB \Rightarrow \angle ABC = \angle ACB$.

This contradicts the given hypothesis that $\angle ABC > \angle ACB$.

$\therefore \quad AC \neq AB$.

CASE II

If possible, let $AC < AB$. Then, $AB > AC$.

Since the angle opposite to the longer side is larger, so

$$AB > AC \Rightarrow \angle ACB > \angle ABC.$$

This contradicts the given hypothesis that $\angle ABC > \angle ACB$.

$\therefore \quad AC$ cannot be less than AB.

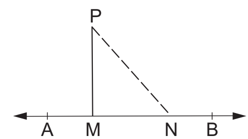
CASE III

Now, we are left with the only possibility that $AC > AB$, which must be true.

Hence, $AC > AB$.

THEOREM 3 *Prove that, of all the line segments that can be drawn to a given line, from a point not lying on it, the perpendicular line segment is the shortest.*

GIVEN A line AB and a point P outside it. $PM \perp AB$ and N is a point, other than M, on AB.



TO PROVE $PM < PN$.

PROOF In $\triangle PMN$, we have $\angle M = 90°$.

But, in a right-angled triangle, each one of the angles other than the right angle is an acute angle.

$\therefore \quad \angle PNM < \angle PMN$.

But, the side opposite to the smaller angle in a triangle is shorter.

$\therefore \quad PM < PN$.

Hence, the perpendicular from P to the given line is shortest of all line segments from P to AB.

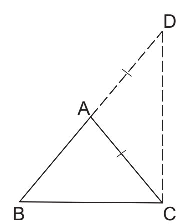
DISTANCE BETWEEN A LINE AND A POINT

The distance between a line and a point, not on it, is the length of perpendicular from the point to the given line.

NOTE The distance between a line and a point lying on it, is zero.

THEOREM 4 *Prove that the sum of any two sides of a triangle is greater than the third side.*

GIVEN A $\triangle ABC$.



TO PROVE (i) $AB + AC > BC$,

(ii) $AB + BC > AC$,

(iii) $BC + AC > AB$.

CONSTRUCTION Produce BA to D such that $AD = AC$. Join CD.

PROOF (i) In $\triangle ACD$, we have

$AD = AC$ [by construction]

$\Rightarrow \angle ACD = \angle ADC$ [$\angle$s opposite to equal sides]

$\Rightarrow \angle BCD > \angle ADC$ [$\because \angle BCD > \angle ACD$]

$\Rightarrow \angle BCD > \angle BDC$ [$\because \angle ADC = \angle BDC$]

$\Rightarrow BD > BC$ [side opposite to larger angle is larger]

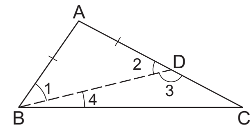
$\Rightarrow BA + AD > BC$

$\Rightarrow BA + AC > BC$ [$\because AD = AC$].

$\therefore \quad AB + AC > BC$.

Similarly, $AB + BC > AC$ and $BC + AC > AB$.

THEOREM 5 *Prove that the difference between any two sides of a triangle is less than its third side.*



GIVEN A $\triangle ABC$.

TO PROVE (i) $AC - AB < BC$,

(ii) $BC - AC < AB$,

(iii) $BC - AB < AC$.

CONSTRUCTION Let $AC > AB$. Then, along AC, set off $AD = AB$. Join BD.

PROOF $AB = AD \Rightarrow \angle 1 = \angle 2$. ... (i)

Side CD of $\triangle BCD$ has been produced to A.

$\therefore \quad \angle 2 > \angle 4$ ... (ii) [$\because$ ext. angle > each int. opp. angle].

Side AD of $\triangle ABD$ has been produced to C.

$\therefore \quad \angle 3 > \angle 1$ ... (iii) [ext. angle > each int. opp. angle]

$\Rightarrow \quad \angle 3 > \angle 2$ ... (iv) [using (i)].

From (ii) and (iv), we get $\angle 3 > \angle 4 \Rightarrow \angle 4 < \angle 3$.

Now, $\angle 4 < \angle 3$

$\Rightarrow \quad CD < BC$

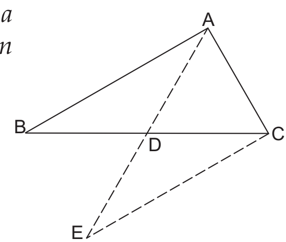
$\Rightarrow \quad AC - AD < BC$

$\Rightarrow \quad AC - AB < BC \quad [\because AD = AB]$.

Hence, $AC - AB < BC$.

Similarly, $BC - AC < AB$ and $BC - AB < AC$.

THEOREM 6 *Prove that the sum of any two sides of a triangle is greater than twice the median drawn to the third side.*



GIVEN A $\triangle ABC$ in which AD is a median.

TO PROVE $AB + AC > 2AD$.

CONSTRUCTION Produce AD to E such that $AD = DE$. Join EC.

PROOF In $\triangle ADB$ and $\triangle EDC$, we have

$AD = DE$ (by construction)

$\angle ADB = \angle EDC$ (vert. opp. angles)

$BD = CD$ ($\because$ D is the midpoint of BC)

$\therefore \quad \triangle ADB \cong \triangle EDC$ (by SAS-criteria).

$\therefore \quad AB = EC$ (c.p.c.t.).

We know that the sum of any two sides of a triangle is greater than the third side. So, in $\triangle ACE$, we have

$$EC + AC > AE.$$

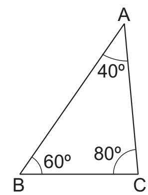
$\therefore \quad AB + AC > 2AD \quad [\because EC = AB \text{ and } AE = 2AD]$.

Hence, $AB + AC > 2AD$.

SOLVED EXAMPLES

EXAMPLE 1 *In a $\triangle ABC$, if $\angle A = 40°$ and $\angle B = 60°$ then which side of the triangle is longest and which is shortest?*

SOLUTION Here, $\angle A = 40°$ and $\angle B = 60°$.



$\therefore \quad \angle C = 180° - (40° + 60°) = 80°$.

Thus, $\angle C$ is largest and $\angle A$ is smallest.

So, the side opposite to $\angle C$ is longest.

Hence, the longest side is AB.

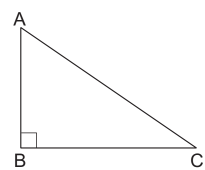
Also, the side opposite to $\angle A$ is shortest.

$\therefore \quad BC$ is the shortest side.

EXAMPLE 2 *In a right-angled triangle, prove that the hypotenuse is the longest side.*

SOLUTION Let ABC be a right triangle in which $\angle B = 90°$.

Then, $\angle A + \angle C = 90°$.



$\therefore \quad \angle B > \angle A$ and $\angle B > \angle C$

$\Rightarrow \quad AC > BC$ and $AC > AB$

[side opp. to larger angle is longer].

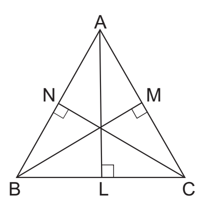
$\therefore \quad AC$ is the longest side.

Hence, in a right triangle, the hypotenuse is the longest side.

EXAMPLE 3 *Show that the sum of three altitudes of a triangle is less than the sum of the three sides of the triangle.*

SOLUTION Let AL, BM, CN be the three altitudes of $\triangle ABC$.

We know that, of all line segments drawn from a point outside a line to the line, the perpendicular is the shortest.

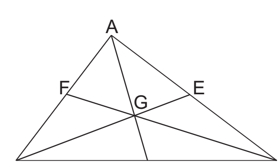


$\therefore \quad AL < AB$, $BM < BC$ and $CN < AC$.

$\therefore \quad AL + BM + CN < AB + BC + AC$.

EXAMPLE 4 *Prove that the perimeter of a triangle is greater than the sum of its three medians.*

SOLUTION Let AD, BE and CF be the three medians of a $\triangle ABC$.



We know that the sum of any two sides of a triangle is greater than twice the median drawn to the third side.

$\therefore \quad AB + AC > 2AD,$ ... (i)

$AB + BC > 2BE$ ... (ii)

and $BC + AC > 2CF.$ ... (iii)

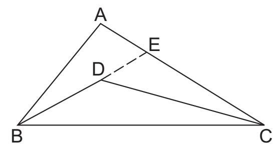
Adding, the corresponding sides of (i), (ii) and (iii), we get

$$2(AB + BC + AC) > 2(AD + BE + CF)$$

$\therefore \quad (AB + BC + AC) > (AD + BE + CF).$

Hence, the perimeter of a triangle is greater than the sum of its three medians.

EXAMPLE 5 *In the adjoining figure, ABC is a triangle and D is any point in its interior. Show that* $(BD + DC) < (AB + AC)$.



SOLUTION Produce BD to meet AC at E.

We know that in a triangle, the sum of any two sides is always greater than the third side.

In $\triangle ABE$, we have

$AB + AE > BE \Rightarrow AB + AE > BD + DE.$ ... (i)

In $\triangle CDE$, we have

$DE + EC > DC.$ ... (ii)

From (i) and (ii), we get

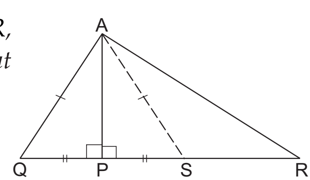
$$AB + AE + DE + EC > BD + DE + DC$$

$\Rightarrow \quad AB + (AE + EC) > BD + DC$

$\Rightarrow \quad (AB + AC) > (BD + DC).$

Hence, $(BD + DC) < (AB + AC)$.

EXAMPLE 6 *In the given figure,* $AP \perp QR$, $PR > PQ$ *and* $PQ = PS$. *Show that* $AR > AQ$.



SOLUTION In $\triangle APQ$ and $\triangle APS$, we have

$PQ = PS$ (given)

$\angle APQ = \angle APS$ (each equal to 90°)

$AP = AP$ (common)

$\therefore \quad \triangle APQ \cong \triangle APS$ (SAS-criteria).

$\therefore \quad \angle AQP = \angle ASP \Rightarrow \angle AQS = \angle ASQ.$ ... (i)

Now, side RS of $\triangle ARS$ has been produced to Q.

$\therefore \quad \angle ASQ > \angle ARS \quad$ [ext. angle > int. opp. angle]

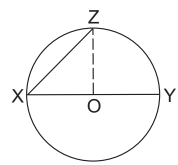
$\Rightarrow \quad \angle AQS > \angle ARS \quad$ [using (i)]

$\Rightarrow \quad \angle AQR > \angle ARQ$

$\Rightarrow \quad AR > AQ \quad$ [the side opposite to greater angle is larger].

Hence, $AR > AQ$.

EXAMPLE 7 *In the given figure, O is the centre of the circle and XOY is a diameter. If XZ is any other chord, show that XY > XZ.*



GIVEN A circle with centre O in which XOY is a diameter and XZ is another chord.

TO PROVE $XY > XZ$.

CONSTRUCTION Join OZ.

PROOF We know that in a triangle, the sum of any two sides is always greater than the third side.

So, in $\triangle XOZ$, we have

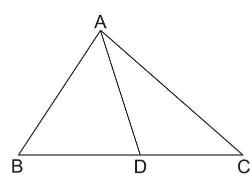
$XO + OZ > XZ$

$\Rightarrow \quad XO + OY > XZ \quad [\because\ OZ = OY = \text{radius of the circle}]$

$\Rightarrow \quad XY > XZ \quad [\because\ XO + OY = XY]$

Hence, $XY > XZ$.

EXAMPLE 8 *In $\triangle ABC$, if D is any point on the side BC, show that $(AB + BC + AC) > 2AD$.*



GIVEN A $\triangle ABC$ in which D is a point on BC and AD is drawn.

TO PROVE $(AB + BC + AC) > 2AD$.

PROOF We know that in a triangle, the sum of any two sides is always greater than the third side.

So, in $\triangle ABD$, we have $AB + BD > AD$. … (i)

And, in $\triangle ACD$, we have $AC + CD > AD$. … (ii)

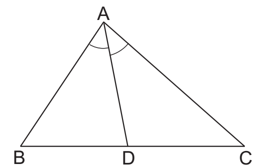
Adding the corresponding sides of (i) and (ii), we get

$AB + (BD + CD) + AC > 2AD$

$\Rightarrow \quad (AB + BC + AC) > 2AD \quad [\because\ BD + CD = BC]$.

Hence, $(AB + BC + AC) > 2AD$.

EXAMPLE 9 *In $\triangle ABC$, if AD is the bisector of $\angle A$, show that $AB > BD$ and $AC > DC$.*



GIVEN A $\triangle ABC$ in which AD is the bisector of $\angle A$.

TO PROVE $AB > BD$ and $AC > DC$.

PROOF In $\triangle ACD$, side CD has been produced to B.

$\therefore$ ext. $\angle ADB >$ int. opp. $\angle CAD$

$\Rightarrow$ $\angle ADB > \angle CAD$

$\Rightarrow$ $\angle ADB > \angle BAD$ [$\because$ $\angle CAD = \angle BAD$]

$\Rightarrow$ $AB > BD$ [side opposite to larger angle is larger].

Again, in $\triangle ABD$, the side BD has been produced to C.

$\therefore$ ext. $\angle ADC >$ int. opp. $\angle BAD$

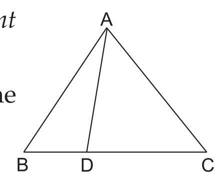
$\Rightarrow$ $\angle ADC > \angle BAD$

$\Rightarrow$ $\angle ADC > \angle CAD$ [$\because$ $\angle BAD = \angle CAD$]

$\Rightarrow$ $AC > DC$ [side opposite to larger angle is larger].

Hence, $AB > BD$ and $AC > DC$.

EXAMPLE 10 *In the given figure, $AB > AC$. If D is any point on BC, show that $AB > AD$.*



SOLUTION We know that the angle opposite to the larger side is larger.

$\therefore$ $AB > AC \Rightarrow \angle ACB > \angle ABC$

$\Rightarrow \angle ACD > \angle ABD.$ ... (i)

Again, side CD of $\triangle ACD$ has been produced to B.

$\therefore$ ext. $\angle ADB > \angle ACD.$ ... (ii)

From (i) and (ii), we get

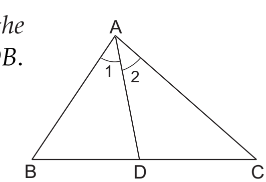
$\angle ADB > \angle ACD > \angle ABD$

$\Rightarrow$ $\angle ADB > \angle ABD$

$\Rightarrow$ $AB > AD$ [side opposite to larger angle is larger].

Hence, $AB > AD$.

EXAMPLE 11 *In the given figure, $AC > AB$ and AD is the bisector of $\angle A$. Show that $\angle ADC > \angle ADB$.*



SOLUTION $AC > AB$

$\Rightarrow$ $\angle B > \angle C$

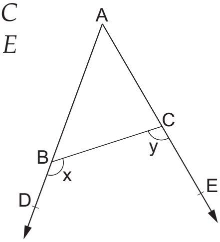
$\Rightarrow$ $\angle B + \angle 1 > \angle C + \angle 2$ [$\because$ $\angle 1 = \angle 2$]

$\Rightarrow$ $\angle ADC > \angle ADB$

[$\because$ $\angle B + \angle 1 = \angle ADC$ and $\angle C + \angle 2 = \angle ADB$].

Hence, $\angle ADC > \angle ADB$.

EXAMPLE 12 *In the given figure, the sides AB and AC of $\triangle ABC$ have been extended to D and E respectively. If $x > y$, show that $AB > AC$.*



SOLUTION $x > y \Rightarrow -x < -y$

$\Rightarrow (180 - x) < (180 - y)$

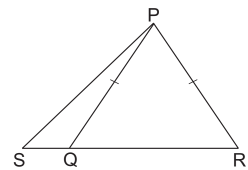
$\Rightarrow \angle ABC < \angle ACB$

$\Rightarrow \angle ACB > \angle ABC$

$\Rightarrow AB > AC$ [side opposite to larger angle is larger].

Hence, $AB > AC$.

EXAMPLE 13 *In the given figure, Q is a point on the side SR of $\triangle PSR$ such that $PQ = PR$. Prove that $PS > PQ$.*



SOLUTION We know that the angles opposite to equal sides are equal.

$\therefore \quad PQ = PR \Rightarrow \angle PQR = \angle PRQ.$ … (i)

In $\triangle PSQ$, the side SQ has been produced to R.

$\therefore \quad \angle PQR > \angle PSQ$ [exterior angle is greater than int. opp. $\angle$s]

$\Rightarrow \quad \angle PRQ > \angle PSQ$ [using (i)]

$\Rightarrow \quad \angle PRS > \angle PSR$ [$\because \angle PRQ = \angle PRS$ and $\angle PSQ = \angle PSR$]

$\Rightarrow \quad PS > PR$

$\Rightarrow \quad PS > PQ$ [$\because PQ = PR$].

Hence, $PS > PQ$.

EXERCISE 9B

1. Is it possible to construct a triangle with lengths of its sides as given below? Give reason for your answer.

(i) 5 cm, 4 cm, 9 cm (ii) 8 cm, 7 cm, 4 cm

(iii) 10 cm, 5 cm, 6 cm (iv) 2.5 cm, 5 cm, 7 cm

(v) 3 cm, 4 cm, 8 cm

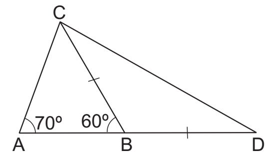
2. In $\triangle ABC$, $\angle A = 50°$ and $\angle B = 60°$. Determine the longest and shortest sides of the triangle.

3. (i) In $\triangle ABC$, $\angle A = 90°$. Which is its longest side?

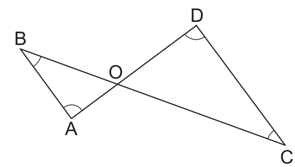
(ii) In $\triangle ABC$, $\angle A = \angle B = 45°$. Which is its longest side?

(iii) In $\triangle ABC$, $\angle A = 100°$ and $\angle C = 50°$. Which is its shortest side?

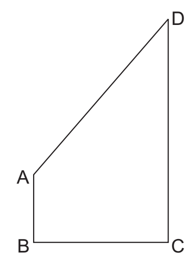
4. In $\triangle ABC$, side AB is produced to D such that $BD = BC$. If $\angle A = 70°$ and $\angle B = 60°$, prove that (i) $AD > CD$ (ii) $AD > AC$.



5. In the given figure, $\angle B < \angle A$ and $\angle C < \angle D$. Show that $AD < BC$.



6. AB and CD are respectively the smallest and largest sides of a quadrilateral $ABCD$. Show that $\angle A > \angle C$ and $\angle B > \angle D$.



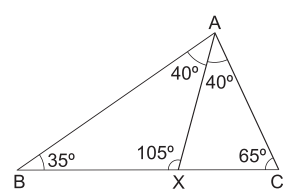
7. In a quadrilateral $ABCD$, show that

$$(AB + BC + CD + DA) > (AC + BD).$$

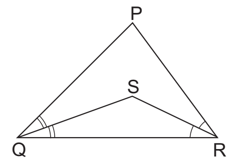
8. In a quadrilateral $ABCD$, show that

$$(AB + BC + CD + DA) < 2(BD + AC).$$

9. In $\triangle ABC$, $\angle B = 35°$, $\angle C = 65°$ and the bisector of $\angle BAC$ meets BC in X. Arrange AX, BX and CX in descending order.



10. In the given figure, $PQ > PR$ and QS and RS are the bisectors of $\angle Q$ and $\angle R$ respectively. Show that $SQ > SR$.



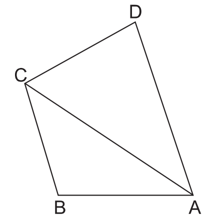
11. D is any point on the side AC of $\triangle ABC$ with $AB = AC$. Show that $CD < BD$.

12. Prove that in a triangle, other than an equilateral triangle, angle opposite the longest side is greater than $\frac{2}{3}$ of a right angle.

13. In the given figure, prove that

(i) $CD + DA + AB > BC$

(ii) $CD + DA + AB + BC > 2AC$.



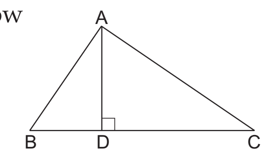
14. If O is a point within $\triangle ABC$, show that

(i) $AB + AC > OB + OC$

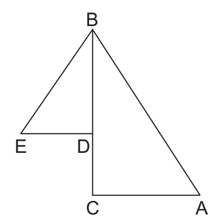
(ii) $AB + BC + CA > OA + OB + OC$

(iii) $OA + OB + OC > \frac{1}{2}(AB + BC + CA)$.

15. In the given figure, $AD \perp BC$ and $CD > BD$. Show that $AC > AB$.



16. In the given figure, D is a point on side BC of a $\triangle ABC$ and E is a point such that $CD = DE$. Prove that $AB + AC > BE$.



ANSWERS (EXERCISE 9B)

1. (i) No (ii) Yes (iii) Yes (iv) Yes (v) No

2. Longest side is AB, shortest side is BC

3. (i) BC (ii) AB (iii) AC

9. $BX > AX > CX$

HINTS TO SOME SELECTED QUESTIONS

1. For construction of a triangle, the sum of two sides must be greater than the third side.

4. $\angle ACB = 180^\circ - (70^\circ + 60^\circ) = 50^\circ$. Let $\angle BCD = \angle BDC = x^\circ$.

   $\angle CBD = (180^\circ - 60^\circ) = 120^\circ$.

   $\therefore \quad 120 + x + x = 180 \Rightarrow x = 30$.

5. In $\triangle OAB$, $\angle B < \angle A \Rightarrow OA < OB$.

   In $\triangle OCD$, $\angle C < \angle D \Rightarrow OD < OC$.

   $\therefore \quad (OA + OD) < (OB + OC) \Rightarrow AD < BC$.

6. Join AC. In $\triangle ABC$, we have

$$AB < BC \Rightarrow BC > AB$$
$$\Rightarrow \angle BAC > \angle ACB. \quad \ldots \text{(i)}$$

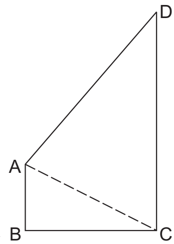
In $\triangle ACD$, we have

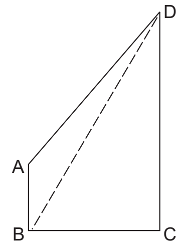
$$CD > AD \Rightarrow \angle CAD > \angle ACD. \quad \ldots \text{(ii)}$$

$\therefore \quad (\angle BAC + \angle CAD) > (\angle ACB + \angle ACD)$

$\Rightarrow \quad \angle A > \angle C.$

Similarly, by joining B and D, we may prove that $\angle B > \angle D$.





7. Join AC. Then,

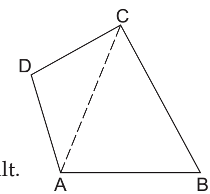
$$AB + BC > AC \text{ and } CD + DA > AC.$$

$\therefore \quad (AB + BC + CD + DA) > 2AC. \quad \ldots$ (i)

Similarly, by joining BD, we get

$$(AB + BC + CD + DA) > 2BD. \quad \ldots \text{(ii)}$$

Adding respective sides of (i) and (ii), we get the required result.



8. $OA + OB > AB, OB + OC > BC,$

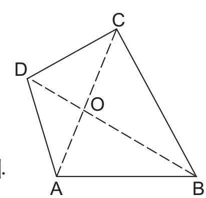
$OC + OD > CD, OD + OA > DA.$

Adding, we get

$$2(OA + OC + OB + OD) > (AB + BC + CD + DA)$$

$\Rightarrow \quad 2(AC + BD) > (AB + BC + CD + DA)$

$[\because \; OA + OC = AC, OB + OD = BD].$



9. $\angle A + 35^\circ + 65^\circ = 180^\circ \Rightarrow \angle A = 80^\circ.$

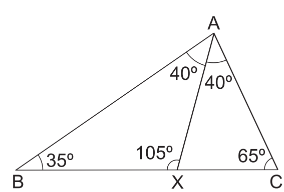
$\therefore \quad \angle BAX = \angle CAX = 40^\circ, \angle ABX = 35^\circ, \angle ACX = 65^\circ.$

$\angle AXB = 180^\circ - (40^\circ + 35^\circ) = 105^\circ.$

$\angle BAX > \angle ABX \Rightarrow BX > AX.$

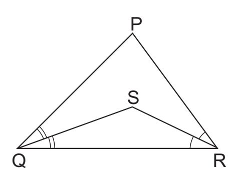
$\angle ACX > \angle CAX \Rightarrow AX > CX.$

$\therefore \quad BX > AX > CX.$



10. $PQ > PR \Rightarrow \angle PRQ > \angle PQR$

$$\Rightarrow \frac{1}{2}\angle PRQ > \frac{1}{2}\angle PQR$$
$$\Rightarrow \angle SRQ > \angle SQR$$
$$\Rightarrow SQ > SR.$$



12. Let AB be the longest side. Then,

$$AB > BC \text{ and } AB > CA$$

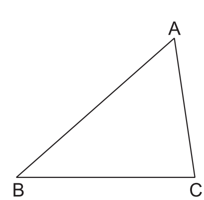
$\Rightarrow \quad \angle C > \angle A$ and $\angle C > \angle B$

[$\because$ angle opposite to longer side is larger]

$\Rightarrow \quad 2\angle C > (\angle A + \angle B)$

$\Rightarrow \quad 3\angle C > (\angle A + \angle B + \angle C)$

$\Rightarrow \quad 3\angle C > 180^\circ$



$\Rightarrow \quad \angle C > 60°$

$\Rightarrow \quad \angle C > \left(\frac{2}{3} \text{ of a right angle}\right).$

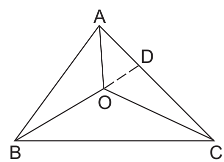
13. (i) $CD + DA > CA \Rightarrow (CD + DA) + AB > (CA + AB)$

$\Rightarrow (CD + DA + AB) > CB.$

(ii) $(CD + DA) > CA$ and $(AB + BC) > AC$

$\Rightarrow \quad (CD + DA + AB + BC) > 2AC.$

14. (i) In $\triangle ABD$, we have



$AB + AD > BD \Rightarrow AB + AD > OB + OD.$ ... (i)

In $\triangle ODC$, we have $OD + DC > OC.$ ... (ii)

$\therefore \quad AB + AD + OD + DC > OB + OD + OC$ [adding (i) and (ii)]

$\Rightarrow \quad AB + AD + DC > OB + OC$

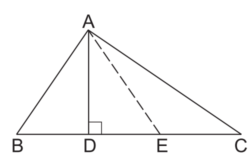
$\Rightarrow \quad AB + AC > OB + OC \quad [\because AD + DC > AC].$

(ii) $AB + AC > OB + OC.$
Similarly, $BC + BA > OC + OA$ and $CA + CB > OA + OB.$
Adding, we get $(AB + BC + CA) > (OA + OB + OC).$

(iii) In $\triangle OAB$, $\triangle OBC$ and $\triangle OCA$, we have
$OA + OB > AB$, $OB + OC > BC$ and $OC + OA > CA.$

$\therefore \quad (OA + OB + OC) > \frac{1}{2}(AB + BC + CA).$

15. Set off $DE = BD$. Join AE.



$\triangle ADB \cong \triangle ADE$ [SAS-congruence]
as $BD = ED$, $\angle ADB = \angle ADE = 90°$ and $AD = AD.$

$\therefore \quad AB = AE \Rightarrow \angle ABE = \angle AED > \angle C$

$\Rightarrow \angle B > \angle C \Rightarrow AC > AB.$

16. $AB + AC > BC$

$\Rightarrow \quad AB + AC > BD + DC = BD + DE > BE.$

$\therefore \quad (AB + AC) > BE.$

MULTIPLE-CHOICE QUESTIONS (MCQ)

Choose the correct answer in each of the following:

1. Which of the following is not a criterion for congruence of triangles?
(a) SSA (b) SAS (c) ASA (d) SSS

2. If $AB = QR$, $BC = RP$ and $CA = PQ$ then which of the following holds?
(a) $\triangle ABC \cong \triangle PQR$ (b) $\triangle CBA \cong \triangle PQR$
(c) $\triangle CAB \cong \triangle PQR$ (d) $\triangle BCA \cong \triangle PQR$

3. If $\triangle ABC \cong \triangle PQR$ then which of the following is not true?

(a) $BC = PQ$ (b) $AC = PR$ (c) $BC = QR$ (d) $AB = PQ$

4. In $\triangle ABC$, $AB = AC$ and $\angle B = 50°$. Then, $\angle A = ?$

(a) 40° (b) 50° (c) 80° (d) 130°

5. In $\triangle ABC$, $BC = AB$ and $\angle B = 80°$. Then, $\angle A = ?$

(a) 50° (b) 40° (c) 100° (d) 80°

6. In $\triangle ABC$, $\angle C = \angle A$, $BC = 4$ cm and $AC = 5$ cm. Then, $AB = ?$

(a) 4 cm (b) 5 cm (c) 8 cm (d) 2.5 cm

7. Two sides of a triangle are of length 4 cm and 2.5 cm. The length of the third side of the triangle cannot be

(a) 6 cm (b) 6.5 cm (c) 5.5 cm (d) 6.3 cm

8. In $\triangle ABC$, if $\angle C > \angle B$, then

(a) $BC > AC$ (b) $AB > AC$ (c) $AB < AC$ (d) $BC < AC$

9. It is given that $\triangle ABC \cong \triangle FDE$ in which $AB = 5$ cm, $\angle B = 40°$, $\angle A = 80°$ and $FD = 5$ cm. Then, which of the following is true?

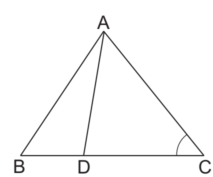
(a) $\angle D = 60°$ (b) $\angle E = 60°$ (c) $\angle F = 60°$ (d) $\angle D = 80°$

10. In $\triangle ABC$, $\angle A = 40°$ and $\angle B = 60°$. Then, the longest side of $\triangle ABC$ is

(a) BC (b) AC

(c) AB (d) cannot be determined

11. In the given figure, $AB > AC$. Then, which of the following is true?



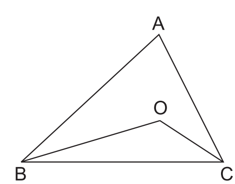
(a) $AB < AD$

(b) $AB = AD$

(c) $AB > AD$

(d) Cannot be determined

12. In the given figure, $AB > AC$. If BO and CO are the bisectors of $\angle B$ and $\angle C$ respectively then

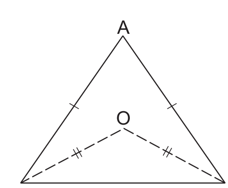


(a) $OB = OC$

(b) $OB > OC$

(c) $OB < OC$

13. In the given figure, $AB = AC$ and $OB = OC$. Then, $\angle ABO : \angle ACO = ?$



(a) 1 : 1

(b) 2 : 1

(c) 1 : 2

(d) none of these

14. If the altitudes from two vertices of a triangle to the opposite sides are equal then the triangle is

(a) equilateral (b) isosceles

(c) scalene (d) right angled

15. In $\triangle ABC$ and $\triangle DEF$, it is given that $AB = DE$ and $BC = EF$. In order that $\triangle ABC \cong \triangle DEF$, we must have

(a) $\angle A = \angle D$ (b) $\angle B = \angle E$ (c) $\angle C = \angle F$ (d) none of these

16. In $\triangle ABC$ and $\triangle DEF$, it is given that $\angle B = \angle E$ and $\angle C = \angle F$. In order that $\triangle ABC \cong \triangle DEF$, we must have

(a) $AB = DF$ (b) $AC = DE$ (c) $BC = EF$ (d) $\angle A = \angle D$

17. In $\triangle ABC$ and $\triangle PQR$, it is given that $AB = AC$, $\angle C = \angle P$ and $\angle B = \angle Q$. Then, the two triangles are

(a) isosceles but not congruent

(b) isosceles and congruent

(c) congruent but not isosceles

(d) neither congruent nor isosceles

18. Which is true?

(a) A triangle can have two right angles.

(b) A triangle can have two obtuse angles.

(c) A triangle can have two acute angles.

(d) An exterior angle of a triangle is less than either of the interior opposite angles.

19. Fill in the blanks with < or >.

(a) (Sum of any two sides of a triangle) ...... (the third side).

(b) (Difference of any two sides of a triangle) ...... (the third side).

(c) (Sum of three altitudes of a triangle) ...... (sum of its three sides).

(d) (Sum of any two sides of a triangle) ...... (twice the median to the 3rd side).

(e) (Perimeter of a triangle) ...... (sum of its three medians).

20. Fill in the blanks.

(a) Each angle of an equilateral triangle measures ....

(b) Medians of an equilateral triangle are ....

(c) In a right triangle, the hypotenuse is the ...... side.

(d) Drawing a $\triangle ABC$ with $AB = 3$ cm, $BC = 4$ cm and $CA = 7$ cm is ....

ANSWERS (MCQ)

1. (a) **2.** (c) **3.** (a) **4.** (c) **5.** (a) **6.** (a) **7.** (b) **8.** (b)
9. (b) **10.** (c) **11.** (c) **12.** (b) **13.** (a) **14.** (b) **15.** (b) **16.** (c)
17. (a) **18.** (c) **19.** (a) > (b) < (c) < (d) > (e) >
20. (a) 60° (b) equal (c) longest (d) not possible

HINTS TO SOME SELECTED QUESTIONS

1. SSA is not a criterion for congruence of triangles.

2. Clearly, $\begin{matrix} A & B & C \\ \downarrow & \downarrow & \downarrow \\ Q & R & P \end{matrix}$

   So, $\triangle CAB \cong \triangle PQR$.

3. $\triangle ABC \cong \triangle PQR \Rightarrow BC = QR$.

   $\therefore \quad BC = PQ$ is not true.

4. $AB = AC \Rightarrow \angle C = \angle B = 50°$.

   So, $\angle A + 50° + 50° = 180° \Rightarrow \angle A = 80°$.

5. $BC = AB \Rightarrow \angle A = \angle C = x°$ (say).

   Then, $x + 80 + x = 180 \Rightarrow 2x = 100 \Rightarrow x = 50$.

6. $\angle C = \angle A \Rightarrow AB = BC = 4$ cm.

7. Sum of two sides must be greater than the third side.

   So, the third side cannot be 6.5 cm.

8. $\angle C > \angle B \Rightarrow AB > AC$.

9. Given, $\triangle ABC \cong \triangle FDE$.

   $AB = FD = 5$ cm, $\angle B = 40°$ and $\angle A = 80°$.

   $\therefore \quad \angle C = 180° - (80° + 40°) = 60°$.

   So, we must have $\angle E = \angle C = 60°$.

10. $\angle C = 180° - (40° + 60°) = 80°$. So, the longest side is AB.

11. $AB > AC \Rightarrow \angle ACB > \angle ABC$.

    Ext. $\angle ADB > \angle ACD \Rightarrow \angle ADB > \angle ACB > \angle ABC$

    $\Rightarrow \angle ADB > \angle ABD \Rightarrow AB > AD$.

12. $AB > AC \Rightarrow \angle C > \angle B \Rightarrow \frac{1}{2}\angle C > \frac{1}{2}\angle B$

    $\Rightarrow \angle OCB > \angle OBC \Rightarrow OB > OC$.

13. Join OA.

    In $\triangle OAB$ and $\triangle OAC$, we have

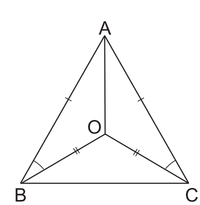
    $AB = AC$ (given),

    $OB = OC$ (given)

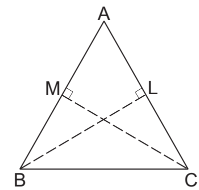
    and $OA = OA$.

    $\therefore \quad \triangle OAB \cong \triangle OAC \Rightarrow \angle ABO = \angle ACO$.

    $\therefore \quad \angle ABO : \angle ACO = 1 : 1$.



14. A $\triangle ABC$ is given in which $BL \perp AC$ and $CM \perp AB$ such that $BL = CM$. Then, we have to prove that $AB = AC$.



In $\triangle ABL$ and $\triangle ACM$, we have:

$BL = CM$ (given),

$\angle BAL = \angle CAM$ (common),

$\angle ALB = \angle AMC$ (each 90°)

$\triangle ABL \cong \triangle ACM$ and hence $AB = AC$.

$\triangle ABC$ is isosceles.

15. For congruence, we must have $\angle B = \angle E$.

16. For congruence, we must have $BC = EF$.

17. $AB = AC \Rightarrow \angle C = \angle B \Rightarrow \angle P = \angle Q$ [$\because \angle C = \angle P$ and $\angle B = \angle Q$]

$\Rightarrow QR = PQ$.

Thus, both the triangles are isosceles but not congruent.

SUMMARY OF IMPORTANT FACTS AND FORMULAE

1. CONGRUENT TRIANGLES *$\triangle ABC$ is said to be congruent to $\triangle DEF$ only when one of them can be made to superpose on the other (and vice versa) so as to cover it exactly. And, we write, $\triangle ABC \cong \triangle DEF$.*

2. CRITERIA FOR CONGRUENCE

(i) **SAS** *(Two sides and the included angle)*

(ii) **AAS** *(Two angles and one side)*

(iii) **SSS** *(All the three corresponding sides are equal)*

(iv) **RHS** *(In two right triangles, hypotenuse and one side)*

3. (i) The angles opposite to two equal sides of a triangle are equal.

(ii) The sides opposite to two equal angles of a triangle are equal.

4. (i) In a triangle, the longer side has the greater angle opposite it.

(ii) In a triangle, the greater angle has the longer side opposite it.

5. (i) The sum of any two sides of a triangle is greater than the third side.

(ii) The difference between any two sides of a triangle is less than the third side.

6. (i) If the bisector of the vertical angle of a triangle bisects the base, the triangle is isosceles.

(ii) In an isosceles triangle, the altitude from the vertex bisects the base.

(iii) If the altitude from the vertex of a triangle bisects the base, the triangle is isosceles.

(iv) The perpendiculars drawn from the vertices of equal angles of an isosceles triangle to the opposite sides are equal.

(v) If the altitudes from two vertices of a triangle to the opposite sides are equal, the triangle is isosceles.

7. (i) Each angle of an equilateral triangle is 60°.

(ii) The medians of an equilateral triangle are equal.

8. (i) The sum of three altitudes of a triangle is less than the sum of three sides of the triangle.

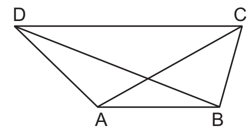
(ii) The sum of any two sides of a triangle is greater than twice the median drawn to the third side.

(iii) The perimeter of a triangle is greater than the sum of its three medians.

9. In a right triangle, the hypotenuse is the longest side.

10 Quadrilaterals

QUADRILATERAL A plane figure bounded by four line segments AB, BC, CD and DA is called a quadrilateral, written as quad. $ABCD$ or $\square ABCD$.



In a quad. $ABCD$, we have:

(i) VERTICES The points A, B, C, D are called the vertices of quad. $ABCD$.

(ii) SIDES The line segments AB, BC, CD and DA are called the sides of quad. $ABCD$.

(iii) DIAGONALS The line segments AC and BD are called the diagonals of quad. $ABCD$.

(iv) ADJACENT SIDES Two sides of a quadrilateral having a common end point are called its consecutive or adjacent sides.

(AB, BC), (BC, CD), (CD, DA) and (DA, AB) are four pairs of adjacent sides of quad. $ABCD$.

(v) OPPOSITE SIDES Two sides of a quadrilateral having no common end point are called its opposite sides.

(AB, CD) and (AD, BC) are two pairs of opposite sides of quad. $ABCD$.

(vi) CONSECUTIVE ANGLES Two angles of a quadrilateral having a common arm are called its consecutive angles.

$(\angle A, \angle B)$, $(\angle B, \angle C)$, $(\angle C, \angle D)$ and $(\angle D, \angle A)$ are four pairs of consecutive angles of a quad. $ABCD$.

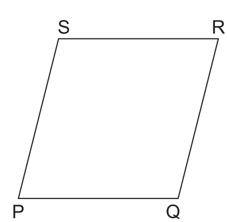
(vii) OPPOSITE ANGLES Two angles of a quadrilateral having no common arm are called its opposite angles.

$(\angle A, \angle C)$ and $(\angle B, \angle D)$ are two pairs of opposite angles of quad. $ABCD$.

VARIOUS TYPES OF QUADRILATERALS

1. PARALLELOGRAM A quadrilateral in which both pairs of opposite sides are parallel is called a parallelogram, written as $\|$gm. In $\|$gm $PQRS$, we have

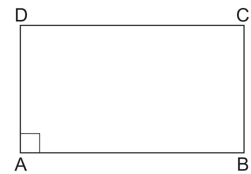
$$PQ \parallel SR, \quad PS \parallel QR.$$



2. RECTANGLE A parallelogram one of whose angles is 90°, is called a rectangle, written as rect. $ABCD$, etc.

   In rect. $ABCD$, we have

   $AB \parallel DC$, $AD \parallel BC$ and $\angle A = 90°$.

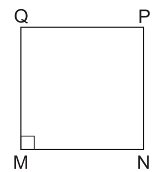


3. SQUARE A parallelogram whose all sides are equal and one of whose angles is 90° is called a square. A square is thus a rectangle having all sides equal.

   In square $MNPQ$, we have

   $MN \parallel QP$, $MQ \parallel NP$ and $MN = NP = PQ = QM$

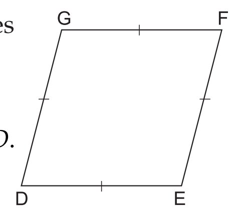
   and $\angle M = 90°$.



4. RHOMBUS A parallelogram having all sides equal is called a rhombus.

   In rhombus $DEFG$, we have

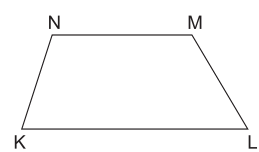
   $DE \parallel GF$, $DG \parallel EF$ and $DE = EF = FG = GD$.



5. TRAPEZIUM A quadrilateral having one pair of opposite sides parallel is called a trapezium.

   In trapezium $KLMN$, we have

   $KL \parallel NM$.

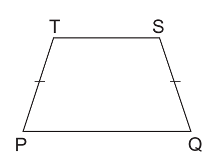


   The line segment joining the midpoints of nonparallel sides of a trapezium is called its median.

6. ISOSCELES TRAPEZIUM If the two nonparallel sides of a trapezium are equal then it is called an isosceles trapezium.

   In isosceles trapezium $PQST$, we have

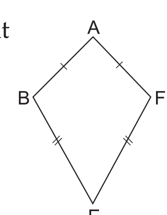
   $PQ \parallel TS$ and $PT = QS$.



7. KITE A quadrilateral in which two pairs of adjacent sides are equal is known as a kite.

   Quad. $ABEF$ is a kite, in which

   $AB = AF$ and $EB = EF$.



From the above definitions it is clear that:

(i) Rectangle, square and rhombus are all parallelograms.

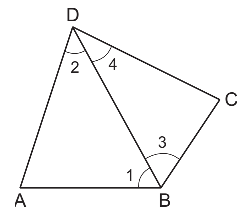
(ii) A parallelogram is a trapezium while a trapezium is not a parallelogram.

(iii) A square is both a rectangle and a rhombus.

(iv) A kite is not a parallelogram.

(v) A rectangle or a rhombus is not necessarily a square.

THEOREM 1 *The sum of all the four angles of a quadrilateral is* 360°.



GIVEN A quad. $ABCD$.

TO PROVE $\angle A + \angle B + \angle C + \angle D = 360^\circ$.

CONSTRUCTION Join BD.

PROOF Since the sum of the angles of a triangle is 180°, we have

$\angle A + \angle 1 + \angle 2 = 180^\circ$ … (i) (sum of $\angle$s of $\triangle ABD$)

and $\angle 3 + \angle C + \angle 4 = 180^\circ$ … (ii) (sum of $\angle$s of $\triangle BCD$).

On adding (i) and (ii), we get

$$\angle A + \angle C + (\angle 1 + \angle 3) + (\angle 2 + \angle 4) = 360^\circ$$

$$\Rightarrow \quad \angle A + \angle C + \angle B + \angle D = 360^\circ$$

$$[\because \angle 1 + \angle 3 = \angle B \text{ and } \angle 2 + \angle 4 = \angle A]$$

$$\Rightarrow \quad \angle A + \angle B + \angle C + \angle D = 360^\circ.$$

SOLVED EXAMPLES

EXAMPLE 1 *Three angles of a quadrilateral measure* 110°, 82° *and* 68°. *Find the measure of the fourth angle.*

SOLUTION Let the measure of the fourth angle be x°.

Then, the sum of the angles of a quadrilateral being 360°, we have: $110 + 82 + 68 + x = 360 \Rightarrow 260 + x = 360 \Rightarrow x = 100$.

Hence, the measure of the fourth angle is 100°.

EXAMPLE 2 *The angles of a quadrilateral are in the ratio* $3 : 5 : 9 : 13$. *Find all the angles of the quadrilateral.*

SOLUTION Let $ABCD$ be the given quadrilateral and let

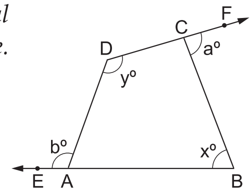
$$\angle A = (3x)^\circ, \angle B = (5x)^\circ, \angle C = (9x)^\circ \text{ and } \angle D = (13x)^\circ.$$

Since the sum of the angles of a quadrilateral is 360°, we have

$$3x + 5x + 9x + 13x = 360 \Rightarrow 30x = 360 \Rightarrow x = 12.$$

$$\therefore \quad \angle A = 36^\circ, \angle B = 60^\circ, \angle C = 108^\circ \text{ and } \angle D = 156^\circ.$$

EXAMPLE 3 *The sides BA and DC of a quadrilateral are produced as shown in the given figure. Prove that $x + y = a + b$.*



SOLUTION We have $\angle A + b^\circ = 180^\circ$ (linear pair)

$\Rightarrow \quad \angle A = (180^\circ - b^\circ)$. ... (i)

Also, $\angle C + a^\circ = 180^\circ$ (linear pair)

$\Rightarrow \quad \angle C = (180^\circ - a^\circ)$. ... (ii)

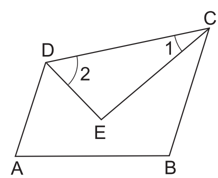
Now, $\angle A + \angle B + \angle C + \angle D = 360^\circ$

(sum of the $\angle$s of a quad. is 360°)

$\Rightarrow \quad (180^\circ - b^\circ) + x^\circ + (180^\circ - a^\circ) + y^\circ = 360^\circ$ [using (i) and (ii)]

$\Rightarrow \quad x + y = a + b$.

EXAMPLE 4 *In a quadrilateral ABCD, the line segments bisecting $\angle C$ and $\angle D$ meet at E. Prove that $\angle A + \angle B = 2\angle CED$.*



SOLUTION Let CE and DE be the bisectors of $\angle C$ and $\angle D$ respectively. Then, $\angle 1 = \frac{1}{2}\angle C$ and $\angle 2 = \frac{1}{2}\angle D$.

In $\triangle DEC$, we have $\angle 1 + \angle 2 + \angle CED = 180^\circ$

(sum of the $\angle$s of a $\triangle$ is 180°)

$\Rightarrow \quad \angle CED = 180^\circ - (\angle 1 + \angle 2)$. ... (i)

Again, the sum of the angles of a quadrilateral is 360°.

$\therefore \quad \angle A + \angle B + \angle C + \angle D = 360^\circ$

$\Rightarrow \quad \frac{1}{2}(\angle A + \angle B) + \frac{1}{2}\angle C + \frac{1}{2}\angle D = 180^\circ$

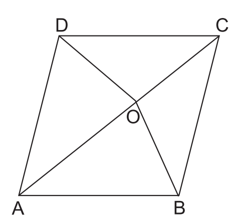
$\Rightarrow \quad \frac{1}{2}(\angle A + \angle B) + \angle 1 + \angle 2 = 180^\circ$

$\Rightarrow \quad \frac{1}{2}(\angle A + \angle B) = 180^\circ - (\angle 1 + \angle 2)$. ... (ii)

From (i) and (ii), we get $\frac{1}{2}(\angle A + \angle B) = \angle CED$.

Hence, $\angle A + \angle B = 2\angle CED$.

EXAMPLE 5 *In the adjoining figure, a point O is taken inside an equilateral quad. ABCD such that $OB = OD$. Show that A, O and C are in the same straight line.*



GIVEN A quad. $ABCD$ in which $AB = BC = CD = DA$ and O is a point within it such that $OB = OD$.

TO PROVE $\angle AOB + \angle COB = 180^\circ$.

PROOF In $\triangle$s OAB and OAD, we have

$AB = AD$ (given), $OA = OA$ (common) and $OB = OD$ (given)

$\therefore \quad \triangle OAB \cong \triangle OAD$.

$\therefore \quad \angle AOB = \angle AOD$. ... (i) (c.p.c.t.)

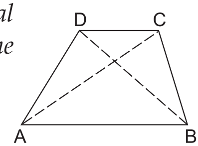
Similarly, $\triangle OBC \cong \triangle ODC$.

$\therefore \quad \angle COB = \angle COD$. ... (ii)

Now, $\angle AOB + \angle COB + \angle COD + \angle AOD = 360^\circ$ [$\angle$s at a point]

$\Rightarrow \quad 2(\angle AOB + \angle COB) = 360^\circ \Rightarrow \angle AOB + \angle COB = 180^\circ$.

EXAMPLE 6 *In the adjoining figure, ABCD is a quadrilateral in which AB is the longest side and CD is the shortest side.*
Prove that (i) $\angle C > \angle A$, *(ii)* $\angle D > \angle B$.



SOLUTION Join AC and BD.

In $\triangle ABC$, $AB > BC \Rightarrow \angle ACB > \angle BAC$.

In $\triangle ADC$, $AD > DC \Rightarrow \angle ACD > \angle CAD$

$\left[\because \text{ in a triangle, the longer side has the greater angle opposite it}\right]$.

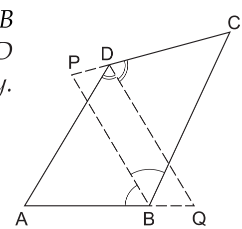
$\therefore \quad \angle ACB + \angle ACD > \angle BAC + \angle CAD$.

$\therefore \quad \angle C > \angle A$.

Similarly, $\angle D > \angle B$.

EXAMPLE 7 *In the adjoining figure, the bisectors of* $\angle B$ *and* $\angle D$ *of a quadrilateral ABCD meet CD and AB produced at P and Q respectively. Prove that*

$$\angle P + \angle Q = \frac{1}{2}(\angle B + \angle D).$$



SOLUTION In $\angle ADQ$, we have

$\angle A + \angle ADQ + \angle Q = 180^\circ$ (sum of the $\angle$s of a $\triangle$)

$\Rightarrow \quad \angle A + \frac{1}{2}\angle D + \angle Q = 180^\circ$. ... (i)

In $\triangle CBP$, we have

$\angle C + \angle CBP + \angle P = 180^\circ$ (sum of the $\angle$s of a $\triangle$)

$\Rightarrow \quad \angle C + \frac{1}{2}\angle B + \angle P = 180^\circ$. ... (ii)

Adding (i) and (ii), we get

$$\angle A + \angle C + \frac{1}{2}\angle B + \frac{1}{2}\angle D + \angle P + \angle Q = 360^\circ$$

$$\Rightarrow \quad \angle A + \angle C + \angle B + \angle D + \angle P + \angle Q = 360^\circ + \frac{1}{2}\angle B + \frac{1}{2}\angle D$$

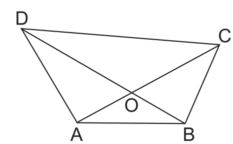
[adding $\left(\frac{1}{2}\angle B + \frac{1}{2}\angle D\right)$ on both sides]

$$\Rightarrow \quad 360^\circ + \angle P + \angle Q = 360^\circ + \frac{1}{2}(\angle B + \angle D)$$

[$\because$ sum of all the angles of a quadrilateral is 360°]

$$\Rightarrow \quad \angle P + \angle Q = \frac{1}{2}(\angle B + \angle D).$$

EXAMPLE 8 *If ABCD is a quadrilateral whose diagonals AC and BD intersect at O, prove that*



(i) $(AB + BC + CD + DA) > (AC + BD)$,

(ii) $(AB + BC + CD + DA) < 2(AC + BD)$.

SOLUTION (i) We know that the sum of any two sides of a triangle is greater than the third.

In $\triangle ABC$, we have $AB + BC > AC$.

In $\triangle ACD$, we have $CD + DA > AC$.

In $\triangle BCD$, we have $BC + CD > BD$.

In $\triangle ABD$, we have $DA + AB > BD$.

Adding these inequalities, we get

$$2(AB + BC + CD + DA) > 2(AC + BD).$$

$$\therefore \quad AB + BC + CD + DA > AC + BD.$$

(ii) In $\triangle AOB$, we have $OA + OB > AB$.

In $\triangle BOC$, we have $OB + OC > BC$.

In $\triangle COD$, we have $OC + OD > CD$.

In $\triangle DOA$, we have $OD + OA > DA$.

Adding these inequalities, we get

$$2(OA + OC) + 2(OB + OD) > AB + BC + CD + DA$$

$$\Rightarrow \quad 2(AC + BD) > AB + BC + CD + DA$$

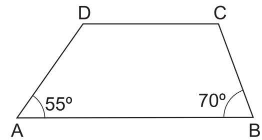
$$\Rightarrow \quad AB + BC + CD + DA < 2(AC + BD).$$

EXERCISE 10A

1. Three angles of a quadrilateral are 75°, 90° and 75°. Find the measure of the fourth angle.

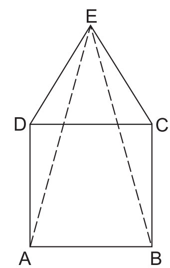
2. The angles of a quadrilateral are in the ratio 2 : 4 : 5 : 7. Find the angles.

3. In the adjoining figure, $ABCD$ is a trapezium in which $AB \parallel DC$. If $\angle A = 55^\circ$ and $\angle B = 70^\circ$, find $\angle C$ and $\angle D$.

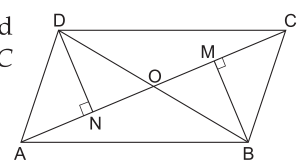


4. In the adjoining figure, $ABCD$ is a square and $\triangle EDC$ is an equilateral triangle. Prove that

(i) $AE = BE$, (ii) $\angle DAE = 15^\circ$.



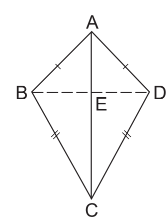
5. In the adjoining figure, $BM \perp AC$ and $DN \perp AC$. If $BM = DN$, prove that AC bisects BD.



6. In the given figure, $ABCD$ is a quadrilateral in which $AB = AD$ and $BC = DC$. Prove that

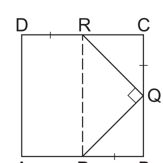
(i) AC bisects $\angle A$ and $\angle C$, (ii) $BE = DE$,

(iii) $\angle ABC = \angle ADC$.



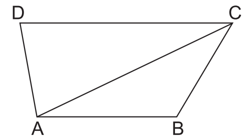
7. In the given figure, $ABCD$ is a square and $\angle PQR = 90^\circ$. If $PB = QC = DR$, prove that

(i) $QB = RC$, (ii) $PQ = QR$, (iii) $\angle QPR = 45^\circ$.



8. If O is a point within a quadrilateral $ABCD$, show that $OA + OB + OC + OD > AC + BD$.

9. In the adjoining figure, $ABCD$ is a quadrilateral and AC is one of its diagonals. Prove that



(i) $AB + BC + CD + DA > 2AC$

(ii) $AB + BC + CD > DA$

(iii) $AB + BC + CD + DA > AC + BD$.

10. Prove that the sum of all the angles of a quadrilateral is 360°.

ANSWERS (EXERCISE 10A)

1. 120° **2.** 40°, 80°, 100°, 140° **3.** $\angle C = 110^\circ, \angle D = 125^\circ$

HINTS TO SOME SELECTED QUESTIONS

4. $\triangle ADE \cong \triangle BCE$ [$\because$ $AD = BC, DE = CE, \angle ADE = \angle BCE = (90^\circ + 60^\circ)$]

$\therefore$ $AE = BE$.

Now, $\angle ADE = 150^\circ$ and $DA = DE \Rightarrow \angle DAE = \angle DEA = 15^\circ$.

5. Let AC and BD intersect at O.

Now, $\triangle OND \cong \triangle OMB$ [$\because$ $\angle OND = \angle OMB, \angle DON = \angle BOM$ and $DN = BN$].

$\therefore$ $OD = OB$.

6. $\triangle ABC \cong \triangle ADC$

$\Rightarrow$ $\angle BAC = \angle DAC, \angle BCA = \angle DCA$ and $\angle ABC = \angle ADC$.

$\triangle ABE \cong \triangle ADE \Rightarrow BE = DE$.

7. $BC = DC, CQ = DR \Rightarrow BC - CQ = DC - DR \Rightarrow QB = RC$.

From $\triangle CQR, \angle RQB = \angle QCR + \angle QRC$

$\Rightarrow$ $\angle RQP + \angle PQB = 90^\circ + \angle QRC$

$\Rightarrow$ $90^\circ + \angle PQB = 90^\circ + \angle QRC \Rightarrow \angle PQB = \angle QRC$.

Now, $\triangle RCQ \cong \triangle QBP$ and therefore, $QR = PQ$.

$PQ = QR \Rightarrow \angle QPR = \angle PRQ$.

But, $\angle QPR + \angle PRQ = 90^\circ$. So, $\angle QPR = 45^\circ$.

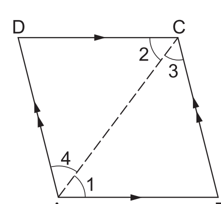
RESULTS ON PARALLELOGRAMS

THEOREM 1 *Prove that in a parallelogram*

(i) each diagonal divides the parallelogram into two congruent triangles;

(ii) opposite sides are equal;

(iii) opposite angles are equal.



GIVEN A ‖gm $ABCD$ in which $AB \parallel DC$ and $AD \parallel BC$.

TO PROVE (i) $\triangle ABC \cong \triangle CDA$ and $\triangle ABD \cong \triangle CDB$;

(ii) $AB = CD$ and $BC = AD$;

(iii) $\angle B = \angle D$ and $\angle A = \angle C$.

CONSTRUCTION Join A and C.

PROOF (i) In $\triangle$s ABC and CDA, we have

$\angle 1 = \angle 2$ (alt. interior $\angle$s, as $AB \parallel DC$ and CA cuts them)

$\angle 3 = \angle 4$ (alt. interior $\angle$s, as $BC \parallel AD$ and CA cuts them)

$AC = CA$ (common)

$\therefore \triangle ABC \cong \triangle CDA$ (AAS-criteria).

Similarly, $\triangle ABD \cong \triangle CDB$.

(ii) $\triangle ABC \cong \triangle CDA$ (proved)

$\therefore AB = CD$ and $BC = AD$ (c.p.c.t.).

(iii) $\triangle ABC \cong \triangle CDA$ (proved)

$\therefore \angle B = \angle D$ (c.p.c.t.).

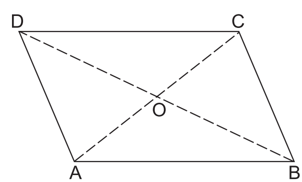
Also, $\angle 1 = \angle 2$ and $\angle 3 = \angle 4$.

$\therefore \angle 1 + \angle 4 = \angle 2 + \angle 3 \Rightarrow \angle A = \angle C$.

Hence, $\angle B = \angle D$ and $\angle A = \angle C$.

THEOREM 2 *Prove that the diagonals of a parallelogram bisect each other.*

GIVEN A $\parallel$gm $ABCD$ in which $AB \parallel DC$ and $BC \parallel AD$. Its diagonals AC and BD intersect each other at a point O.



TO PROVE $OA = OC$ and $OB = OD$.

PROOF In $\triangle$s AOB and COD, we have

$AB = CD$ [opposite sides of a $\parallel$gm]

$\angle OAB = \angle OCD$ (alt. interior $\angle$s, as $AB \parallel DC$ and CA cuts them)

$\angle OBA = \angle ODC$ (alt. interior $\angle$s, as $AB \parallel DC$ and DB cuts them)

$\therefore \triangle AOB \cong \triangle COD$ (AAS-criteria).

Hence, $OA = OC$ and $OB = OD$ (c.p.c.t.).

SUMMARY In a parallelogram

(i) the opposite sides are equal

(ii) the opposite angles are equal

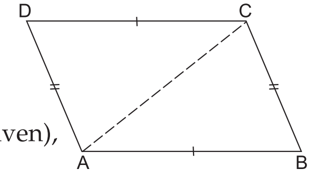
(iii) each diagonal bisects the parallelogram

(iv) the diagonals bisect each other.

CONVERSE OF THE ABOVE THEOREMS

THEOREM 3 *If each pair of opposite sides of a quadrilateral are equal then it is a parallelogram.*

GIVEN A quad. $ABCD$ in which $AB = CD$ and $AD = BC$.



TO PROVE $ABCD$ is a $\|$gm.

CONSTRUCTION Join A and C.

PROOF In $\triangle$s ABC and CDA, we have

$AB = CD$ (given), $BC = AD$ (given),

$AC = CA$ (common).

$\therefore \quad \triangle ABC \cong \triangle CDA$.

$\therefore \quad \angle BAC = \angle DCA$.

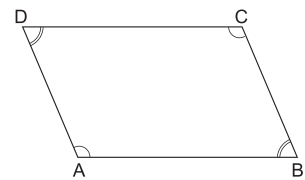
But, these are alternate interior angles.

$\therefore \quad AB \| DC$. Similarly, $AD \| BC$.

Hence, $ABCD$ is a $\|$gm.

THEOREM 4 *If in a quadrilateral, each pair of opposite angles are equal then it is a parallelogram.*

GIVEN A quad. $ABCD$ in which $\angle A = \angle C$ and $\angle B = \angle D$.



TO PROVE $ABCD$ is a $\|$gm.

PROOF We have

$\angle A = \angle C$ and $\angle B = \angle D$ (given)

$\Rightarrow \quad \angle A + \angle B = \angle C + \angle D$

$\Rightarrow \quad \angle A + \angle B = \angle C + \angle D = 180^\circ$ [$\because$ $\angle A + \angle B + \angle C + \angle D = 360^\circ$].

Now, the line segments BC and AD are cut by the transversal AB such that $\angle A + \angle B = 180^\circ$.

$\therefore \quad AD \| BC$ [$\because$ $\angle A$ and $\angle B$ are co-interior $\angle$s].

Again, $\angle A = \angle C$ and $\angle D = \angle B \Rightarrow \angle A + \angle D = \angle C + \angle B$

$\Rightarrow \quad \angle A + \angle D = \angle C + \angle B = 180^\circ$ [$\because$ $\angle A + \angle B + \angle C + \angle D = 360^\circ$].

Now, the line segments DC and AB are cut by the transversal DA such that $\angle A + \angle D = 180^\circ$.

$\therefore \quad AB \| DC$ [$\because$ $\angle A$ and $\angle D$ are co-interior $\angle$s].

Thus, $AB \| DC$ and $AD \| BC$.

Hence, $ABCD$ is a $\|$gm.

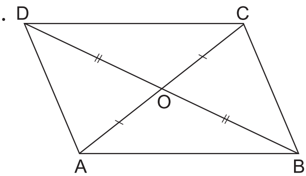
THEOREM 5 *If the diagonals of a quadrilateral bisect each other then prove that the quadrilateral is a parallelogram.*

GIVEN A quad. $ABCD$ whose diagonals AC and BD intersect at a point O such that $OA = OC$ and $OB = OD$.

TO PROVE $ABCD$ is a $\|$gm.

PROOF In $\triangle$s OAB and OCD, we have

$OA = OC$ (given), $OB = OD$ (given)



and $\angle AOB = \angle COD$ (vert. opp. $\angle$s).

$\therefore$ $\triangle OAB \cong \triangle OCD$ (SAS-criteria).

$\therefore$ $\angle BAO = \angle DCO$ (c.p.c.t.).

But, these are alternate interior angles.

$\therefore$ $AB \parallel DC$.

Again, in $\triangle$s OAD and OCB, we have

$OA = OC$ (given), $OD = OB$ (given)

and $\angle DOA = \angle BOC$ (vert. opp. $\angle$s).

$\therefore$ $\triangle OAD \cong \triangle OCB$ (SAS-criteria).

$\therefore$ $\angle ADO = \angle CBO$ (c.p.c.t.).

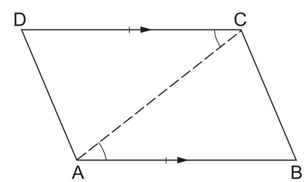
But, these are alternate interior angles.

$\therefore$ $AD \parallel BC$.

Thus, $AB \parallel DC$ and $AD \parallel BC$.

Hence, $ABCD$ is a $\parallel$gm.

THEOREM 6 *Prove that a quadrilateral is a parallelogram, if its one pair of opposite sides are equal and parallel.*



GIVEN A quad. $ABCD$ in which $AB = DC$ and $AB \parallel DC$.

TO PROVE $ABCD$ is a $\parallel$gm.

CONSTRUCTION Join A and C.

PROOF In $\triangle$s ABC and CDA, we have

$AB = DC$ (given), $AC = CA$ (common),

and $\angle BAC = \angle DCA$ [alt. interior $\angle$s, as $AB \parallel DC$ and CA cuts them].

$\therefore$ $\triangle ABC \cong \triangle CDA$ (SAS-criteria).

$\therefore$ $\angle BCA = \angle DAC$ (c.p.c.t.).

But, these are alternate interior angles.

$\therefore$ $AD \parallel BC$.

Now, $AB \parallel DC$ and $AD \parallel BC$.

$\therefore$ $ABCD$ is a $\parallel$gm.

SUMMARY A quadrilateral is a parallelogram

(i) if both pairs of opposite sides are equal

or (ii) if both pairs of opposite angles are equal

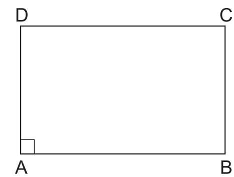
or (iii) if the diagonals bisect each other

or (iv) if a pair of opposite sides are equal and parallel.

SOME RESULTS ON RECTANGLE, RHOMBUS AND SQUARE

THEOREM 7 *Prove that each angle of a rectangle is a right angle.*

GIVEN A rectangle $ABCD$ in which $\angle A = 90^\circ$.



TO PROVE $\angle A = \angle B = \angle C = \angle D = 90^\circ$.

PROOF $ABCD$ is a rectangle

$\Rightarrow$ $ABCD$ is a $\parallel$gm

$\Rightarrow$ $AB \parallel DC$ and $AD \parallel BC$.

Now, $AD \parallel BC$ and AB is a transversal.

$\Rightarrow$ $\angle A + \angle B = 180^\circ$ [$\because$ $\angle A$ and $\angle B$ are co-interior $\angle$s]

$\Rightarrow$ $\angle B = 180^\circ - \angle A = 180^\circ - 90^\circ = 90^\circ$.

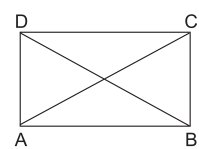
Also, $\angle C = \angle A = 90^\circ$ and $\angle D = \angle B = 90^\circ$.

[$\because$ opposite $\angle$s of a $\parallel$gm are equal].

Thus, each of the angles of rectangle is a right angle.

THEOREM 8 *Prove that the diagonals of a rectangle are equal.*

GIVEN A rect. $ABCD$ in which AC and BD are the diagonals.



TO PROVE $AC = BD$.

PROOF In $\triangle$s ABD and BAC, we have

$AB = BA$ (common)

$\angle A = \angle B$ (each equal to 90°)

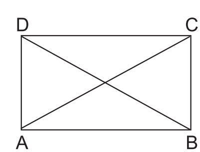
$AD = BC$ (opposite sides of a $\parallel$gm)

$\therefore$ $\triangle ABD \cong \triangle BAC$ (SAS-criteria).

Hence, $BD = AC$.

THEOREM 9 (Converse of Theorem 8) *If the two diagonals of a parallelogram are equal, prove that the parallelogram is a rectangle.*

GIVEN A $\parallel$gm $ABCD$ in which $AC = BD$.



TO PROVE $ABCD$ is a rectangle.

PROOF In $\triangle$s ABC and DCB, we have

$AB = DC$ (opposite sides of a $\parallel$gm),

$BC = CB$ (common) and $AC = DB$ (given).

$\therefore$ $\triangle ABC \cong \triangle DCB$ (SSS-criteria).

$\therefore$ $\angle ABC = \angle DCB$. … (i)

But, $DC \parallel AB$ and CB cuts them.

$\therefore$ $\angle ABC + \angle DCB = 180^\circ$ (co-interior $\angle$s).

$\therefore$ $\angle ABC = \angle DCB = 90^\circ$ [using (i)].

Thus, $ABCD$ is a $\parallel$gm one of whose angles is 90°.

Hence, $ABCD$ is a rectangle.

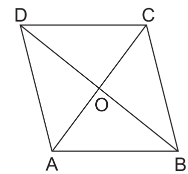
THEOREM 10 *Prove that the diagonals of a rhombus bisect each other at right angles.*

GIVEN A rhombus $ABCD$ whose diagonals AC and BD intersect at a point O.

TO PROVE (i) $OA = OC$ and $OB = OD$

(ii) $\angle BOC = \angle DOC = \angle AOD = \angle AOB = 90°$.

PROOF (i) Clearly, $ABCD$ is a ||gm, in which $AB = BC = CD = DA$.



Also, we know that the diagonals of a ||gm bisect each other.

$\therefore$ $OA = OC$ and $OB = OD$.

(ii) Now, in $\triangle$s BOC and DOC, we have

$OB = OD$, $BC = DC$ and $OC = OC$ (common).

$\therefore$ $\triangle BOC \cong \triangle DOC$.

$\therefore$ $\angle BOC = \angle DOC$ (c.p.c.t.).

But, $\angle BOC + \angle DOC = 180°$ (linear pair)

$\therefore$ $\angle BOC = \angle DOC = 90°$.

Similarly, $\angle AOB = \angle AOD = 90°$.

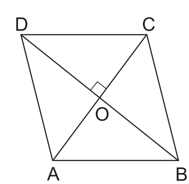
Hence, the diagonals of a rhombus bisect each other at right angles.

THEOREM 11 (Converse of Theorem 10) *If the diagonals of a quadrilateral bisect each other at right angles, prove that it is a rhombus.*

GIVEN A quadrilateral $ABCD$ whose diagonals AC and BD intersect at O such that $OA = OC$ and $OB = OD$ and $AC \perp BD$.

TO PROVE $ABCD$ is a rhombus.

PROOF Since the diagonals of quad. $ABCD$ bisect each other, therefore $ABCD$ is a ||gm.



Now, in $\triangle$s AOD and COD, we have

$OA = OC$ (given),

$\angle AOD = \angle COD = 90°$ [$\because$ $AC \perp BD$]

and $OD = OD$ (common).

$\therefore$ $\triangle AOD \cong \triangle COD$.

And so, $AD = CD$ (c.p.c.t.).

Now, $AB = CD$ and $AD = BC$ (opp. sides of a ||gm)

and $AD = CD$ (proved).

$\therefore$ $AB = CD = AD = BC$.

Hence, $ABCD$ is a rhombus.

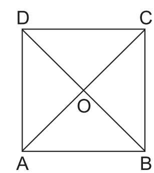
THEOREM 12 *Prove that the diagonals of a square are equal and bisect each other at right angles.*

GIVEN A square $ABCD$ whose diagonals AC and BD intersect at O.

TO PROVE (i) $AC = BD$

(ii) $OA = OC$ and $OB = OD$

(iii) $AC \perp BD$.



PROOF (i) In $\triangle ABC$ and BAD, we have

$AB = BA$ (common)

$BC = AD$ (sides of a square)

$\angle ABC = \angle BAD$ (each equal to 90°)

$\therefore \triangle ABC \cong \triangle BAD$ (SAS-criterion).

And so, $AC = BD$ (c.p.c.t.).

(ii) Now, $ABCD$ is a square and therefore a ||gm.

And so, $OA = OC$ and $OB = OD$

[$\because$ diagonals of a ||gm bisect each other].

(iii) Now, in $\triangle AOB$ and AOD, we have

$OB = OD$ [$\because$ diagonals of a ||gm bisect each other]

$AB = AD$ (sides of a square)

$AO = AO$ (common)

$\therefore \triangle AOB \cong \triangle AOD$ (SSS-criterion).

$\therefore \angle AOB = \angle AOD$.

But, $\angle AOB + \angle AOD = 180^\circ$ (linear pair)

$\therefore \angle AOB = \angle AOD = 90^\circ$.

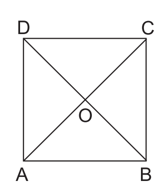
Thus, $AO \perp BD$, i.e., $AC \perp BD$.

THEOREM 13 (Converse of Theorem 12) *If the diagonals of a quadrilateral are equal and bisect each other at right angles then prove that the quadrilateral is a square.*

GIVEN A quad. $ABCD$ in which the diagonals AC and BD intersect at O such that $AC = BD$; $OA = OC$ and $OB = OD$; $AC \perp BD$.

TO PROVE $ABCD$ is a square.

PROOF Since the diagonals of quad. $ABCD$ bisect each other, therefore $ABCD$ is a ||gm.



Now, in $\triangle ABO$ and ADO, we have

$OB = OD$ (given)

$OA = OA$ (common)

$\angle AOB = \angle AOD = 90^\circ$ [$\because$ $AC \perp BD$]

$\therefore$ $\triangle ABO \cong \triangle ADO$ (SAS-criterion).

And so, $AB = AD$ (c.p.c.t.).

Now, $AB = CD$ and $AD = BC$ [$\because$ opp. sides of a ‖gm are equal].

$\therefore$ $AB = BC = CD = AD$.

Again, in $\triangle$s ABC and BAD, we have

$AB = BA$ (common)

$AC = BD$ (given)

$BC = AD$ (proved)

$\therefore$ $\triangle ABC \cong \triangle BAD$ (SSS-criterion).

And so, $\angle ABC = \angle BAD$ (c.p.c.t.).

But, $\angle ABC + \angle BAD = 180^\circ$ (co-interior $\angle$s).

$\therefore$ $\angle ABC = \angle BAD = 90^\circ$.

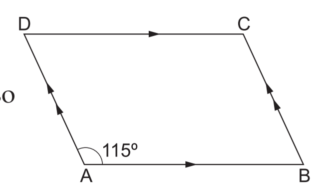
Thus, $AB = BC = CD = AD$ and $\angle A = 90^\circ$.

$\therefore$ $ABCD$ is a square.

SOLVED EXAMPLES

EXAMPLE 1 *In a ‖gm ABCD, if $\angle A = 115^\circ$, find $\angle B$, $\angle C$ and $\angle D$.*

SOLUTION Let $ABCD$ be a ‖gm in which $\angle A = 115^\circ$.



Since $AD \parallel BC$ and AB cuts them, so

$\angle A + \angle B = 180^\circ$

$\Rightarrow$ $115^\circ + \angle B = 180^\circ$

$\Rightarrow$ $\angle B = (180^\circ - 115^\circ) = 65^\circ$.

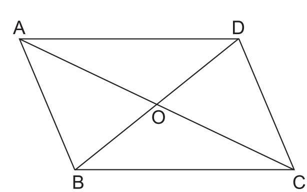
Since the opposite angles of a ‖gm are equal, we have

$\angle C = \angle A = 115^\circ$ and $\angle D = \angle B = 65^\circ$.

Hence, $\angle B = 65^\circ$, $\angle C = 115^\circ$ and $\angle D = 65^\circ$.

EXAMPLE 2 *Diagonals AC and BD of a parallelogram ABCD intersect at O. If* $OA = 3$ cm *and* $OD = 2$ cm, *determine the lengths of AC and BD.*

SOLUTION The diagonals of a parallelogram bisect each other.



$\therefore$ $OA = OC = 3$ cm

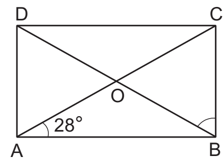
and $OB = OD = 2$ cm.

And so,

$AC = OA + OC = (3 + 3)$ cm $= 6$ cm

and $BD = OB + OD = (2 + 2)$ cm $= 4$ cm.

EXAMPLE 3 *In the adjoining figure, ABCD is a rectangle whose diagonals AC and BD intersect at O. If $\angle OAB = 28°$, find $\angle OBC$.*



SOLUTION We know that the diagonals of a rectangle are equal and bisect each other.

$\therefore \quad OA = OB \Rightarrow \angle OBA = \angle OAB = 28°$.

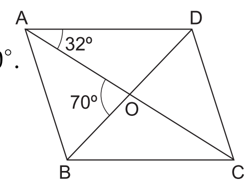
Also, each angle of a rectangle measures 90°.

$\therefore \quad \angle ABC = 90° \Rightarrow \angle OBA + \angle OBC = 90°$

$\therefore \quad 28° + \angle OBC = 90° \Rightarrow \angle OBC = (90° - 28°) = 62°$.

EXAMPLE 4 *The diagonals AC and BD of a parallelogram ABCD intersect each other at the point O. If $\angle DAC = 32°$ and $\angle AOB = 70°$, find $\angle DBC$.*

SOLUTION $\angle AOB + \angle AOD = 180°$ (linear pair)



$\Rightarrow \quad 70° + \angle AOD = 180° \Rightarrow \angle AOD = 110°$.

In $\triangle AOD$, we have

$\angle DAO + \angle AOD + \angle ADO = 180°$ (sum of $\angle$s of a $\triangle$)

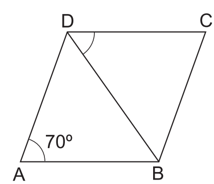
$\Rightarrow \quad 32° + 110° + \angle ADO = 180°$

$\Rightarrow \quad \angle ADO = (180° - 32° - 110°) = 38°$.

Now, $\angle OBC = \angle ADO = 38°$ (alt. interior $\angle$s).

$\therefore \quad \angle DBC = \angle OBC = 38°$.

EXAMPLE 5 *In the adjoining figure, ABCD is a rhombus. If $\angle A = 70°$, find $\angle CDB$.*



SOLUTION We have $\angle C = \angle A = 70°$

(opposite $\angle$s of a $\|$gm).

Let $\angle CDB = x°$.

In $\triangle CDB$, we have

$CD = CB \Rightarrow \angle CBD = \angle CDB = x°$.

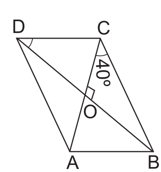
$\therefore \quad \angle CDB + \angle CBD + \angle DCB = 180°$ (sum of $\angle$s of a $\triangle$)

$\Rightarrow \quad x° + x° + 70° = 180° \Rightarrow 2x = 110$, i.e., $x = 55$.

Hence, $\angle CDB = 55°$.

EXAMPLE 6 *ABCD is a rhombus such that $\angle ACB = 40°$. Find $\angle ADB$.*

SOLUTION Let the diagonals AC and BD intersect at O.



Then, $\angle BOC = 90°$.

[$\because$ diagonals of a rhombus intersect at right angles]

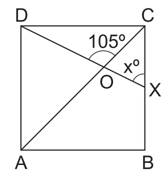
In $\triangle BOC$, we have

$$\angle OCB + \angle BOC + \angle CBO = 180^\circ \qquad \text{(sum of } \angle s \text{ of a } \triangle)$$

$\Rightarrow \quad 40^\circ + 90^\circ + \angle CBO = 180^\circ \Rightarrow \angle CBO = 50^\circ.$

Now, $\angle ADB = \angle CBO = 50^\circ$ (alt. interior $\angle s$).

EXAMPLE 7 *In the adjoining figure, ABCD is a square. A line segment DX cuts the side BC at X and the diagonal AC at O such that* $\angle COD = 105^\circ$ *and* $\angle OXC = x^\circ$. *Find the value of x.*



SOLUTION The angles of a square are bisected by the diagonals.

$\therefore \quad \angle OCX = 45^\circ \quad [\because \angle DCB = 90^\circ$ and CA bisects $\angle DCB]$.

Also, $\angle COD + \angle COX = 180^\circ$ (linear pair)

$\Rightarrow \quad 105^\circ + \angle COX = 180^\circ \Rightarrow \angle COX = (180^\circ - 105^\circ) = 75^\circ.$

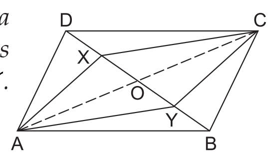
Now, in $\triangle COX$, we have

$$\angle OCX + \angle COX + \angle OXC = 180^\circ$$

$\Rightarrow \quad 45^\circ + 75^\circ + \angle OXC = 180^\circ \Rightarrow \angle OXC = (180^\circ - 120^\circ) = 60^\circ.$

Hence, $x = 60$.

EXAMPLE 8 *In the adjoining figure, ABCD is a parallelogram and X, Y are the points on the diagonal BD such that* $DX = BY$. *Prove that*



(i) CXAY is a parallelogram,

(ii) $\triangle ADX \cong \triangle CBY$ *and* $\triangle ABY \cong \triangle CDX$, *and*

(iii) $AX = CY$ and $CX = AY$.

SOLUTION (i) Join AC, meeting BD at O.

Since the diagonals of a parallelogram bisect each other, we have $OA = OC$ and $OD = OB$.

Now, $OD = OB$ and $DX = BY$

$\Rightarrow \quad OD - DX = OB - BY \Rightarrow OX = OY.$

Now, $OA = OC$ and $OX = OY$.

$CXAY$ is a quadrilateral whose diagonals bisect each other.

$\therefore \quad CXAY$ is a $\|$gm.

(ii) and (iii) In $\triangle s$ ADX and CBY, we have

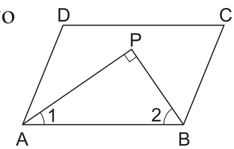
$AD = BC$	(opp. sides of a $\|$gm)
$\angle ADX = \angle CBY$	(alt. interior $\angle s$)
$DX = BY$	(given)

$\therefore \quad \triangle ADX \cong \triangle CBY$ (SAS-criterion).

And so, $AX = CY$ (c.p.c.t.).

Similarly, $\triangle ABY \cong \triangle CDX$ and so $CX = AY$ (c.p.c.t.).

EXAMPLE 9 *Prove that in a parallelogram, the bisectors of any two consecutive angles intersect at right angles.*



GIVEN A ||gm $ABCD$ in which the bisectors of two consecutive angles $\angle A$ and $\angle B$ intersect at a point P.

TO PROVE $\angle APB = 90°$.

PROOF $AD \parallel BC$ and AB is a transversal [$\because$ $ABCD$ is a ||gm].

$$\therefore \quad \angle A + \angle B = 180°$$

$$\Rightarrow \quad \frac{1}{2}\angle A + \frac{1}{2}\angle B = 90°$$

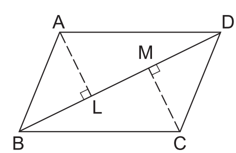
$$\Rightarrow \quad \angle 1 + \angle 2 = 90° \quad \left[\begin{array}{l}\because AP \text{ and } BP \text{ are bisectors of} \\ \angle A \text{ and } \angle B \text{ respectively}\end{array}\right].$$

In $\triangle APB$, we have

$\angle 1 + \angle 2 + \angle APB = 180°$ (sum of $\angle$s of a $\triangle$)

$\Rightarrow \quad 90° + \angle APB = 180° \Rightarrow \angle APB = 90°$.

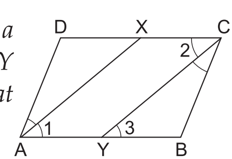
EXAMPLE 10 *ABCD is a parallelogram and AL and CM are perpendiculars from vertices A and C on diagonal BD, as shown in the adjoining figure. Show that (i) $\triangle ALB \cong \triangle CMD$ and (ii) $AL = CM$.*



SOLUTION In $\triangle$s ALB and CMD, we have

$AB = CD$	(opp. sides of a ||gm)
$\angle ALB = \angle CMD = 90°$	[$\because$ $AL \perp BD$, $CM \perp BD$]
$\angle ABL = \angle CDM$	(alt. interior $\angle$s)
$\therefore \quad \triangle ALB \cong \triangle CMD$	(AAS-criterion).
And so, $AL = CM$	(c.p.c.t.).

EXAMPLE 11 *In the adjoining figure, ABCD is a parallelogram and line segments AX and CY bisect $\angle A$ and $\angle C$ respectively. Prove that $AX \parallel CY$.*



SOLUTION We have: $\angle A = \angle C$ (opp. $\angle$s of a ||gm)

$$\Rightarrow \quad \frac{1}{2}\angle A = \frac{1}{2}\angle C \Rightarrow \angle 1 = \angle 2. \qquad \ldots \text{(i)}$$

Now, $DC \parallel AB$ and CY is a transversal

$\Rightarrow \quad \angle 2 = \angle 3 \qquad \ldots$ (ii) (alt. interior $\angle$s).

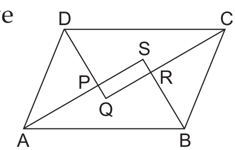
From (i) and (ii), we get $\angle 1 = \angle 3$.

But, these are corresponding angles formed when AX and CY are cut by transversal AB.

$\therefore \quad AX \parallel CY$.

EXAMPLE 12 *Show that the bisectors of the angles of a parallelogram enclose a rectangle.*

SOLUTION Since $DC \parallel AB$ and DA cuts them, we have



$\angle A + \angle D = 180^\circ$ (co-interior $\angle$s)

$\Rightarrow \quad \frac{1}{2}\angle A + \frac{1}{2}\angle D = 90^\circ$

$\Rightarrow \quad \angle PAD + \angle PDA = 90^\circ$

$\Rightarrow \quad \angle APD = 90^\circ$ [$\because$ sum of the $\angle$s of a $\triangle$ is 180°]

$\Rightarrow \quad \angle SPQ = 90^\circ$ (vert. opp. $\angle$s).

Now, $AD \parallel BC$ and DC cuts them.

$\therefore \quad \angle D + \angle C = 180^\circ$ (co-interior $\angle$s)

$\Rightarrow \quad \frac{1}{2}\angle D + \frac{1}{2}\angle C = 90^\circ$

$\Rightarrow \quad \angle QDC + \angle QCD = 90^\circ$

$\Rightarrow \quad \angle DQC = 90^\circ$ [$\because$ sum of the $\angle$s of a $\triangle$ is 180°]

$\Rightarrow \quad \angle PQR = 90^\circ$.

Similarly, $\angle QRS = 90^\circ$ and $\angle RSP = 90^\circ$.

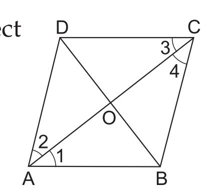
Thus, $PQRS$ is a quadrilateral each of whose angles is 90°.

Hence, $PQRS$ is a rectangle.

EXAMPLE 13 *If a diagonal of a parallelogram bisects one of the angles of the parallelogram, prove that it also bisects the angle opposite it, and that the two diagonals are perpendicular to each other.*

Also, prove that it is a rhombus.

GIVEN A ‖gm $ABCD$ whose diagonals AC and BD intersect at O. Also, AC bisects $\angle A$, i.e., $\angle 1 = \angle 2$.



TO PROVE (i) AC bisects $\angle C$, i.e., $\angle 3 = \angle 4$.

(ii) $AC \perp BD$.

(iii) $ABCD$ is a rhombus.

PROOF In ‖gm $ABCD$, we have

$AB \parallel DC$ and CA cuts them

$\therefore \quad \angle 1 = \angle 3$ (alt. interior ∠s)

and $AD \parallel BC$ and CA cuts them

$\therefore \quad \angle 2 = \angle 4$ (alt. interior ∠s).

Now, $\angle 1 = \angle 2 \Rightarrow \angle 3 = \angle 4$.

This shows that AC bisects $\angle C$ also.

Now, $\angle A = \angle C$ (opp. ∠s of a ‖gm).

$$\Rightarrow \quad \frac{1}{2}\angle A = \frac{1}{2}\angle C \Rightarrow \angle 2 = \angle 3 \Rightarrow AD = DC.$$

But, $AB = DC$ and $AD = BC$.

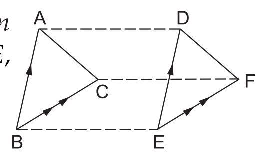
$\therefore \quad AB = BC = DC = AD$.

So, $ABCD$ is a rhombus.

But, the diagonals of a rhombus bisect each other at right angles.

$\therefore \quad AC \perp BD$.

EXAMPLE 14 *Let $\triangle ABC$ and $\triangle DEF$ be two triangles given in such a way that $AB \parallel DE$, $AB = DE$, $BC \parallel EF$ and $BC = EF$.*



Prove that

(i) $AC \parallel DF$ and $AC = DF$,

(ii) $\triangle ABC \cong \triangle DEF$.

SOLUTION $AB \parallel DE$ and $AB = DE$

$\Rightarrow \quad ABED$ is a ‖gm $\Rightarrow AD \parallel BE$ and $AD = BE$. … (i)

Again, $BC \parallel EF$ and $BC = EF$

$\Rightarrow \quad BEFC$ is a ‖gm $\Rightarrow CF \parallel BE$ and $CF = BE$. … (ii)

From (i) and (ii), we get

$AD \parallel CF$ and $AD = CF$.

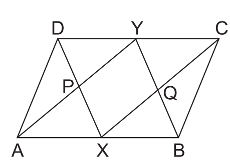
$\therefore \quad ACFD$ is a ‖gm.

Consequently, $AC \parallel DF$ and $AC = DF$.

Also, $AB = DE$, $BC = EF$ and $AC = DF$.

$\therefore \quad \triangle ABC \cong \triangle DEF$ (SSS-criterion).

EXAMPLE 15 *In the adjoining figure, $ABCD$ is a parallelogram in which X and Y are the midpoints of AB and DC respectively. If AY and DX intersect in P while CX and BY intersect in Q, show that*



(i) *quad. AXCY is a parallelogram,*
(ii) *quad. XBYD is a parallelogram,*
(iii) *quad. PXQY is a parallelogram.*

SOLUTION $AB \parallel DC \Rightarrow AX \parallel YC$.

Also, $AB = DC \Rightarrow \frac{1}{2}AB = \frac{1}{2}DC \Rightarrow AX = YC$.

$\therefore$ $AX \parallel YC$ and $AX = YC \Rightarrow AXCY$ is a $\parallel$gm.

$\therefore$ $AY \parallel XC$. In particular, $PY \parallel XQ$.

Again, $AB \parallel DC \Rightarrow XB \parallel DY$.

Also, $AB = DC \Rightarrow \frac{1}{2}AB = \frac{1}{2}DC \Rightarrow XB = DY$.

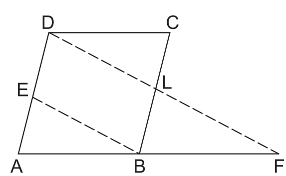
$\therefore$ $XB \parallel DY$ and $XB = DY \Rightarrow XBYD$ is a $\parallel$gm.

$\therefore$ $XD \parallel BY$. In particular, $XP \parallel QY$.

Thus, $PY \parallel XQ$ and $XP \parallel QY$.

$\therefore$ $PXQY$ is a parallelogram.

EXAMPLE 16 *In the adjoining figure, ABCD is a parallelogram and E is the midpoint of AD. A line through D, drawn parallel to EB, meets AB produced at F and BC at L. Prove that*



(i) *$AF = 2DC$ and (ii) $DF = 2DL$.*

SOLUTION $EB \parallel DL$ and $ED \parallel BL \Rightarrow EBLD$ is a $\parallel$gm.

$\therefore$ $BL = ED = \frac{1}{2}AD = \frac{1}{2}BC = CL$.

Now, in $\triangle$s DCL and FBL, we have

$CL = BL$ (proved), $\angle DLC = \angle FLB$ (vert. opp. $\angle$s)

and $\angle CDL = \angle BFL$ (alt. int. $\angle$s).

$\therefore$ $\triangle DCL \cong \triangle FBL$.

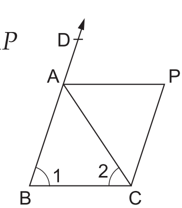
$\therefore$ $DC = BF$ and $DL = FL$.

Now, $BF = DC = AB \Rightarrow 2AB = 2DC \Rightarrow AF = 2DC$.

$DL = FL \Rightarrow DF = 2DL$.

EXAMPLE 17 *In the adjoining figure, $AB = AC$; $CP \parallel BA$ and AP is the bisector of $\angle CAD$. Prove that*

(i) *$\angle PAC = \angle BCA$ and*
(ii) *ABCP is a parallelogram.*



SOLUTION In $\triangle ABC$, $AB = AC \Rightarrow \angle 1 = \angle 2$.

Ext. $\angle CAD = \angle 1 + \angle 2$

$\Rightarrow \quad \angle CAD = 2(\angle 2) \quad [\because \angle 1 = \angle 2]$

$\Rightarrow \quad 2\angle PAC = 2(\angle 2) \Rightarrow \angle PAC = \angle 2 \Rightarrow \angle PAC = \angle BCA.$

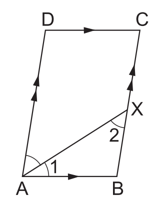
Now, $\angle PAC = \angle BCA \Rightarrow AP \parallel BC$

($\because$ $\angle PAC$ and $\angle BCA$ are alt. interior $\angle$s).

Also, $CP \parallel BA$ (given).

$\therefore$ $ABCP$ is a parallelogram.

EXAMPLE 18 *In the adjoining figure, ABCD is a parallelogram and the bisector of $\angle A$ bisects BC at X. Prove that $AD = 2AB$.*



SOLUTION $ABCD$ is a parallelogram.

$\therefore$ $AD \parallel BC$ and AX cuts them.

$\therefore \quad \angle BXA = \angle DAX = \frac{1}{2}\angle A$ (alt. interior $\angle$s).

$\therefore \quad \angle 2 = \frac{1}{2}A$. Also, $\angle 1 = \frac{1}{2}A$.

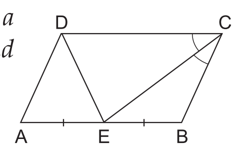
$\therefore \quad \angle 2 = \angle 1 \Rightarrow AB = BX = \frac{1}{2}BC = \frac{1}{2}AD.$

Hence, $AD = 2AB$.

EXAMPLE 19 *In the adjoining figure, ABCD is a parallelogram, E is the midpoint of AB and CE bisects $\angle BCD$. Prove that*

(i) $AE = AD$, (ii) DE bisects $\angle ADC$ and

(iii) $\angle DEC = 90°$.



SOLUTION $AB \parallel DC$ and EC cuts them $\Rightarrow \angle BEC = \angle ECD$

$\Rightarrow \quad \angle BEC = \angle ECB \quad [\because \angle ECD = \angle ECB]$

$\Rightarrow \quad EB = BC \Rightarrow AE = AD.$

Now, $AE = AD \Rightarrow \angle ADE = \angle AED \Rightarrow \angle ADE = \angle EDC$

[$\because$ $\angle AED = \angle EDC$ (alt. interior $\angle$s)].

$\therefore$ DE bisects $\angle ADC$.

Further, $\angle ADC + \angle BCD = 180°$ [co-interior $\angle$s]

$\Rightarrow \quad \frac{1}{2}\angle ADC + \frac{1}{2}\angle BCD = 90° \Rightarrow \angle EDC + \angle DCE = 90°.$

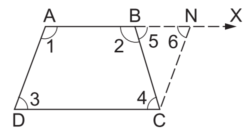
But, $\angle EDC + \angle DCE + \angle DEC = 180°$ [sum of the $\angle$s of a $\triangle$]

$\therefore \quad \angle DEC = 90°.$

EXAMPLE 20 *ABCD is a trapezium in which $AB \parallel CD$ and $AD = BC$. Show that*
(i) $\angle A = \angle B$ and $\angle C = \angle D$, (ii) $\triangle ABC \cong \triangle BAD$ and
(iii) $AC = BD$.

GIVEN A trapezium $ABCD$ in which $AB \parallel CD$ and $AD = BC$.

TO PROVE (i) $\angle A = \angle B$ and $\angle C = \angle D$,
(ii) $\triangle ABC \cong \triangle BAD$ and
(iii) $AC = BD$.



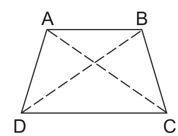
CONSTRUCTION Produce AB to X. Draw $CN \parallel DA$ such that CN meets AX at N.

PROOF We have

$AD \parallel NC$ and $AN \parallel DC$ (since $AB \parallel DC$) $\Rightarrow$ $ADCN$ is a ∥gm.

$\therefore$ $NC = AD = BC \Rightarrow \angle 5 = \angle 6$. … (i)

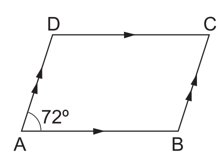
Now, $\angle 1 + \angle 6 = 180^\circ$ (co-interior ∠s)
and $\angle 2 + \angle 5 = 180^\circ$ (linear pair)
$\therefore$ $\angle 1 = \angle 2$ [using (i)].
Now, $\angle 3 = \angle 6$ (opp. ∠s of a ∥gm)
and $\angle 4 = \angle 5$ (alt. interior ∠s)
$\therefore$ $\angle 3 = \angle 4$ [using (i)].



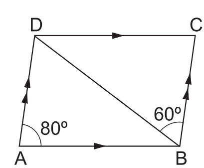
Thus, $\angle A = \angle B$ and $\angle C = \angle D$.
Now, in △s ABC and BAD, we have:
$AB = BA$ (common)
$\angle ABC = \angle BAD$ (prove that $\angle A = \angle B$)
$AD = BC$ (given)
$\therefore$ $\triangle ABC \cong \triangle BAD$ (SAS-criterion).
And so, $AC = BD$ (c.p.c.t.).

EXERCISE 10B

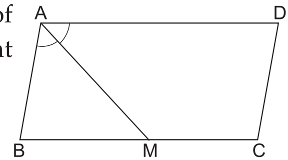
1. In the adjoining figure, $ABCD$ is a parallelogram in which $\angle A = 72^\circ$. Calculate $\angle B$, $\angle C$ and $\angle D$.



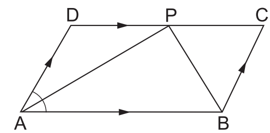
2. In the adjoining figure, $ABCD$ is a parallelogram in which $\angle DAB = 80^\circ$ and $\angle DBC = 60^\circ$. Calculate $\angle CDB$ and $\angle ADB$.



3. In the adjoining figure, M is the midpoint of side BC of a parallelogram $ABCD$ such that $\angle BAM = \angle DAM$. Prove that $AD = 2CD$.

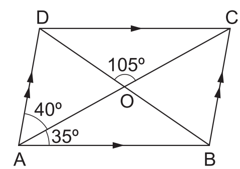


4. In the adjoining figure, $ABCD$ is a parallelogram in which $\angle A = 60°$. If the bisectors of $\angle A$ and $\angle B$ meet DC at P, prove that (i) $\angle APB = 90°$, (ii) $AD = DP$ and $PB = PC = BC$, (iii) $DC = 2AD$.



5. In the adjoining figure, $ABCD$ is a parallelogram in which $\angle BAO = 35°$, $\angle DAO = 40°$ and $\angle COD = 105°$.

Calculate (i) $\angle ABO$, (ii) $\angle ODC$, (iii) $\angle ACB$ and (iv) $\angle CBD$.



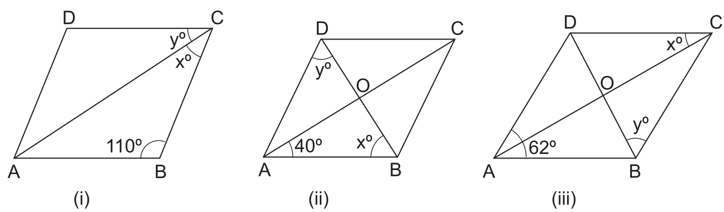
6. In a ||gm $ABCD$, if $\angle A = (2x + 25)°$ and $\angle B = (3x - 5)°$, find the value of x and the measure of each angle of the parallelogram.

7. If an angle of a parallelogram is four fifths of its adjacent angle, find the angles of the parallelogram.

8. Find the measure of each angle of a parallelogram, if one of its angles is 30° less than twice the smallest angle.

9. $ABCD$ is a parallelogram in which $AB = 9.5$ cm and its perimeter is 30 cm. Find the length of each side of the parallelogram.

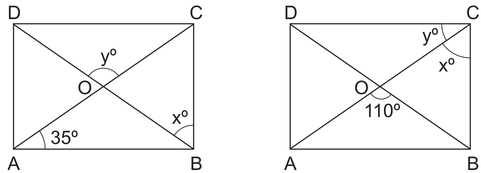
10. In each of the figures given below, $ABCD$ is a rhombus. Find the value of x and y in each case.



11. The lengths of the diagonals of a rhombus are 24 cm and 18 cm respectively. Find the length of each side of the rhombus.

12. Each side of a rhombus is 10 cm long and one of its diagonals measures 16 cm. Find the length of the other diagonal and hence find the area of the rhombus.

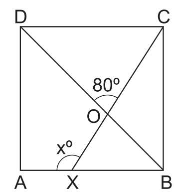
13. In each of the figures given below, $ABCD$ is a rectangle. Find the values of x and y in each case.



(i) (ii)

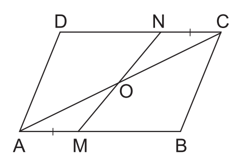
14. In a rhombus $ABCD$, the altitude from D to the side AB bisects AB. Find the angles of the rhombus.

15. In the adjoining figure, $ABCD$ is a square. A line segment CX cuts AB at X and the diagonal BD at O such that $\angle COD = 80^\circ$ and $\angle OXA = x^\circ$. Find the value of x.

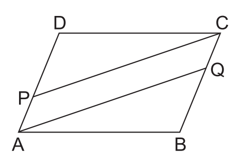


16. In a rhombus $ABCD$ show that diagonal AC bisects $\angle A$ as well as $\angle C$ and diagonal BD bisects $\angle B$ as well as $\angle D$.

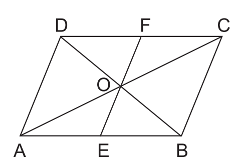
17. In a parallelogram $ABCD$, points M and N have been taken on opposite sides AB and CD respectively such that $AM = CN$. Show that AC and MN bisect each other.



18. In the adjoining figure, $ABCD$ is a parallelogram. If P and Q are points on AD and BC respectively such that $AP = \frac{1}{3}AD$ and $CQ = \frac{1}{3}BC$, prove that $AQCP$ is a parallelogram.



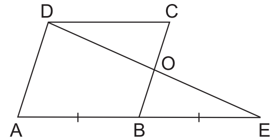
19. In the adjoining figure, $ABCD$ is a parallelogram whose diagonals intersect each other at O. A line segment EOF is drawn to meet AB at E and DC at F. Prove that $OE = OF$.



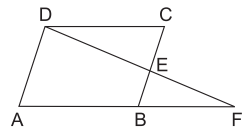
20. The angle between two altitudes of a parallelogram through the vertex of an obtuse angle of the parallelogram is 60°. Find the angles of the parallelogram.

21. $ABCD$ is a rectangle in which diagonal AC bisects $\angle A$ as well as $\angle C$. Show that (i) $ABCD$ is a square, (ii) diagonal BD bisects $\angle B$ as well as $\angle D$.

22. In the adjoining figure, $ABCD$ is a parallelogram in which AB is produced to E so that $BE = AB$. Prove that ED bisects BC.



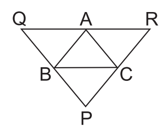
23. In the adjoining figure, $ABCD$ is a parallelogram and E is the midpoint of side BC. If DE and AB when produced meet at F, prove that $AF = 2AB$.



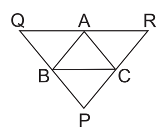
24. Two parallel lines l and m are intersected by a transversal t. Show that the quadrilateral formed by the bisectors of interior angles is a rectangle.

25. K, L, M and N are points on the sides AB, BC, CD and DA respectively of a square $ABCD$ such that $AK = BL = CM = DN$. Prove that $KLMN$ is a square.

26. A $\triangle ABC$ is given. If lines are drawn through A, B, C, parallel respectively to the sides BC, CA and AB, forming $\triangle PQR$, as shown in the adjoining figure, show that $BC = \frac{1}{2}QR$.



27. In the adjoining figure, $\triangle ABC$ is a triangle and through A, B, C lines are drawn, parallel respectively to BC, CA and AB, intersecting at P, Q and R. Prove that the perimeter of $\triangle PQR$ is double the perimeter of $\triangle ABC$.



ANSWERS (EXERCISE 10B)

1. $\angle B = 108°, \angle C = 72°, \angle D = 108°$ **2.** $\angle CDB = 40°, \angle ADB = 60°$

5. (i) $\angle ABO = 40°$ (ii) $\angle ODC = 40°$ (iii) $\angle ACB = 40°$ (iv) $\angle CBD = 65°$

6. $x = 32$; $\angle A = 89°, \angle B = 91°, \angle C = 89°$ and $\angle D = 91°$

7. $80°, 100°, 80°, 100°$ **8.** $70°, 110°, 70°, 110°$

9. $AB = 9.5$ cm $= DC$, $BC = 5.5$ cm $= DA$

10. (i) $x = 35, y = 35$ (ii) $x = 50, y = 50$ (iii) $x = 31, y = 59$

11. 15 cm **12.** 12 cm, 96 cm^2

13. (i) $x = 55, y = 110$ (ii) $x = 55, y = 35$ **14.** 60°, 120°, 60°, 120°

15. $x = 130$ **20.** 60°, 120°, 60°, 120°

HINTS TO SOME SELECTED QUESTIONS

3. $\angle AMB = \angle DAM \quad (\because\ AD \parallel BC)$

$\Rightarrow \quad \angle AMB = \angle BAM \Rightarrow AB = BM \Rightarrow CD = \frac{1}{2}(BC) \Rightarrow CD = \frac{1}{2}(AD) \Rightarrow AD = 2CD.$

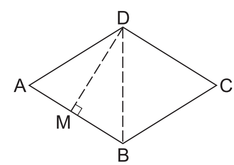
7. $x + \frac{4}{5}x = 180°.$

8. $(2x - 30) + x = 180.$

10. (i) $AB = BC \Rightarrow \angle BAC = \angle BCA = x°$. And, $x + x + 110 = 180.$
(ii) $\angle AOB = 90°$ (ii) $\angle DCB = \angle BAD = 62°$. So, $\angle DCO = 31°.$

13. (i) Diagonals of a rectangle are equal and bisect each other.
$\therefore \quad OA = OB$ and so $\angle OAB = \angle OBA = 35°$. Also, $\angle ABC = 90°.$

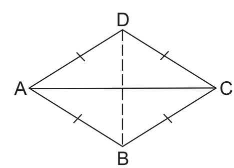
14. $DM \perp AB$ and $AM = MB.$



$\triangle AMD \cong \triangle BMD$
$(\because\ AM = MB, \angle AMD = \angle BMD = 90°, DM = DM).$
$\therefore \quad AB = AD = DB \Rightarrow \triangle ABD$ is equilateral.
So, $\angle A = 60° \Rightarrow \angle B = 120°, \angle C = 60°, \angle D = 120°.$

15. $\angle OBX = 45°$ and $\angle BOX = 80°.$

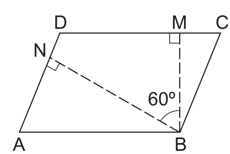
16. $\triangle ABC \cong \triangle ADC \quad (\because\ AD = AB, CD = CB, AC = AC)$
$\therefore \quad \angle BAC = \angle DAC.$



17. $\triangle AMO \cong \triangle CNO \quad (\because\ \angle OAM = \angle OCN, AM = CN, \angle MOA = \angle NOC).$

19. $\triangle ODF \cong \triangle OBE \quad (\because\ OD = OB, \angle DOF = \angle BOE$ and $\angle ODF = \angle OBE).$
$\therefore \quad OF = OE.$

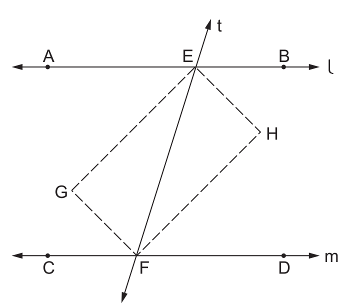
20. $\angle ABM = 90°$ and $\angle MBN = 60° \Rightarrow \angle ABN = 30°.$
In $\triangle ABN$, we have $\angle A = 60°.$
And so, $\angle B = 120°, \angle C = 60°, \angle D = 120°.$



22. $\triangle ODC \cong \triangle OEB \quad (\because\ DC = BE, \angle OCD = \angle OBE, \angle COD = \angle BOE).$
$\therefore \quad OC = OB.$

23. $\triangle DEC = \triangle FEB$. So, $BF = DC = AB.$

24. $\angle AEF + \angle BEF = 180° \quad$ (linear pair)
$\Rightarrow \quad \frac{1}{2}(\angle AEF) + \frac{1}{2}(\angle BEF) = 90°$
$\Rightarrow \quad \angle GEF + \angle FEH = 90° \Rightarrow \angle GEH = 90°.$
Now, $\angle AEF + \angle CFE = 180°$ (co-interior $\angle$s)
$\Rightarrow \quad \frac{1}{2}(\angle AEF) + \frac{1}{2}(\angle CFE) = 90°$
$\Rightarrow \quad \angle GEF + \angle GFE = 90° \Rightarrow \angle EGF = 90°$
(sum of $\angle$s of a $\triangle$ is 180°).
Similarly, $\angle GFH = \angle EHF = 90°.$



26. $BC \parallel QA$ and $CA \parallel QB \Rightarrow BCAQ$ is a ||gm $\Rightarrow BC = QA$.

Similarly, $BCRA$ is a ||gm $\Rightarrow BC = AR$.

$\therefore \quad 2BC = QA + AR = QR$.

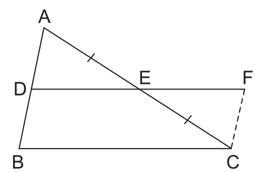
27. As proved in prev. question: $BC = \frac{1}{2}QR$, $CA = \frac{1}{2}PQ$, $AB = \frac{1}{2}PR$

$\Rightarrow \quad AB + BC + CA = \frac{1}{2}(PR + QR + PQ)$.

MIDPOINT THEOREM

THEOREM 1 (Midpoint Theorem) *The line segment joining the midpoints of any two sides of a triangle is parallel to the third side and equal to half of it.*

GIVEN A $\triangle ABC$ in which D and E are the midpoints of AB and AC respectively. DE is joined.



TO PROVE $DE \parallel BC$ and $DE = \frac{1}{2}BC$.

CONSTRUCTION Draw $CF \parallel BA$, meeting DE produced in F.

PROOF In $\triangle$s AED and CEF, we have:

$\angle AED = \angle CEF$ (vert. opp. $\angle$s)

$AE = CE$ [$\because$ E is the midpoint of AC]

$\angle DAE = \angle FCE$ (alt. interior $\angle$s)

$\therefore \quad \triangle AED \cong \triangle CEF$ (ASA-criterion).

And so, $AD = CF$ and $DE = EF$ (c.p.c.t.).

But, $AD = BD$ [$\because$ D is the midpoint of AB]

and $BD \parallel CF$ (by construction)

$\therefore \quad BD = CF$ and $BD \parallel CF$

$\Rightarrow \quad BCFD$ is a ||gm

$\Rightarrow \quad DF \parallel BC$ and $DF = BC$

$\Rightarrow \quad DE \parallel BC$ and $DE = \frac{1}{2}DF = \frac{1}{2}BC$ [$\because$ $DE = EF$].

Hence, $DE \parallel BC$ and $DE = \frac{1}{2}BC$.

THEOREM 2 (Converse of Midpoint Theorem) *The line drawn through the midpoint of one side of a triangle, parallel to another side, bisects the third side.*

GIVEN A $\triangle ABC$ in which D is the midpoint of AB and $DE \parallel BC$.

TO PROVE E is the midpoint of AC.

CONSTRUCTION Draw $CF \parallel BA$, meeting DE produced in F.

PROOF We have

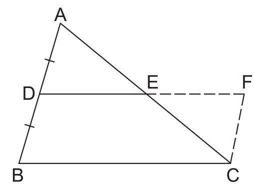
$DF \parallel BC$ [$\because$ $DE \parallel BC$]

$BD \parallel CF$ [$\because$ $CF \parallel BA$]

$\therefore$ $DBCF$ is a ∥gm and so,

$CF = DB = AD$ [$\because$ D is the midpoint of AB].

Now, in $\triangle$s ADE and CFE, we have



$\angle EAD = \angle ECF$ (alt. interior $\angle$s),

$AD = CF$ (proved)

and $\angle ADE = \angle CFE$ (alt. interior $\angle$s).

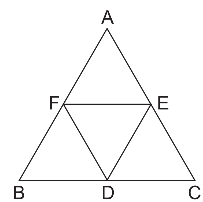
$\therefore$ $\triangle ADE \cong \triangle CFE$ (ASA-criterion)

and so, $AE = CE$ (c.p.c.t.).

Hence, E is the midpoint of AC.

SOME SOLVED PROBLEMS ON MIDPOINT THEOREM

EXAMPLE 1 *If D, E and F are respectively the midpoints of the sides BC, CA and AB of an equilateral triangle ABC, prove that $\triangle DEF$ is also an equilateral triangle.*



SOLUTION Since D and E are the midpoints of sides BC and CA respectively, we have $DE = \frac{1}{2}AB$ (by midpoint theorem).

Similarly, $FE = \frac{1}{2}BC$ and $DF = \frac{1}{2}CA$.

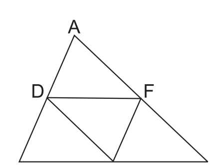
$\therefore$ $AB = BC = CA \Rightarrow \frac{1}{2}AB = \frac{1}{2}BC = \frac{1}{2}CA \Rightarrow DE = FE = DF.$

Thus, all the sides of $\triangle DEF$ are equal.

Hence, $\triangle DEF$ is an equilateral triangle.

EXAMPLE 2 *In $\triangle ABC$, D, E and F are respectively the midpoints of sides AB, BC and CA. Show that $\triangle ABC$ is divided into four congruent triangles by joining D, E and F in pairs.*

SOLUTION D and F are the midpoints of sides AB and AC respectively of $\triangle ABC$.



$\therefore$ $DF \parallel BC$.

Similarly, $DE \parallel AC$ and $EF \parallel AB$.

Now, $DF \parallel BE$ and $EF \parallel BD$

$\Rightarrow$ $DBEF$ is a ∥gm

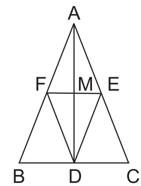
$\Rightarrow \quad \triangle EDB \cong \triangle DEF \qquad$ [$\because$ DE is a diagonal of ||gm $DBEF$].

[NOTE Any diagonal of a ||gm divides it into two congruent triangles.]

Similarly, $\triangle CFE \cong \triangle DEF$ and $\triangle FAD \cong \triangle DEF$.

Hence, all the four triangles are congruent.

EXAMPLE 3 *In the adjoining figure, $\triangle ABC$ is an isosceles triangle in which $AB = AC$ and D, E, F are the midpoints of BC, CA and AB respectively. Show that $AD \perp FE$ and AD is bisected by FE.*



SOLUTION Let AD intersect FE at M. Join DE and DF.

Now, D and E being the midpoints of the sides BC and CA respectively, we have

$$DE \parallel AB \text{ and } DE = \frac{1}{2}AB \quad \text{(by midpoint theorem).}$$

Similarly, $DF \parallel AC$ and $DF = \frac{1}{2}AC$.

$\therefore \quad AB = AC \Rightarrow \frac{1}{2}AB = \frac{1}{2}AC \Rightarrow DE = DF. \qquad \ldots$ (i)

Now, $DE \parallel FA$ and $DE = FA \quad$ [$\because$ $DE \parallel AB$ and $DE = \frac{1}{2}AB = FA$]

$\Rightarrow \quad DEAF$ is a ||gm $\Rightarrow$ $DEAF$ is a rhombus

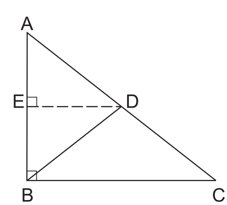
[$\because$ $DE = DF$ from (i), $DE = FA$ and $DF = EA$].

But, the diagonals of a rhombus bisect each other at right angles.

$\therefore \quad AD \perp FE$ and $AM = MD$.

Hence, $AD \perp FE$ and AD is bisected by FE.

EXAMPLE 4 *Let ABC be a triangle, right-angled at B and D be the midpoint of AC. Show that $DA = DB = DC$.*



SOLUTION Through D, draw $DE \parallel BC$, meeting AB at E.

Now, $\angle AED = \angle ABC = 90°$ [corres. $\angle$s]

$\therefore \quad \angle BED = \angle AED = 90° \qquad$ [$\because$ $\angle AED + \angle BED = 180°$].

Now, in $\triangle ABC$, it is given that D is the midpoint of AC and $DE \parallel BC$ (by construction).

$\therefore$ E must be the midpoint of AB

(by converse of midpoint theorem).

$\therefore$ $AE = BE$.

Now, in $\triangle$s AED and BED, we have

$AE = BE$ (proved), $ED = ED$ (common),

$\angle AED = \angle BED$ (each equal to 90°).

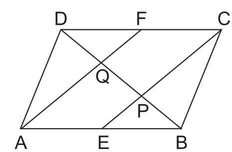
$\therefore$ $\triangle AED \cong \triangle BED$

$\therefore$ $DA = DB$.

But, $DA = DC$ [$\because$ D is the midpoint of AC].

Hence, $DA = DB = DC$.

EXAMPLE 5 *ABCD is a parallelogram in which E and F are the midpoints of the sides AB and CD respectively. Prove that the line segments CE and AF trisect the diagonal BD.*



SOLUTION Let BD be intersected by CE and AF at P and Q respectively.

$AB \parallel DC$ and $AB = DC$ (opposite sides of a ‖gm)

$\Rightarrow$ $AE \parallel FC$ and $\frac{1}{2}AB = \frac{1}{2}DC$ $\Rightarrow$ $AE \parallel FC$ and $AE = FC$

$\Rightarrow$ $AECF$ is a ‖gm $\Rightarrow$ $AF \parallel CE$ $\Rightarrow$ $EP \parallel AQ$ and $FQ \parallel CP$.

In $\triangle BAQ$, E is the midpoint of AB and $EP \parallel AQ$, so P is the midpoint of BQ.

$\therefore$ $BP = PQ$.

Again, in $\triangle DPC$, F is the midpoint of DC and $FQ \parallel CP$, so Q is the midpoint of DP.

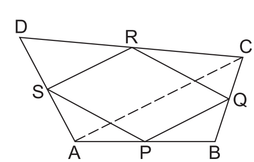
$\therefore$ $PQ = QD$.

$\therefore$ $BP = PQ = QD$.

Hence, CE and AF trisect the diagonal BD.

EXAMPLE 6 *Show that the figure formed by joining the midpoints of the adjacent sides of a quadrilateral is a parallelogram.*

SOLUTION Let $ABCD$ be a quadrilateral in which P, Q, R, S are the midpoints of AB, BC, CD and DA respectively. Join AC.



In $\triangle ABC$, the points P and Q are the midpoints of AB and BC respectively.

$\therefore \quad PQ \parallel AC$ and $PQ = \frac{1}{2}AC$ (by midpoint theorem).

Again, in $\triangle DAC$, the points S and R are the midpoints of AD and DC respectively.

$\therefore \quad SR \parallel AC$ and $SR = \frac{1}{2}AC$ (by midpoint theorem).

Now, $PQ \parallel AC$ and $SR \parallel AC \Rightarrow PQ \parallel SR$.

Also, $PQ = SR$ (each equal to $\frac{1}{2}AC$).

$\therefore \quad PQ \parallel SR$ and $PQ = SR$.

Hence, $PQRS$ is a parallelogram.

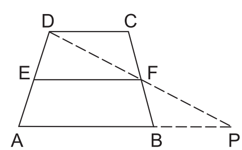
EXAMPLE 7 *E and F are respectively the midpoints of the nonparallel sides AD and BC of a trapezium ABCD. Prove that*

(i) $EF \parallel AB$, *(ii)* $EF = \frac{1}{2}(AB + CD)$.

GIVEN A trapezium $ABCD$ in which E and F are midpoints of sides AD and BC respectively.

TO PROVE (i) $EF \parallel AB$, (ii) $EF = \frac{1}{2}(AB + CD)$.

CONSTRUCTION Join DF and produce it to meet AB produced in P.



PROOF In $\triangle$s DCF and PBF we have

$\angle DFC = \angle PFB$ (vert. opposite $\angle$s)

$CF = BF$ ($\because$ F is the midpoint of BC)

$\angle DCF = \angle PBF$ (alt. interior $\angle$s)

$\therefore \quad \triangle DCF \cong \triangle PBF$ (ASA-criterion).

And so, $DF = PF$ and $CD = BP$ (c.p.c.t).

Now, in $\triangle DAP$, we have

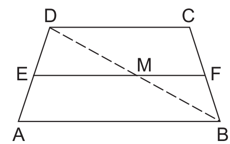
E is the midpoint of AD and F is the midpoint of DP.
[$\because$ $DF = PF$]

$\therefore \quad EF \parallel AP$ and $EF = \frac{1}{2}AP$ [by midpoint theorem]

$\Rightarrow \quad EF \parallel AB$ and $EF = \frac{1}{2}(AB + BP)$.

Hence, $EF \parallel AB$ and $EF = \frac{1}{2}(AB + CD)$.

EXAMPLE 8 *Let ABCD be a trapezium in which $AB \parallel DC$ and let E be the midpoint of AD. Let F be a point on BC such that $EF \parallel AB$. Prove that*



(i) F is the midpoint of BC, (ii) $EF = \frac{1}{2}(AB + DC)$.

SOLUTION Join BD, cutting EF at M.

Now, in $\triangle DAB$, E is the midpoint of AD and $EM \parallel AB$.

$\therefore$ M is the midpoint of BD.

$\therefore$ $EM = \frac{1}{2}AB$. … (i)

Again, in $\triangle BDC$, M is the midpoint of BD and $MF \parallel DC$.

$\therefore$ F is the midpoint of BC.

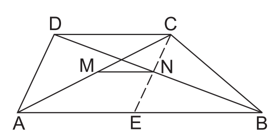
$\therefore$ $MF = \frac{1}{2}DC$. … (ii)

$\therefore$ $EF = EM + MF = \frac{1}{2}(AB + DC)$ [from (i) and (ii)].

Hence, F is the midpoint of BC and $EF = \frac{1}{2}(AB + DC)$.

EXAMPLE 9 *Prove that the line segment joining the midpoints of the diagonals of a trapezium is parallel to the parallel sides and equal to half their difference.*

SOLUTION Let $ABCD$ be a trapezium in which $AB \parallel DC$, and let M and N be the midpoints of the diagonals AC and BD respectively.



Join CN and produce it to meet AB at E.

In $\triangle$s CDN and EBN, we have:

$DN = BN$ [$\because$ N is midpoint of BD]

$\angle DCN = \angle BEN$ (alt. interior $\angle$s)

$\angle CDN = \angle EBN$ (alt. interior $\angle$s)

$\therefore$ $\triangle CDN \cong \triangle EBN$ [SAA-criteria].

$\therefore$ $DC = EB$ and $CN = NE$ (c.p.c.t.).

Thus, in $\triangle CAE$, the points M and N are the midpoints of AC and CE respectively.

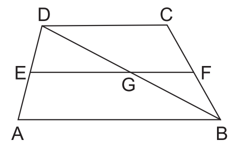
$\therefore$ $MN \parallel AE$ and $MN = \frac{1}{2}AE \Rightarrow MN \parallel AB \parallel DC$

and $MN = \frac{1}{2}AE = \frac{1}{2}(AB - EB) = \frac{1}{2}(AB - DC)$ [$\because$ $EB = DC$].

EXAMPLE 10 *ABCD is a trapezium in which $AB \parallel DC$ and E is the midpoint of AD. A line is drawn through E parallel to AB intersecting BC at F. Show that F is the midpoint of BC.*

GIVEN A trapezium $ABCD$ in which $AB \parallel DC$. E is the midpoint of side AD. A line EF is drawn parallel to AB intersecting BC at F.

TO PROVE F is the midpoint of BC.



CONSTRUCTION Join BD. Let BD intersect EF at G.

PROOF In $\triangle DAB$, we have

E is the midpoint of AD and $EG \parallel AB$.

$\therefore$ G is the midpoint of BD [by converse of midpoint theorem].

Now, in $\triangle BCD$, we have

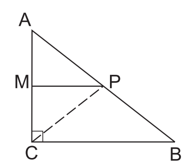
G is the midpoint of BD and $GF \parallel DC$

[$\because$ $EF \parallel AB$ and $AB \parallel DC \Rightarrow EF \parallel DC$].

Hence, F is the midpoint of BC [by converse of midpoint theorem].

EXAMPLE 11 *ABC is a triangle right angled at C. A line through the midpoint P of hypotenuse AB and parallel to BC intersects AC at M. Show that (i) M is the midpoint of AC, (ii) $MP \perp AC$, (iii) $CP = AP = \frac{1}{2}AB$.*

SOLUTION In $\triangle ACB$, we have



P is the midpoint of AB and $PM \parallel BC$.

$\therefore$ M is the midpoint of AC.

[by converse of midpoint theorem].

Now, $PM \parallel BC$.

$\therefore$ $\angle PMC + \angle BCM = 180^\circ$ (co-interior $\angle$s)

$\Rightarrow$ $\angle PMC + 90^\circ = 180^\circ \Rightarrow \angle PMC = 90^\circ$.

Thus, $MP \perp AC$.

Join PC.

In $\triangle$s PMA and PMC, we have:

$MA = MC$ [$\because$ M is the midpoint of AC]

$\angle PMA = \angle PMC$ (each equal to 90°)

$PM = PM$ (common)

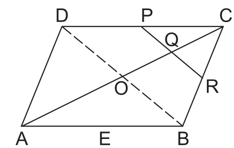
$\therefore$ $\triangle PMA \cong \triangle PMC$ (SAS-criterion).

And so, $AP = CP$ (c.p.c.t.).

Now, P is the midpoint of AB.

$\therefore \quad CP = AP = \frac{1}{2}AB.$

EXAMPLE 12 *In the adjoining figure, ABCD is a ∥gm in which P is the midpoint of DC and Q is a point on AC such that* $CQ = \frac{1}{4}AC$. *Also, PQ when produced meets BC at R. Prove that R is the midpoint of BC.*



SOLUTION Join BD, intersecting AC at O.

Then, $AO = OC$ [$\because$ diagonals of a ∥gm bisect each other].

$\therefore \quad CQ = \frac{1}{4}AC = \frac{1}{4} \times (2OC) = \frac{1}{2}OC.$

Thus, Q is the midpoint of OC.

Now, in $\triangle CDO$, P and Q are the midpoints of CD and CO respectively.

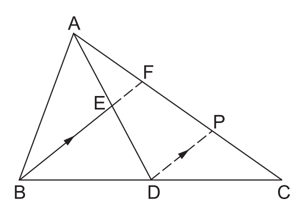
$\therefore \quad PQ \parallel DO$ and therefore, $QR \parallel OB$

[$\because PQ \parallel DO \Rightarrow PQR \parallel DOB$].

Now, in $\triangle COB$, Q is the midpoint of CO and $QR \parallel OB$.

$\therefore \quad R$ must be the midpoint of BC.

EXAMPLE 13 *In the adjoining figure, AD is a median of* $\triangle ABC$ *and E is the midpoint of AD. Also, BE produced meets AC in F. Prove that* $AF = \frac{1}{3}AC$.



SOLUTION Draw $DP \parallel EF$.

In $\triangle ADP$, E is the midpoint of AD and $EF \parallel DP$.

$\therefore \quad F$ is the midpoint of AP, i.e., $AF = FP$.

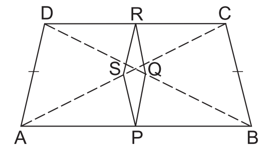
In $\triangle FBC$, D is the midpoint of BC and $DP \parallel BF$.

$\therefore \quad P$ is the midpoint of FC, i.e., $FP = PC$.

Thus, $AF = FP = PC$.

Hence, $AF = \frac{1}{3}AC.$

<u>EXAMPLE 14</u> *In the adjoining figure, ABCD is a trapezium in which $AB \parallel DC$ and $AD = BC$. If P, Q, R, S be respectively the midpoints of BA, BD, CD, CA then show that PQRS is a rhombus.*



SOLUTION In $\triangle BDC$, Q and R are the midpoints of BD and CD respectively.

$\therefore \quad QR \parallel BC$ and $QR = \frac{1}{2}BC$.

Similarly, $PS \parallel BC$ and $PS = \frac{1}{2}BC$.

$\therefore \quad PS \parallel QR$ and $PS = QR \quad$ (each equal to $\frac{1}{2}BC$).

$\therefore \quad PQRS$ is a parallelogram.

In $\triangle ACD$, S and R are the midpoints of AC and CD respectively.

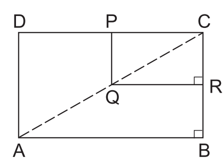
$\therefore \quad SR \parallel AD$ and $SR = \frac{1}{2}AD = \frac{1}{2}BC \qquad [\because \; AD = BC]$.

$\therefore \quad PS = QR = SR = PQ$.

Hence, $PQRS$ is a rhombus.

<u>EXAMPLE 15</u> *In the adjoining figure, ABCD and PQRC are rectangles, where Q is the midpoint of AC. Prove that*

(i) $DP = PC$, (ii) $PR = \frac{1}{2}AC$.



SOLUTION $\angle CRQ = \angle CBA = 90^\circ \Rightarrow QR \parallel AB$.

In $\triangle ABC$, Q is the midpoint of AC and $QR \parallel AB$.

$\therefore \quad R$ is the midpoint of BC, i.e., $BR = RC$.

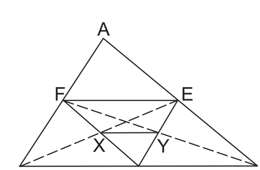
Similarly, P is the midpoint of DC.

$\therefore \quad DP = PC$.

In $\triangle CDB$, P is the midpoint of DC and R is the midpoint of BC.

$\therefore \quad PR \parallel DB$ and $PR = \frac{1}{2}DB = \frac{1}{2}AC \qquad [\because \; AC = BD]$.

<u>EXAMPLE 16</u> *In the adjoining figure, D, E, F are the midpoints of the sides BC, CA and AB of $\triangle ABC$. If BE and DF intersect at X while CF and DE intersect at Y, prove that $XY = \frac{1}{4}BC$.*



SOLUTION In $\triangle ABC$, F and E are the midpoints of AB and AC respectively.

$\therefore \quad FE \parallel BC$ and $FE = \frac{1}{2}BC = BD$.

$\therefore \quad FE \parallel BD$ and $FE = BD$.

So, $BDEF$ is a parallelogram whose diagonals BE and DF intersect each other at X.

$\therefore \quad X$ is the midpoint of DF.

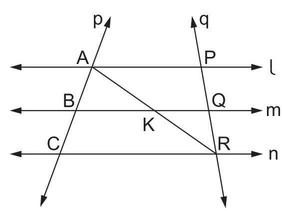
Similarly, Y is the midpoint of DE.

Thus, in $\triangle DEF$, X and Y are the midpoints of DF and DE respectively.

So, $XY \parallel FE$ and $XY = \frac{1}{2}FE = \frac{1}{2} \times \frac{1}{2}BC = \frac{1}{4}BC$.

THEOREM 3 (Intercept Theorem) *If there are three parallel lines and the intercepts made by them on one transversal are equal then the intercepts on any other transversal are also equal.*

GIVEN Three parallel lines l, m and n are cut by a transversal p at A, B, C respectively such that $AB = BC$. Also, q is another transversal, cutting l, m, n at P, Q, R respectively.



TO PROVE $PQ = QR$.

CONSTRUCTION Join AR. Let AR intersect m at K.

PROOF In $\triangle ACR$, we have

B is the midpoint of AC and $BK \parallel CR$ [$\because m \parallel n$].

$\therefore \quad K$ is the midpoint of AR (by converse of midpoint theorem).

Now, in $\triangle RPA$, we have

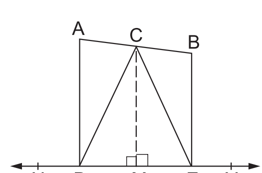
K is the midpoint of RA and $KQ \parallel AP$ [$\because m \parallel l$].

$\therefore \quad Q$ is the midpoint of PR (by converse of midpoint theorem).

Hence, $PQ = QR$.

SOME SOLVED PROBLEMS ON INTERCEPT THEOREM

EXAMPLE 1 *In the adjoining figure, two points A and B lie on the same side of a line XY. If $AD \perp XY$, $BE \perp XY$ and C is the midpoint of AB, prove that $CD = CE$.*



SOLUTION Draw $CM \perp XY$.

Now, $AD \perp XY$, $CM \perp XY$ and $BE \perp XY \Rightarrow AD \parallel CM \parallel BE$.

Thus, AD, CM, BE are three parallel lines, cut by the transversal ACB at A, C and B respectively such that $AC = CB$.

These lines AD, CM, BE are also cut by the transversal XY at D, M and E respectively.

$\therefore \quad DM = ME \quad$ (by intercept theorem).

Now, in $\triangle$s CDM and CEM, we have

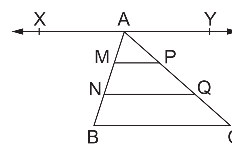
$DM = ME \quad$ (proved), $\qquad CM = CM \quad$ (common),

$\angle CMD = \angle CME = 90°$.

$\therefore \quad \triangle CDM \cong \triangle CEM$.

Hence, $CD = CE \quad$ (c.p.c.t.).

EXAMPLE 2 *In the adjoining figure, points M and N divide the side AB of $\triangle ABC$ into three equal parts. Line segments MP and NQ are both parallel to BC and meet AC in P and Q respectively. Prove that P and Q divide AC into three equal parts.*



SOLUTION Through A, draw $XAY \parallel BC$.

Now, $XY \parallel MP \parallel NQ$ are cut by the transversal AB at A, M, N respectively such that $AM = MN$.

Also, $XY \parallel MP \parallel NQ$ are cut by the transversal AC at A, P, Q respectively.

$\therefore \quad AP = PQ \quad$ … (i) (by intercept theorem).

Again, $MP \parallel NQ \parallel BC$ are cut by the transversal AB at M, N, B respectively such that $MN = NB$.

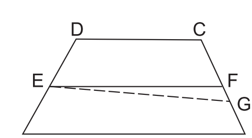
Also, $MP \parallel NQ \parallel BC$ are cut by the transversal AC at P, Q, C respectively.

$\therefore \quad PQ = QC \quad$ … (ii) (by intercept theorem).

Thus, from (i) and (ii), we get $AP = PQ = QC$.

Hence, P and Q divide AC into three equal parts.

EXAMPLE 3 *E and F are respectively the midpoints of nonparallel sides AD and BC of a trapezium ABCD. Prove that $EF \parallel AB$.*



SOLUTION Let $ABCD$ be a trapezium in which $AB \parallel DC$. Let E and F be the midpoints of AD and BC respectively. E and F are joined.

We have to show that $EF \parallel AB$.

If possible, let EF be not parallel to AB then draw $EG \parallel AB$, meeting BC at G.

Now, $AB \parallel EG \parallel DC$ and the transversal AD cuts them at A, E, D respectively such that $AE = ED$.

Also, BGC is the other transversal cutting AB, EG and DC at B, G and C respectively.

$\therefore \quad BG = GC \quad$ (by intercept theorem).

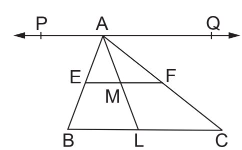
This shows that G is the midpoint of BC.

Hence, G must coincide with F $\quad [\because F$ is the midpoint of $BC]$.

Thus, our supposition is wrong.

Hence, $EF \parallel AB$.

EXAMPLE 4 *Prove that any line segment drawn from the vertex of a triangle to the base is bisected by the line segment joining the midpoints of the other sides of the triangle.*



SOLUTION Let $\triangle ABC$ be a given triangle in which E and F are the midpoints of AB and AC respectively. Let AL be a line segment drawn from vertex A to the base BC, meeting BC at L and EF at M.

We have to show that $AM = ML$.

Through A, draw $PAQ \parallel BC$.

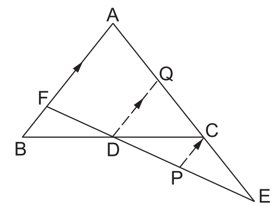
In $\triangle ABC$; E and F being the midpoints of AB and AC respectively, we have $EF \parallel BC$.

Now, PAQ, EF and BC are three parallel lines such that the intercepts AE and EB made by them on transversal AEB are equal.

$\therefore \quad$ the intercepts AM and ML made by them on transversal AML must be equal.

Hence, $AM = ML$.

EXAMPLE 5 *In the adjoining figure, the side AC of $\triangle ABC$ is produced to E such that $CE = \frac{1}{2}AC$. If D is the midpoint of BC and ED produced meets AB at F, and CP, DQ are drawn parallel to BA, prove that $FD = \frac{1}{3}FE$.*



SOLUTION In $\triangle ABC$, D is the midpoint of BC and $DQ \parallel BA$.

$\therefore$ Q is the midpoint of AC.

$\therefore$ $AQ = QC$.

Now, $FA \parallel DQ \parallel PC$, and AQC is the transversal such that $AQ = QC$ and FDP is the other transversal on them.

$\therefore$ $FD = DP$ … (i) (by intercept theorem).

Now, $EC = \frac{1}{2}AC = QC$.

$\therefore$ in $\triangle EQD$, C is the midpoint of EQ and $CP \parallel DQ$.

$\therefore$ P must be the midpoint of DE.

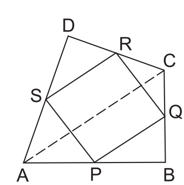
$\therefore$ $DP = PE$. … (ii)

Thus, $FD = DP = PE$ [from (i) and (ii)].

Hence, $FD = \frac{1}{3}FE$.

EXERCISE 10C

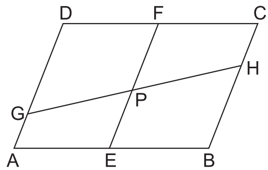
1. P, Q, R and S are respectively the midpoints of the sides AB, BC, CD and DA of a quadrilateral $ABCD$. Show that



   (i) $PQ \parallel AC$ and $PQ = \frac{1}{2}AC$
   (ii) $PQ \parallel SR$
   (iii) $PQRS$ is a parallelogram.

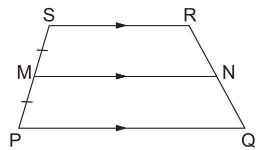
2. A square is inscribed in an isosceles right triangle so that the square and the triangle have one angle common. Show that the vertex of the square opposite the vertex of the common angle bisects the hypotenuse.

3. In the adjoining figure, $ABCD$ is a ||gm in which E and F are the midpoints of AB and CD respectively. If GH is a line segment that cuts AD, EF and BC at G, P and H respectively, prove that $GP = PH$.



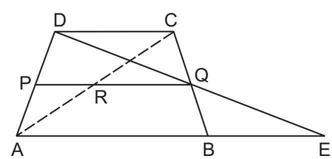
4. M and N are points on opposite sides AD and BC of a parallelogram $ABCD$ such that MN passes through the point of intersection O of its diagonals AC and BD. Show that MN is bisected at O.

5. In the adjoining figure, $PQRS$ is a trapezium in which $PQ \parallel SR$ and M is the midpoint of PS. A line segment $MN \parallel PQ$ meets QR at N. Show that N is the midpoint of QR.

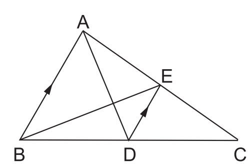


6. In a parallelogram $PQRS$, $PQ = 12$ cm and $PS = 9$ cm. The bisector of $\angle P$ meets SR in M. PM and QR both when produced meet at T. Find the length of RT.

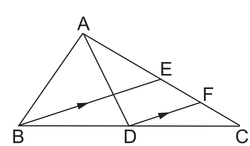
7. In the adjoining figure, $ABCD$ is a trapezium in which $AB \parallel DC$ and P, Q are the midpoints of AD and BC respectively. DQ and AB when produced meet at E. Also, AC and PQ intersect at R. Prove that (i) $DQ = QE$, (ii) $PR \parallel AB$ and (iii) $AR = RC$.



8. In the adjoining figure, AD is a median of $\triangle ABC$ and $DE \parallel BA$. Show that BE is also a median of $\triangle ABC$.

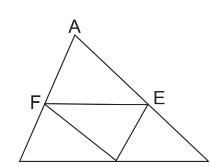


9. In the adjoining figure, AD and BE are the medians of $\triangle ABC$ and $DF \parallel BE$. Show that $CF = \frac{1}{4}AC$.



10. Prove that the line segments joining the middle points of the sides of a triangle divide it into four congruent triangles.

11. In the adjoining figure, D, E, F are the midpoints of the sides BC, CA and AB respectively, of $\triangle ABC$. Show that $\angle EDF = \angle A$, $\angle DEF = \angle B$ and $\angle DFE = \angle C$.



12. Show that the quadrilateral formed by joining the midpoints of the pairs of adjacent sides of a rectangle is a rhombus.

13. Show that the quadrilateral formed by joining the midpoints of the pairs of adjacent sides of a rhombus is a rectangle.

14. Show that the quadrilateral formed by joining the midpoints of the pairs of adjacent sides of a square is a square.

15. Prove that the line segments joining the midpoints of opposite sides of a quadrilateral bisect each other.

16. The diagonals of a quadrilateral $ABCD$ are equal. Prove that the quadrilateral formed by joining the midpoints of its sides is a rhombus.

17. The diagonals of a quadrilateral $ABCD$ are perpendicular to each other. Prove that the quadrilateral formed by joining the midpoints of its sides is a rectangle.

18. The midpoints of the sides AB, BC, CD and DA of a quadrilateral $ABCD$ are joined to form a quadrilateral. If $AC = BD$ and $AC \perp BD$ then prove that the quadrilateral formed is a square.

ANSWERS (EXERCISE 10C)

6. 3 cm

HINTS TO SOME SELECTED QUESTIONS

1. In $\triangle ABC$, P and Q are the midpoints of AB and BC respectively.

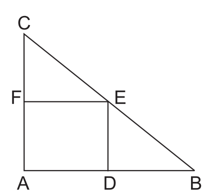
   $\therefore \quad PQ \parallel AC$ and $PQ = \frac{1}{2}AC$. Similarly, $SR \parallel AC$.

2. $AB = AC$ and $AD = AF \Rightarrow AB - AD = AC - AF \Rightarrow DB = CF$.

   We have $\triangle EFC \cong \triangle EDB$

   [$\because CF = DB, \angle EFC = \angle EDB = 90^\circ, EF = ED$].

   $\therefore \quad CE = EB$ (c.p.c.t.).



3. $FC = EB$ and $FC \parallel EB \Rightarrow EBCF$ is a ∥gm

   $\Rightarrow AD \parallel EF \parallel BC$.

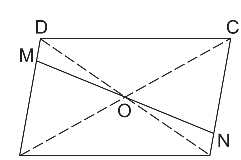
   $\therefore \quad DF = FC \Rightarrow GP = PH$ (by intercept theorem).

4. $\triangle AOM \cong \triangle CON$

   [$\because \angle MAO = \angle NCO, AO = CO, \angle MOA = \angle NOC$]

   [Note Diagonals of a ∥gm bisect each other].

   $\therefore \quad MO = NO$ (c.p.c.t.).



5. $AB \parallel DC$ and $EF \parallel AB \Rightarrow AB \parallel EF \parallel DC$.

   E is the midpoint of $AD \Rightarrow F$ is the midpoint of BC (by intercept theorem).

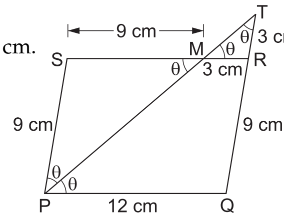
6. $\angle QPM = \angle SPM = \theta$ ($\because$ PM is bisector of $\angle P$)

$\angle QPM = \angle PMS = \theta$ (alt. interior $\angle$s) $\Rightarrow$ $PS = SM = 9$ cm.

$\angle RMT = \angle PMS = \theta$ (vert. opposite $\angle$s)

$\angle RTM = \angle SPM = \theta$ (alt. interior $\angle$s)

$\Rightarrow$ $RT = RM = SR - SM = (12 - 9)$ cm $= 3$ cm.



7. $\triangle QCD \cong \triangle QBE$. So, $DQ = QE$.

$PQ \parallel AE \Rightarrow PQ \parallel AB \parallel DC \Rightarrow AB \parallel PR \parallel DC$.

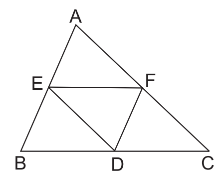
Now, $AB \parallel PR \parallel DC$ are cut by AD and AC. Use intercept theorem.

10. Let D, E, F be the midpoints of BC, CA and AB.

Then, $DE = \frac{1}{2}AB = AF$, $DF = \frac{1}{2}AC = AE$.

$\therefore$ $\triangle AFE \cong \triangle DEF$ [$\because$ $AF = DE$, $AE = DF$, $FE = EF$].

Similarly, $\triangle DEF \cong \triangle FBD$ and $\triangle DEF \cong \triangle EDC$.



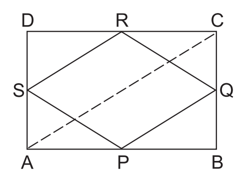
11. $DE \parallel BA$ and $DE = \frac{1}{2}BA \Rightarrow DE \parallel FA$ and $DE = FA$.

$\therefore$ $DEAF$ is a $\parallel$gm.

So, $\angle EDF = \angle A$ [opposite $\angle$s of a $\parallel$gm].

Similarly, $\angle DEF = \angle B$ and $\angle DFE = \angle C$.

12. Let $ABCD$ be a rectangle and P, Q, R, S be the midpoints of AB, BC, CD and DA respectively. We have shown earlier that $PQRS$ is a $\parallel$gm.



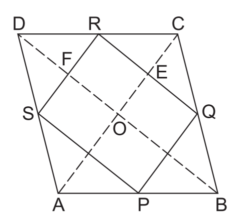
Now, $\triangle SAP \cong \triangle QBP$

[$\because$ $AS = BQ$, $\angle A = \angle B = 90^\circ$ and $AP = BP$].

$\therefore$ $PS = PQ$.

Hence, $PQRS$ is a rhombus.

13. Let $ABCD$ be a rhombus and P, Q, R, S be the midpoints of AB, BC, CD and DA respectively. Then, $PQRS$ is a $\parallel$gm [prove it].



Diagonals of a rhombus bisect each other at right angles.

$\therefore$ $\angle EOF = 90^\circ$.

Now, $RQ \parallel DB \Rightarrow RE \parallel FO$, $SR \parallel AC \Rightarrow FR \parallel OE$.

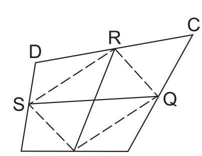
$\therefore$ $OERF$ is a $\parallel$gm. So, $\angle FRE = \angle EOF = 90^\circ$.

Thus, $PQRS$ is a $\parallel$gm with $\angle R = 90^\circ$.

Hence, $PQRS$ is a rectangle.

15. The diagonals of a $\parallel$gm bisect each other.

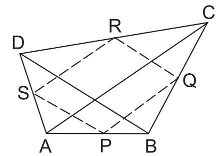
[Note $PQRS$ is a $\parallel$gm.]



16. $SR = \frac{1}{2}AC$; $PQ = \frac{1}{2}AC$;

$SP = \frac{1}{2}BD$ and $QR = \frac{1}{2}BD$.

Since $AC = BD$; we have $PQ = QR = RS = SP$.



17. P, Q, R and S are midpoints of sides AB, BC, CD and DA respectively.

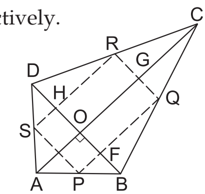
$\therefore$ $PS \parallel BD$ and $PQ \parallel AC$

$\Rightarrow$ $PE \parallel FO$ and $PF \parallel EO$, i.e., $EPFO$ is a $\parallel$gm.

And so, $\angle EPF = \angle EOF = 90^\circ$, i.e., $\angle SPQ = 90^\circ$.

Now, $PQRS$ is a $\parallel$gm with $\angle SPQ = 90^\circ$.

$\therefore$ $PQRS$ is a rectangle.



18. Solve using the solutions of Question Nos. 16 and 17.

MULTIPLE-CHOICE QUESTIONS (MCQ)

Choose the correct answer in each of the following questions:

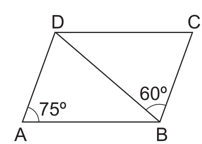
1. Three angles of a quadrilateral are 80°, 95° and 112°. Its fourth angle is

(a) 78° (b) 73° (c) 85° (d) 100°

2. The angles of a quadrilateral are in the ratio 3 : 4 : 5 : 6. The smallest of these angles is

(a) 45° (b) 60° (c) 36° (d) 48°

3. In the adjoining figure, $ABCD$ is a parallelogram in which $\angle BAD = 75^\circ$ and $\angle CBD = 60^\circ$. Then, $\angle BDC = ?$



(a) 60° (b) 75°

(c) 45° (d) 50°

4. $ABCD$ is a rhombus such that $\angle ACB = 50^\circ$. Then, $\angle ADB = ?$

(a) 40° (b) 25° (c) 65° (d) 130°

5. In which of the following figures are the diagonals equal?

(a) Parallelogram (b) Rhombus

(c) Trapezium (d) Rectangle

6. If the diagonals of a quadrilateral bisect each other at right angles then the figure is a

(a) trapezium (b) parallelogram

(c) rectangle (d) rhombus

7. The lengths of the diagonals of a rhombus are 16 cm and 12 cm. The length of each side of the rhombus is

(a) 10 cm (b) 12 cm (c) 9 cm (d) 8 cm

8. The length of each side of a rhombus is 10 cm and one of its diagonals is of length 16 cm. The length of the other diagonal is

(a) 13 cm (b) 12 cm (c) $2\sqrt{39}$ cm (d) 6 cm

9. A diagonal of a rectangle is inclined to one side of the rectangle at 35°. The acute angle between the diagonals is

(a) 55° (b) 70° (c) 45° (d) 50°

10. If $ABCD$ is a parallelogram with two adjacent angles $\angle A = \angle B$ then the parallelogram is a

(a) rhombus (b) trapezium (c) rectangle (d) none of these

11. In a quadrilateral $ABCD$, AO and BO are the bisectors of $\angle A$ and $\angle B$ respectively, $\angle C = 70°$ and $\angle D = 30°$. Then, $\angle AOB = ?$

(a) 40° (b) 50° (c) 80° (d) 100°

12. The bisectors of any two adjacent angles of a parallelogram intersect at

(a) 30° (b) 45° (c) 60° (d) 90°

13. The bisectors of the angles of a parallelogram enclose a

(a) rhombus (b) square (c) rectangle (d) parallelogram

14. If bisectors of $\angle A$ and $\angle B$ of a quadrilateral $ABCD$ intersect each other at P, of $\angle B$ and $\angle C$ at Q, of $\angle C$ and $\angle D$ at R and of $\angle D$ and $\angle A$ at S then $PQRS$ is a

(a) rectangle
(b) parallelogram
(c) rhombus
(d) quadrilateral whose opposite angles are supplementary

15. The figure formed by joining the midpoints of the adjacent sides of a quadrilateral is a

(a) rhombus (b) square (c) rectangle (d) parallelogram

16. The figure formed by joining the midpoints of the adjacent sides of a square is a

(a) rhombus (b) square (c) rectangle (d) parallelogram

17. The figure formed by joining the midpoints of the adjacent sides of a parallelogram is a

(a) rhombus (b) square (c) rectangle (d) parallelogram

18. The figure formed by joining the midpoints of the adjacent sides of a rectangle is a

(a) rhombus (b) square (c) rectangle (d) parallelogram

19. The figure formed by joining the midpoints of the adjacent sides of a rhombus is a

(a) rhombus (b) square (c) rectangle (d) parallelogram

20. The quadrilateral formed by joining the midpoints of the sides of a quadrilateral *ABCD*, taken in order, is a rectangle, if

(a) *ABCD* is a parallelogram
(b) *ABCD* is a rectangle
(c) diagonals of *ABCD* are equal
(d) diagonals of *ABCD* are perpendicular to each other

21. The quadrilateral formed by joining the midpoints of the sides of a quadrilateral *ABCD*, taken in order, is a rhombus, if

(a) *ABCD* is a parallelogram
(b) *ABCD* is a rhombus
(c) diagonals of *ABCD* are equal
(d) diagonals of *ABCD* are perpendicular to each other

22. The figure formed by joining the midpoints of the sides of a quadrilateral *ABCD*, taken in order, is a square, only if

(a) *ABCD* is a rhombus
(b) diagonals of *ABCD* are equal
(c) diagonals of *ABCD* are perpendicular
(d) diagonals of *ABCD* are equal and perpendicular

23. If an angle of a parallelogram is two thirds of its adjacent angle, the smallest angle of the parallelogram is

(a) $108°$ (b) $54°$ (c) $72°$ (d) $81°$

24. If one angle of a parallelogram is $24°$ less than twice the smallest angle then the largest angle of the parallelogram is

(a) $68°$ (b) $102°$ (c) $112°$ (d) $136°$

25. If $\angle A$, $\angle B$, $\angle C$ and $\angle D$ of a quadrilateral *ABCD,* taken in order, are in the ratio $3 : 7 : 6 : 4$ then *ABCD* is a

(a) rhombus (b) kite (c) trapezium (d) parallelogram

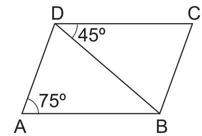
26. Which of the following is not true for a parallelogram?

(a) Opposite sides are equal.
(b) Opposite angles are equal.
(c) Opposite angles are bisected by the diagonals.
(d) Diagonals bisect each other.

27. If *APB* and *CQD* are two parallel lines then the bisectors of $\angle APQ$, $\angle BPQ$, $\angle CQP$ and $\angle PQD$ enclose a

(a) square (b) rhombus (c) rectangle (d) kite

28. In the given figure, $ABCD$ is a parallelogram in which $\angle BDC = 45°$ and $\angle BAD = 75°$. Then, $\angle CBD = ?$

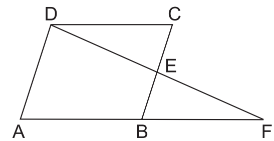


(a) 45° (b) 55°
(c) 60° (d) 75°

29. If area of a $\parallel$gm with sides a and b is A and that of a rectangle with sides a and b is B then

(a) $A > B$ (b) $A = B$ (c) $A < B$ (d) $A \geq B$

30. In the given figure, $ABCD$ is a $\parallel$gm and E is the midpoint of BC. Also, DE and AB when produced meet at F. Then,



(a) $AF = \frac{3}{2}AB$
(b) $AF = 2AB$
(c) $AF = 3AB$
(d) $AF^2 = 2AB^2$

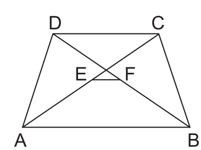
31. P is any point on the side BC of a $\triangle ABC$. P is joined to A. If D and E are the midpoints of the sides AB and AC respectively and M and N are the midpoints of BP and CP respectively then quadrilateral $DENM$ is

(a) a trapezium (b) a parallelogram
(c) a rectangle (d) a rhombus

32. The parallel sides of a trapezium are a and b respectively. The line joining the midpoints of its nonparallel sides will be

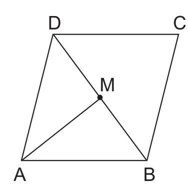
(a) $\frac{1}{2}(a-b)$ (b) $\frac{1}{2}(a+b)$ (c) $\frac{2ab}{(a+b)}$ (d) $\sqrt{ab}$

33. In a trapezium $ABCD$, E and F be the midpoints of the diagonals AC and BD respectively. Then, $EF = ?$



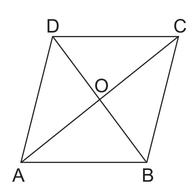
(a) $\frac{1}{2}AB$ (b) $\frac{1}{2}CD$
(c) $\frac{1}{2}(AB+CD)$ (d) $\frac{1}{2}(AB-CD)$

34. In the given figure, $ABCD$ is a parallelogram, M is the midpoint of BD and BD bisects $\angle B$ as well as $\angle D$. Then, $\angle AMB = ?$



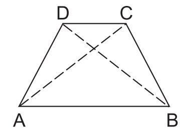
(a) 45° (b) 60°
(c) 90° (d) 30°

35. In the given figure, $ABCD$ is a rhombus. Then,



(a) $AC^2 + BD^2 = AB^2$
(b) $AC^2 + BD^2 = 2AB^2$
(c) $AC^2 + BD^2 = 4AB^2$
(d) $2(AC^2 + BD^2) = 3AB^2$

36. In a trapezium $ABCD$, if $AB \parallel CD$ then $(AC^2 + BD^2) = ?$

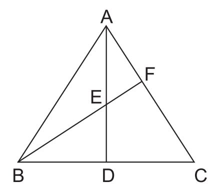


(a) $BC^2 + AD^2 + 2BC \cdot AD$
(b) $AB^2 + CD^2 + 2AB \cdot CD$
(c) $AB^2 + CD^2 + 2AD \cdot BC$
(d) $BC^2 + AD^2 + 2AB \cdot CD$

37. Two parallelograms stand on equal bases and between the same parallels. The ratio of their areas is

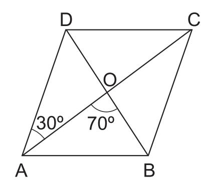
(a) 1 : 2 (b) 2 : 1 (c) 1 : 3 (d) 1 : 1

38. In the given figure, AD is a median of $\triangle ABC$ and E is the midpoint of AD. If BE is joined and produced to meet AC in F then $AF = ?$



(a) $\frac{1}{2}AC$ (b) $\frac{1}{3}AC$
(c) $\frac{2}{3}AC$ (d) $\frac{3}{4}AC$

39. The diagonals AC and BD of a parallelogram $ABCD$ intersect each other at the point O such that $\angle DAC = 30^\circ$ and $\angle AOB = 70^\circ$. Then, $\angle DBC = ?$



(a) 40° (b) 35°
(c) 45° (d) 50°

40. Three statements are given below:

I. In a ||gm, the angle bisectors of two adjacent angles enclose a right angle.
II. The angle bisectors of a ||gm form a rectangle.
III. The triangle formed by joining the midpoints of the sides of an isosceles triangle is not necessarily an isosceles triangle.

Which is true?

(a) I only (b) II only (c) I and II (d) II and III

41. Three statements are given below:

I. In a rectangle $ABCD$, the diagonal AC bisects $\angle A$ as well as $\angle C$.
II. In a square $ABCD$, the diagonal AC bisects $\angle A$ as well as $\angle C$.
III. In a rhombus $ABCD$, the diagonal AC bisects $\angle A$ as well as $\angle C$.

Which is true?

(a) I only (b) II and III (c) I and III (d) I and II

Short-Answer Questions

42. In a quadrilateral $PQRS$, opposite angles are equal. If $SR = 2$ cm and $PR = 5$ cm then determine PQ.

43. Diagonals of a parallelogram are perpendicular to each other. Is this statement true? Give reasons for your answer.

44. What special name can be given to a quadrilateral *PQRS* if $\angle P + \angle S = 180°$?

45. All the angles of a quadrilateral can be acute. Is this statement true? Give reasons for your answer.

46. All the angles of a quadrilateral can be right angles. Is this statement true? Give reasons for your answer.

47. All the angles of a quadrilateral can be obtuse. Is this statement true? Give reasons for your answer.

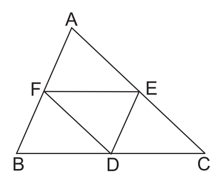
48. Can we form a quadrilateral whose angles are 70°, 115°, 60° and 120°? Give reasons for your answer.

49. What special name can be given to a quadrilateral whose all angles are equal?

50. If *D* and *E* are respectively the midpoints of the sides *AB* and *BC* of $\triangle ABC$ in which $AB = 7.2$ cm, $BC = 9.8$ cm and $AC = 3.6$ cm then determine the length of *DE*.

51. In a quadrilateral *PQRS*, the diagonals *PR* and *QS* bisect each other. If $\angle Q = 56°$, determine $\angle R$.

52. In the adjoining figure, *BDEF* and *AFDE* are parallelograms. Is $AF = FB$? Why or why not?



Data-Sufficiency-Based MCQ

In each of such questions, one question is followed by two statements I and II. The answer is

(a) if the question can be answered by one of the given statements alone and not by the other;

(b) if the question can be answered by either statement alone;

(c) if the question can be answered by both the statements together but not by any one of the two;

(d) if the question cannot be answered by using both the statements together.

53. Is quadrilateral *ABCD* a ||gm?

I. Diagonals *AC* and *BD* bisect each other.

II. Diagonals *AC* and *BD* are equal.

The correct answer is: (a)/(b)/(c)/(d).

54. Is quadrilateral $ABCD$ a rhombus?

I. Quadrilateral $ABCD$ is a ||gm.

II. Diagonals AC and BD are perpendicular to each other.

The correct answer is: (a)/(b)/(c)/(d).

55. Is ||gm $ABCD$ a square?

I. Diagonals of ||gm $ABCD$ are equal.

II. Diagonals of ||gm $ABCD$ intersect at right angles.

The correct answer is: (a)/(b)/(c)/(d).

56. Is quadrilateral $ABCD$ a parallelogram?

I. Its opposite sides are equal.

II. Its opposite angles are equal.

The correct answer is: (a)/(b)/(c)/(d).

Assertion-and-Reason Type MCQ

Each question consists of two statements, namely, Assertion (A) and Reason (R). For selecting the correct answer, use the following code:

(a) Both Assertion (A) and Reason (R) are true and Reason (R) is a correct explanation of Assertion (A).

(b) Both Assertion (A) and Reason (R) are true but Reason (R) is not a correct explanation of Assertion (A).

(c) Assertion (A) is true and Reason (R) is false.

(d) Assertion (A) is false and Reason (R) is true.

57.

Assertion (A)	Reason (R)
If three angles of a quadrilateral are 130°, 70° and 60° then the fourth angle is 100°.	The sum of all the angle of a quadrilateral is 360°.

The correct answer is: (a)/(b)/(c)/(d).

58.

Assertion (A)	Reason (R)
$ABCD$ is a quadrilateral in which P, Q, R and S are the midpoints of AB, BC, CD and DA respectively. Then, $PQRS$ is a parallelogram.	The line segment joining the midpoints of any two sides of a triangle is parallel to the third side and equal to half of it.

The correct answer is: (a)/(b)/(c)/(d).

59.

Assertion (A)	Reason (R)
In a rhombus $ABCD$, the diagonal AC bisects $\angle A$ as well as $\angle C$.	The diagonals of a rhombus bisect each other at right angles.

The correct answer is: (a)/(b)/(c)/(d).

60.

Assertion (A)	Reason (R)
Every parallelogram is a rectangle.	The angle bisectors of a parallelogram form a rectangle.

The correct answer is: (a)/(b)/(c)/(d).

61.

Assertion (A)	Reason (R)
The diagonals of a ∥gm bisect each other.	If the diagonals of a ∥gm are equal and intersect at right angles then the parallelogram is a square.

The correct answer is: (a)/(b)/(c)/(d).

Matching of Columns

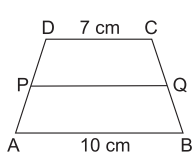
62. Match the following columns:

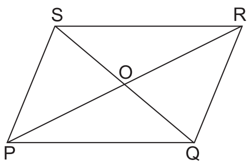
Column I	Column II
(a) Angle bisectors of a parallelogram form a	(p) parallelogram
(b) The quadrilateral formed by joining the midpoints of the pairs of adjacent sides of a square is a	(q) rectangle
(c) The quadrilateral formed by joining the midpoints of the pairs of adjacent sides of a rectangle is a	(r) square
(d) The figure formed by joining the mid-points of the pairs of adjacent sides of a quadrilateral is a	(s) rhombus

The correct answer is:

(a)—……, (b)—……, (c)—……, (d)—……

63. Match the following columns:

Column I	Column II
(a) In the given figure, *ABCD* is a trapezium in which $AB = 10$ cm and $CD = 7$ cm. If *P* and *Q* are the midpoints of *AD* and *BC* respectively then $PQ =$ 	(p) equal

(b) In the given figure, $PQRS$ is a ‖gm whose diagonals intersect at O. If $PR = 13$ cm, then $QR =$ 	(q) at right angles
(c) The diagonals of a square are	(r) 8.5 cm
(d) The diagonals of a rhombus bisect each other	(s) 6.5 cm

The correct answer is:

(a)—……, (b)—……, (c)—……, (d)—……

ANSWERS (MCQ)

1. (b) **2.** (b) **3.** (c) **4.** (a) **5.** (d) **6.** (d) **7.** (a) **8.** (b)
9. (b) **10.** (c) **11.** (b) **12.** (d) **13.** (c) **14.** (d) **15.** (d) **16.** (b)
17. (d) **18.** (a) **19.** (c) **20.** (d) **21.** (c) **22.** (d) **23.** (c) **24.** (c)
25. (c) **26.** (c) **27.** (c) **28.** (c) **29.** (c) **30.** (b) **31.** (b) **32.** (b)
33. (d) **34.** (c) **35.** (c) **36.** (d) **37.** (d) **38.** (b) **39.** (a) **40.** (c)
41. (b) **42.** 2 cm

43. No; the only property of the diagonals of a parallelogram is that they bisect each other.

44. Trapezium

45. No; in that case, the sum of the angles of the quadrilateral will be less than 360°.

46. Yes; in that case, the sum of the angles will be equal to 360°, e.g., square, rectangle.

47. No; in that case, the sum of the angles will be greater than 360°.

48. No; the sum of the given angles is not 360°.

49. Rectangle **50.** 1.8 cm **51.** 124°

52. Yes; $AF = ED$ and $FB = ED$ and so, $AF = FB$

[∵ opposite sides of a ‖gm are equal].

53. (a) **54.** (c) **55.** (c) **56.** (b) **57.** (a) **58.** (a) **59.** (b) **60.** (d)

61. (d) **62.** (a)—(q), (b)—(r), (c)—(s), (d)—(p)

63. (a)—(r), (b)—(s), (c)—(p), (d)—(q)

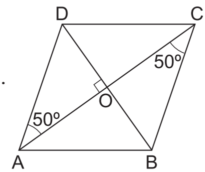
HINTS TO SOME SELECTED QUESTIONS

3. $\angle C = \angle A = 75^\circ$ (opposite $\angle$s of a ||gm).

   In $\triangle BCD$, $\angle CBD + \angle BCD + \angle BDC = 180^\circ$

   $\Rightarrow \quad 60^\circ + 75^\circ + \angle BDC = 180^\circ \Rightarrow 135^\circ + \angle BDC = 180^\circ \Rightarrow \angle BDC = 45^\circ$.

4. $\angle CAD = \angle ACB = 50^\circ$ (alt. interior $\angle$s).

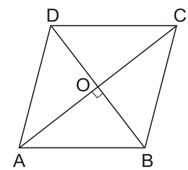


   $\angle AOD = 90^\circ$ (diagonals of a rhombus intersect at 90°).

   $\therefore \quad \angle ADO = 180^\circ - (50^\circ + 90^\circ) = 40^\circ$ (sum of $\angle$s of a $\triangle$ is 180°).

   Thus, $\angle ADB = \angle ADO = 40^\circ$.

7. We know that the diagonals of a rhombus bisect each other at right angles.



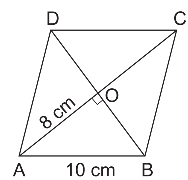
   $\therefore \quad OA = 8$ cm, $OB = 6$ cm and $\angle AOB = 90^\circ$.

   $\therefore \quad AB^2 = (OA^2 + OB^2) = (8^2 + 6^2)\text{ cm}^2 = (64 + 36)\text{ cm}^2 = 100\text{ cm}^2$

   $\Rightarrow \quad AB = \sqrt{100}$ cm $= 10$ cm.

   So, each side = 10 cm.

8. In the given rhombus, we have



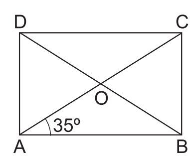
   $AB = 10$ cm and $OA = 8$ cm.

   $\therefore \quad OB^2 = AB^2 - OA^2 = \{(10)^2 - 8^2\}\text{ cm}^2 = 36\text{ cm}^2$

   $\Rightarrow \quad OB = \sqrt{36}$ cm $= 6$ cm.

   $\therefore \quad BD = 2 \times OB = (2 \times 6)$ cm $= 12$ cm.

9. $\angle OAD = 90^\circ - (\angle OAB) = 90^\circ - 35^\circ = 55^\circ$.



   Now, $\angle ODA = \angle OAD = 55^\circ$

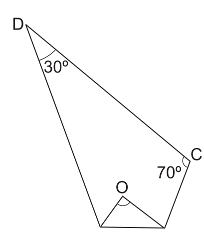
   [$\because$ $OA = OD$ since diagonals of a rectangles are equal and bisect each other].

   $\angle AOD = 180^\circ - (\angle OAD + \angle ODA) = 180^\circ - (55^\circ + 55^\circ) = 70^\circ$.

10. $\angle A + \angle B = 180^\circ$ and $\angle A = \angle B \Rightarrow \angle A = \angle B = 90^\circ$

    $\therefore \quad ABCD$ is a rectangle.

11. Sum of the angles of a quadrilateral is 360°.

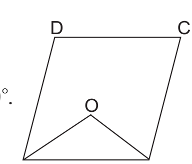


    $\therefore \quad \angle A + \angle B + 30^\circ + 70^\circ = 360^\circ$

    $\Rightarrow \quad \angle A + \angle B = 260^\circ \Rightarrow \frac{1}{2}\angle A + \frac{1}{2}\angle B = 130^\circ$.

    $\Rightarrow \quad \angle AOB = (180^\circ - 130^\circ) = 50^\circ$.

12. $\angle A + \angle B = 180^\circ \Rightarrow \frac{1}{2}\angle A + \frac{1}{2}\angle B = 90^\circ$.

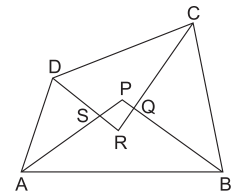


    $\frac{1}{2}\angle A + \frac{1}{2}\angle B + \angle AOB = 180^\circ \Rightarrow 90^\circ + \angle AOB = 180^\circ$.

    $\therefore \quad \angle AOB = 90^\circ$.

14. $\angle APB = 180^\circ - \left(\frac{1}{2}\angle A + \frac{1}{2}\angle B\right)\cdot$

$\angle CRD = 180^\circ - \left(\frac{1}{2}\angle C + \frac{1}{2}\angle D\right)\cdot$



$\therefore \quad \angle SPQ + \angle SRQ = \angle APB + \angle CRD$

$= 360^\circ - \frac{1}{2}(\angle A + \angle B + \angle C + \angle D)$

$= 360^\circ - 180^\circ = 180^\circ.$

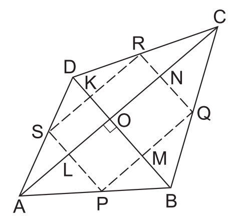
Now, $\angle PSR + \angle PQR = 360^\circ - (\angle SPQ + \angle SRQ)$

$= 360^\circ - 180^\circ = 180^\circ.$

15–19. The figures formed by joining the midpoints of the adjacent sides of different types of quadrilaterals are:

Quadrilateral	Square	$\|$gm	Rectangle	Rhombus
↓	↓	↓	↓	↓
$\|$gm	square	$\|$gm	Rhombus	Rectangle

20. (d) *PQRS* is always a parallelogram.



By midpoint theorem,

$PS \parallel BD$ and $PQ \parallel AC$

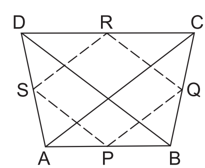
$\Rightarrow \quad LP \parallel OM$ and $PM \parallel LO \;\Rightarrow\; LPMO$ is a $\|$gm.

$\Rightarrow \quad \angle LPM = \angle LOM = 90^\circ \quad [\because\; AC \perp BD \quad \text{(given)}].$

Now, *PQRS* is a $\|$gm with one angle $\angle P = 90^\circ$.

$\therefore \quad PQRS$ is a rectangle if $AC \perp BD$.

21. (c) $PQ = \frac{1}{2}AC,\, QR = \frac{1}{2}BD,\, SR = \frac{1}{2}AC,\, PS = \frac{1}{2}BD.$



If $AC = BD$ then $PQ = QR = SR = PS$,

i.e., *PQRS* will be a rhombus.

22. (d) If diagonals of a quadrilateral are both equal and perpendicular then the quadrilateral formed by joining the midpoints of sides will be both a rectangle and a rhombus, i.e., it will be a square.

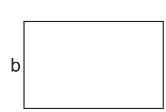
28. $\angle ABD = \angle CDB = 45^\circ \quad$ (alt. interior $\angle$s).

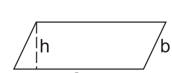
In $\triangle ADB$, $\angle BAD + \angle ABD + \angle ADB = 180^\circ$.

$\therefore \quad 75^\circ + 45^\circ + \angle ADB = 180^\circ \;\Rightarrow\; \angle ADB = 60^\circ.$

$\therefore \quad \angle CBD = \angle ADB = 60^\circ \quad$ (alt. interior $\angle$s).

29. Let h be the height of the $\|$gm.





Then, $h < b$.

$\therefore \quad A = a \times h < a \times b = B.$

Hence, $A < B$.

30. In $\triangle EDC$ and $\triangle EFB$, we have

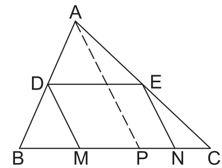
$\angle DCE = \angle EBF \quad$ (alt. interior $\angle$s),

$\angle DEC = \angle FEB \quad$ (vert. opp. $\angle$s) and $EC = EB$.

$\therefore \quad \triangle EDC \cong \triangle EFB$ and therefore, $BF = DC$.

$\therefore \quad AF = (AB + BF) = (AB + DC) = 2AB$.

31. D and E are midpoints of AB and AC respectively.



$\therefore \quad DE \parallel BC$ and $DE = \frac{1}{2}BC$ (midpoint theorem).

$MN = MP + PN = \frac{1}{2}BP + \frac{1}{2}PC = \frac{1}{2}(BP + PC) = \frac{1}{2}BC$.

$\therefore \quad DE \parallel MN$ and $DE = MN$ (each equal to $\frac{1}{2}BC$)

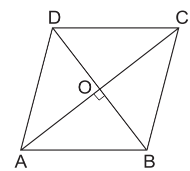
$\Rightarrow \quad DENM$ is a $\parallel$gm.

34. $\angle B = \angle D \Rightarrow \frac{1}{2}\angle B = \frac{1}{2}\angle D \Rightarrow \angle ADB = \angle ABD$.

$\therefore \quad \triangle ABD$ is isosceles and M is the midpoint of BD.

$\therefore \quad AM \perp BD$ and hence $\angle AMB = 90^\circ$.

35. We know that the diagonals of a rhombus bisect each other at right angles.

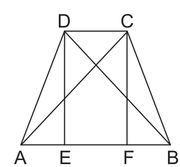


$\therefore \quad OA = \frac{1}{2}AC, OB = \frac{1}{2}BD$ and $\angle AOB = 90^\circ$.

$\therefore \quad AB^2 = OA^2 + OB^2 = \frac{1}{4}AC^2 + \frac{1}{4}BD^2$

$\Rightarrow \quad 4AB^2 = (AC^2 + BD^2)$.

36. In $\triangle ABC$, $\angle B$ is acute.



$\therefore \quad AC^2 = BC^2 + AB^2 - 2AB \cdot AE$.

In $\triangle ABD$, $\angle A$ is acute.

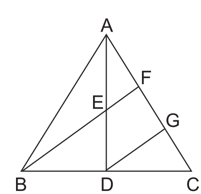
$\therefore \quad BD^2 = AD^2 + AB^2 - 2AB \cdot BF$

$\therefore \quad (AC^2 + BD^2) = (BC^2 + AD^2) + 2AB(AB - AE - BF)$

$$= BC^2 + AD^2 + 2AB \cdot EF = BC^2 + AD^2 + 2AB \cdot CD.$$

37. Two parallelograms on equal bases and between the same parallels are equal in area. So, the ratio of their areas is 1 : 1.

38. Let G be the midpoint of FC. Join DG.



In $\triangle BCF$, D is the midpoint of BC and G is the midpoint of FC.

$\therefore \quad DG \parallel BF \Rightarrow DG \parallel EF$.

In $\triangle ADG$, E is the midpoint of AD and $EF \parallel DG$.

So, F is the midpoint of AG.

$\therefore \quad AF = FG = GC$ [$\because$ G is the midpoint of FC].

$\therefore \quad AF = \frac{1}{3}AC$.

39. $\angle AOC = \angle DAC = 30^\circ$ (alt. interior $\angle$s)

$\Rightarrow \quad \angle OCB = 30^\circ$.

$\angle AOB + \angle BOC = 180^\circ \Rightarrow 70^\circ + \angle BOC = 180^\circ \Rightarrow \angle BOC = 110^\circ$.

In $\triangle OBC$, $\angle BOC + \angle OCB + \angle OBC = 180^\circ$

$\Rightarrow \quad 110^\circ + 30^\circ + \angle OBC = 180^\circ \Rightarrow \angle OBC = 40^\circ$.

42. Opposite angles are equal $\Rightarrow$ $PQRS$ is a $\|$gm $\Rightarrow$ $PQ = SR = 2$ cm.

50. DE is the line joining the midpoints of sides AB and BC of $\triangle ABC$. So, by midpoint theorem,

$$DE = \frac{1}{2}AC = \left(\frac{1}{2} \times 3.6\right) \text{ cm} = 1.8 \text{ cm}.$$

51. $PQRS$ is a $\|$gm since its diagonals bisect each other. Therefore, its adjacent angles are supplementary.

53. We know that if the diagonals of a quadrilateral bisect each other then it is a $\|$gm.

$\therefore$ I gives the answer.

If the diagonals of a quadrilateral are equal then it is not necessarily a $\|$gm.

$\therefore$ II does not give the answer.

Hence, the correct answer is (a).

54. Clearly, I alone is not sufficient to answer the given question.

Also, II alone is not sufficient to answer the given question.

But, both I and II together will give the answer.

$\therefore$ the correct answer is (c).

55. When the diagonals of a $\|$gm are equal, it is either a rectangle or a square.

Also, if the diagonals intersect at right angles then out of rectangle and square, it is a square.

$\therefore$ both I and II will give the answer.

Hence, the correct answer is (c).

56. We know that a quadrilateral $ABCD$ is a parallelogram when either of I and II holds.

So, the correct answer is (b).

REVIEW OF FACTS AND FORMULAE

1. (i) QUADRILATERAL *A plane figure bounded by four line segments AB, BC, CD and DA is called a quadrilateral ABCD.*

(ii) *The sum of all the angles of a quadrilateral is* 360°.

2. (i) PARALLELOGRAM *A quadrilateral in which the opposite sides are parallel, is called a parallelogram.*

(ii) RECTANGLE *A parallelogram one of whose angles is* 90°, *is called a rectangle.*

(iii) TRAPEZIUM *A quadrilateral having one pair of opposite sides parallel is called a trapezium.*

(iv) RHOMBUS *A parallelogram having all sides equal is called a rhombus.*

(v) SQUARE *A quadrilateral in which all sides are equal and the diagonals are equal, is a square.*

3. RESULTS ON PARALLELOGRAM

I. *In a parallelogram*

(i) *opposite sides are equal;*

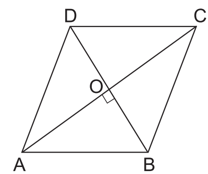
(ii) *opposite angles are equal;*

(iii) *diagonals bisect each other but need not be equal;*

(iv) *sum of adjacent angles is* 180°.

II. *Two parallelograms on the same base and between the same parallels are equal in area.*

4. RESULTS ON RHOMBUS



I. *In a rhombus*

(i) *all sides are equal;*

(ii) *the diagonals bisect each other at right angles;*

(iii) *in a rhombus ABCD, we have*

$$AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2.$$

5. A quadrilateral is a ||gm:

(i) if both pairs of opposite sides are equal,

or (ii) if both pairs of opposite angles are equal,

or (iii) if the diagonals bisect each other,

or (iv) if a pair of opposite sides are equal and parallel.

6. (i) The diagonals of a rectangle are equal and bisect each other.

(ii) If the diagonals of a ||gm are equal then it is a rectangle.

7. (i) The diagonals of a rhombus bisect each other at right angles.

(ii) If the diagonals of a ||gm are perpendicular to each other then it is a rhombus.

8. (i) The diagonals of a square are equal and bisect each other at right angles.

(ii) If the diagonals of a ||gm are equal and intersect at right angles then the ||gm is a square.

9. The quadrilateral formed by joining the midpoints of the pairs of adjacent sides of

(i) a quadrilateral is a ||gm; (ii) a rectangle is a rhombus;

(iii) a rhombus is a rectangle; (iv) a square is a square.

10. MIDPOINT THEOREM The line segments joining the midpoints of any two sides of a triangle is parallel to the third side and equal to half of it.

11. INTERCEPT THEOREM If there are three parallel lines and the intercepts made by them on one transversal are equal then the intercepts on any other transversal are equal.

✿

11 Areas of Parallelograms and Triangles

PLANAR REGION The planar region corresponding to a simple closed figure is the part of the plane enclosed within the boundary of the figure.

AREA The magnitude or measure of the planar region enclosed by a closed figure is called the area of that figure.

CONGRUENT FIGURES Two figures are said to be congruent, if they have the same shape and the same size.

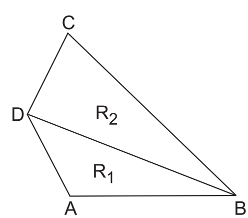
If two congruent figures are cut and one of them is superposed on the other then each figure covers the other completely.

EUCLIDEAN AREA AXIOMS

1. EXISTENCE POSTULATE Corresponding to each polygon P lying in a plane, there exists a real number $\text{ar}(P) \geq 0$, called its area.

2. DOMINANCE POSTULATE For regions R_1 and R_2 in a plane, if $R_1 \subseteq R_2$, i.e., if R_1 is a part of R_2 then $\text{ar}(R_1) \leq \text{ar}(R_2)$.

3. POSTULATE OF ADDITIVITY For any two plane regions R_1 and R_2 such that $\text{ar}(R_1 \cap R_2) = 0$ i.e., if R_1 and R_2 are nonoverlapping regions then $\text{ar}(R_1 \cup R_2) = \text{ar}(R_1) + \text{ar}(R_2)$.



In the adjoining figure, if R represents the total region of a polygon $ABCD$ then $\text{ar}(R)$ is equal to the sum of the areas of regions R_1 and R_2, i.e.,

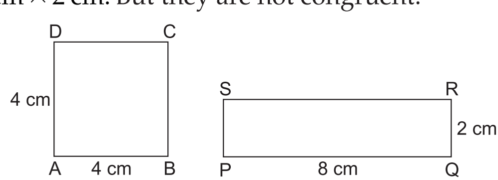
$$\text{ar}(R) = \text{ar}(R_1) + \text{ar}(R_2).$$

4. CONGRUENCE POSTULATE If R_1 and R_2 are two congruent figures then they have equal areas.

Thus, $R_1 \cong R_2 \Rightarrow \text{ar}(R_1) = \text{ar}(R_2)$.

However, the converse is not true. Two figures having equal areas need not be congruent.

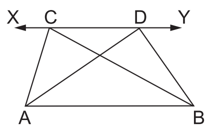
EXAMPLE A square of side 4 cm has the same area as a rectangle $8 \text{ cm} \times 2 \text{ cm}$. But they are not congruent.



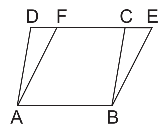
FIGURES ON THE SAME BASE AND BETWEEN THE SAME PARALLELS

Two figures are said to be on the same base and between the same parallels if they have a common side (taken as the base) and the vertices (or the vertex) opposite to the common base of each figure lie (lies) on a line parallel to the base.

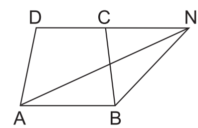
Observe the following figures:



(i)



(ii)



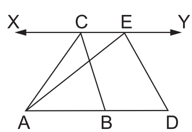
(iii)

In Fig. (i), $\triangle ABC$ and $\triangle ABD$ are on the same base AB and between the same parallels (AB and XY).

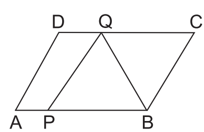
In Fig. (ii), ∥gm $ABCD$ and ∥$ABEF$ are on the same base AB and between the same parallels (AB and DE).

In Fig. (iii), trapezium $ABCD$ and $\triangle ABN$ are on the same base AB and between the same parallels (AB and DN).

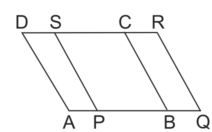
Observe the following figures:



(iv)



(v)



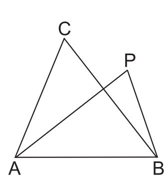
(vi)

In Fig. (iv), $\triangle ABC$ and $\triangle ADE$ are between the same parallels (AD and XY) but not on the same base.

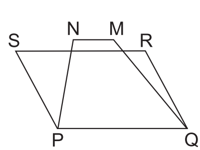
In Fig. (v), ∥gm $ABCD$ and $\triangle PBQ$ are between the same parallels (AB and DC) but not on the same base.

In Fig. (vi), ∥gm $ABCD$ and ∥gm $PQRS$ are between the same parallels (AQ and DR) but not on the same base.

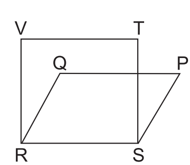
Observe the following figures:



(vii)



(viii)



(ix)

In Fig. (vii), $\triangle ABC$ and $\triangle ABP$ are on the same base AB but not between the same parallels.

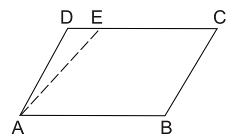
In Fig. (viii), ||gm $PQRS$ and trapezium $PQMN$ are on the same base PQ but not between the same parallels.

In Fig. (ix), square $RSTV$ and ||gm $RSPQ$ are on the same base RS but not between the same parallels.

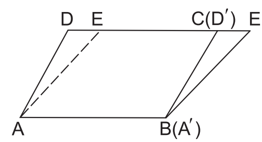
PARALLELOGRAMS ON THE SAME BASE AND BETWEEN THE SAME PARALLELS

Parallelograms on the same base and between the same parallels are equal in area. We shall verify this result through a simple activity.

Activity 1. On a drawing sheet, construct a parallelogram $ABCD$. Draw a line segment AE as shown in Fig. (i). On a separate sheet, construct $\triangle A'D'E'$ congruent to $\triangle ADE$ using a tracing paper. Cut $\triangle A'D'E'$ and place it in such a way that $A'D'$ coincides with side BC of the parallelogram $ABCD$ as shown in Fig. (ii). We thus have two parallelograms $ABCD$ and $ABE'E$ which are *on the same base AB and between the same parallels AB* and DE'.



(i)



(ii)

Now, $\triangle ADE \cong \triangle A'D'E'$

$\Rightarrow \quad \text{ar}(\triangle ADE) = \text{ar}(\triangle A'D'E')$.

Also, $\text{ar}(ABCD) = \text{ar}(\triangle ADE) + \text{ar}(\text{quad. } ABCE)$

$= \text{ar}(\triangle A'D'E') + \text{ar}(\text{quad. } ABCE)$

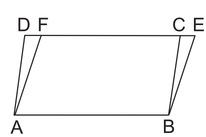
$= \text{ar}(ABE'E)$.

Thus, the two parallelograms $ABCD$ and $ABE'E$ are equal in area.

THEOREM 1 *Parallelograms on the same base and between the same parallels are equal in area.*

GIVEN Two ||gms $ABCD$ and $ABEF$ on the same base AB and between the same parallels AB and FC.

TO PROVE ar(||gm $ABCD$) = ar(||gm $ABEF$).



PROOF In $\triangle ADF$ and $\triangle BCE$, we have:

$AD = BC$ (opp. sides of a ||gm)

$\angle ADF = \angle BCE$ (corresponding $\measuredangle$s)

$DF = CE$ $\left[\begin{array}{l} \because \ DC = FE \text{ (each equal to } AB) \\ \Rightarrow \ DC - FC = FE - FC \ \Rightarrow \ DF = CE \end{array}\right]$

$\therefore \quad \triangle ADF \cong \triangle BCE$ (SAS-criterion).

And so, $\text{ar}(\triangle ADF) = \text{ar}(\triangle BCE)$

(congruent figures have equal areas).

$$\text{Now, ar}(\|\text{gm } ABCD) = \text{ar}(\triangle ADF) + \text{ar}(\text{quad. } ABCF)$$
$$= \text{ar}(\triangle BCE) + \text{ar}(\text{quad. } ABCF)$$
$$= \text{ar}(\|\text{gm } ABEF).$$

Hence, ar($\|$gm $ABCD$) = ar($\|$gm $ABEF$).

Corollary A parallelogram and a rectangle on the same base and between the same parallels are equal in area.

HINT A rectangle is also a parallelogram.

BASE AND ALTITUDE OF A PARALLELOGRAM

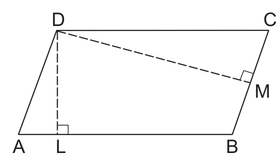
BASE *Any side of a parallelogram can be called its base.*

ALTITUDE *The length of the line segment which is perpendicular to the base from the opposite side is called the altitude or height of the parallelogram corresponding to the given base.*

In the adjoining figure

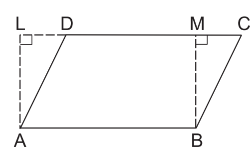
(i) DL is the altitude of $\|$gm $ABCD$, corresponding to the base AB.

(ii) DM is the altitude of $\|$gm $ABCD$, corresponding to the base BC.



THEOREM 2 *The area of a parallelogram is equal to the product of its any side and the corresponding altitude.*

GIVEN A $\|$gm $ABCD$ in which AB is the base and AL is the corresponding height.



TO PROVE area($\|$gm $ABCD$) = $AB \times AL$.

CONSTRUCTION Draw $BM \perp DC$ so that rect. $ABML$ is formed.

PROOF $\|$gm $ABCD$ and rect. $ABML$ are on the same base AB and between the same parallel lines AB and LC.

$\therefore$ ar($\|$gm $ABCD$) = ar(rect. $ABML$) = $AB \times AL$.

$\therefore$ area of a $\|$gm = base $\times$ height.

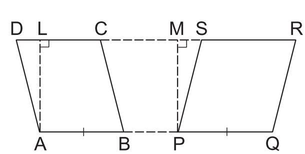
THEOREM 3 *Parallelograms on equal bases and between the same parallels are equal in area.*

GIVEN Two $\|$gms $ABCD$ and $PQRS$ with equal bases AB and PQ and between the same parallels, AQ and DR.

TO PROVE ar($\|$gm $ABCD$) = ar($\|$gm $PQRS$).

CONSTRUCTION Draw $AL \perp DR$ and $PM \perp DR$.

PROOF $AB \parallel DR$, $AL \perp DR$ and $PM \perp DR$.



$\therefore \quad AL = PM.$

Now, ar(||gm $ABCD$) $= AB \times AL$

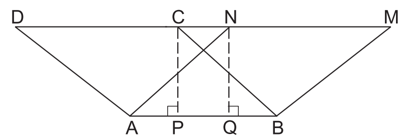
$= PQ \times PM$ [$\because$ $AB = PQ$ and $AL = PM$]

$=$ ar(||gm $PQRS$).

THEOREM 4 (Converse) *Parallelograms on the same base and having equal areas lie between the same parallels.*

GIVEN Two ||gms $ABCD$ and $ABMN$ on the same base AB such that

ar(||gm $ABCD$) = ar(||gm $ABMN$).



TO PROVE $DCNM$ is a straight line parallel to AB, i.e., $CN \parallel AB$.

CONSTRUCTION Draw $CP \perp AB$ and $NQ \perp AB$.

PROOF Since $CP \perp AB$ and $NQ \perp AB$, we have

$CP \parallel NQ$ (lines perpendicular to the same line are parallel).

Now, ar(||gm $ABCD$) = ar(||gm $ABMN$)

$\Rightarrow \quad AB \times CP = AB \times NQ$ (area of a ||gm = base $\times$ height)

$\Rightarrow \quad CP = NQ.$

Now, $CP \parallel NQ$ and $CP = NQ \Rightarrow CPQN$ is a ||gm

$\Rightarrow \quad CN \parallel PQ \Rightarrow CN \parallel AB.$

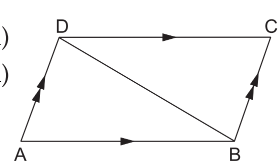
Thus, parallelograms $ABCD$ and $ABMN$ lie between the same parallels.

THEOREM 5 *A diagonal of a parallelogram divides it into two triangles of equal area.*

GIVEN A ||gm $ABCD$ in which BD is a diagonal.

TO PROVE ar($\triangle ABD$) = ar($\triangle CDB$).

PROOF In $\triangle$s ABD and CDB, we have:



$AB = CD$ (opposite sides of a ||gm)

$AD = CB$ (opposite sides of a ||gm)

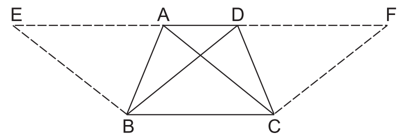
$BD = DB$ (common)

$\therefore \quad \triangle ABD \cong \triangle CDB$ (SSS-criterion).

Hence, ar($\triangle ABD$) = ar($\triangle CDB$).

THEOREM 6 *Triangles on the same base and between the same parallels are equal in area.*

GIVEN Two $\triangle$s ABC and DBC on the same base BC and between the same parallel lines BC and AD.



TO PROVE $\text{ar}(\triangle ABC) = \text{ar}(\triangle DBC)$.

CONSTRUCTION Through B, draw $BE \parallel CA$, meeting DA produced in E. And, through C, draw $CF \parallel BD$, meeting AD produced in F.

PROOF $BE \parallel CA$ and $EA \parallel BC \Rightarrow BCAE$ is a ‖gm.

$CF \parallel BD$ and $DF \parallel BC \Rightarrow BCFD$ is a ‖gm.

Clearly, the ‖gms $BCAE$ and $BCFD$ are on the same base BC and between the same parallel lines BC and EF.

$\therefore \quad \text{ar}(\|\text{gm } BCAE) = \text{ar}(\|\text{gm } BCFD)$. ... (i)

But, a diagonal of a ‖gm divides it into two triangles of equal areas.

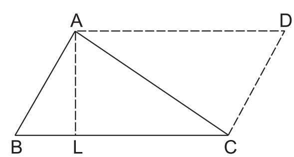
$\therefore \quad \text{ar}(\triangle ABC) = \frac{1}{2}\text{ar}(\|\text{gm } BCAE)$ ... (ii)

and $\text{ar}(\triangle DBC) = \frac{1}{2}\text{ar}(\|\text{gm } BCFD)$. ... (iii)

Thus, from (i), (ii) and (iii), we get

$\text{ar}(\triangle ABC) = \text{ar}(\triangle DBC)$.

THEOREM 7 *Area of a triangle* $= \frac{1}{2} \times$ *base* $\times$ *height*.



GIVEN A $\triangle ABC$ in which BC is the base and AL is the corresponding height.

TO PROVE $\text{ar}(\triangle ABC) = \frac{1}{2} \times BC \times AL$.

CONSTRUCTION Through A and C, draw $AD \parallel BC$ and $CD \parallel BA$, intersecting each other at D.

PROOF $AD \parallel BC$ and $CD \parallel BA \Rightarrow BCDA$ is a ‖gm.

Thus, $BCDA$ is a ‖gm whose diagonal AC divides it into two triangles of equal areas.

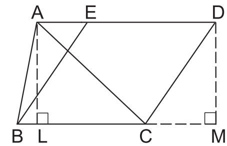
$\therefore \quad \text{ar}(\triangle ABC) = \frac{1}{2} \times \text{ar}(\|\text{gm } BCDA)$

$= \frac{1}{2} \times BC \times AL \quad [\because \text{ar}(\|\text{gm } BCDA) = BC \times AL]$.

$\therefore \quad$ area of a triangle $= \frac{1}{2} \times$ base $\times$ height.

THEOREM 8 *If a triangle and a parallelogram are on the same base, and between the same parallels then the area of the triangle is equal to half the area of the parallelogram.*

GIVEN A $\triangle ABC$ and a ‖gm $BCDE$ on the same base BC and between the same parallel lines BC and AD.



TO PROVE $\text{ar}(\triangle ABC) = \frac{1}{2}\text{ar}(\|\text{gm } BCDE)$.

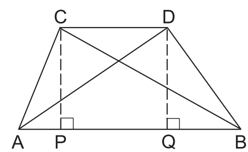
CONSTRUCTION Draw $AL \perp BC$, meeting BC in L, and $DM \perp BC$, meeting BC produced in M.

PROOF Since $BC \parallel AD$, we have $AL = DM$.

$$\therefore \quad \text{ar}(\triangle ABC) = \frac{1}{2} \times BC \times AL = \frac{1}{2} \times BC \times DM \quad [\because AL = DM]$$

$$= \frac{1}{2} \times \text{ar}(\|\text{gm } BCDE).$$

THEOREM 9 *Two triangles having the same base (or equal bases) and equal areas lie between the same parallels.*



GIVEN Two triangles $\triangle ABC$ and $\triangle ABD$ have the same base AB and $\text{ar}(\triangle ABC) = \text{ar}(\triangle ABD)$.

TO PROVE $CD \parallel AB$.

CONSTRUCTION Draw $CP \perp AB$ and $DQ \perp AB$.

PROOF Since $CP \perp AB$ and $DQ \perp AB$, we have

$CP \parallel DQ$ [$\because$ lines perpendicular to same line are parallel].

Now, $\text{ar}(\triangle ABC) = \text{ar}(\triangle ABD)$

$$\Rightarrow \quad \frac{1}{2} \times AB \times CP = \frac{1}{2} \times AB \times DQ$$

$$[\because \text{ area of a } \triangle = \frac{1}{2} \times \text{base} \times \text{height}]$$

$\Rightarrow \quad CP = DQ.$

Now, $CP \parallel DQ$ and $CP = DQ$

$\Rightarrow \quad CPQD$ is a $\|$gm

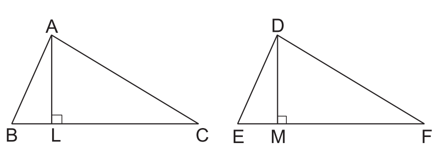
$\Rightarrow \quad CD \parallel PQ \Rightarrow CD \parallel AB.$

Thus, the two triangles lie between the same parallels.

THEOREM 10 *If two triangles have equal areas and one side of the one triangle is equal to one side of the other then their corresponding altitudes are equal.*

GIVEN Two $\triangle$s ABC and DEF such that $\text{ar}(\triangle ABC) = \text{ar}(\triangle DEF)$ and $BC = EF$.

TO PROVE Altitude AL = altitude DM.



PROOF $\text{ar}(\triangle ABC) = \frac{1}{2} \times BC \times AL$ $\quad \left[\because \text{ar}(\triangle) = \frac{1}{2} \times \text{base} \times \text{height}\right]$

and ar($\triangle DEF$) $= \frac{1}{2} \times EF \times DM$.

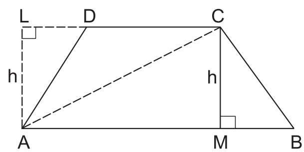
Now, ar($\triangle ABC$) = ar($\triangle DEF$)

$\Rightarrow \quad \frac{1}{2} BC \times AL = \frac{1}{2} \times EF \times DM \Rightarrow AL = DM \qquad [\because BC = EF]$.

Hence, altitude AL = altitude DM.

THEOREM 11 *Area of a trapezium*

$$= \frac{1}{2} \times \textit{(sum of the parallel sides)} \times \textit{(distance between them)}.$$



GIVEN A trapezium $ABCD$ in which $AB \parallel DC$ and $CM \perp AB$. Let $CM = h$.

TO PROVE ar(trap. $ABCD$) $= \frac{1}{2} \times (AB + DC) \times h$.

CONSTRUCTION Draw $AL \perp DC$, meeting CD produced at L. Join AC.

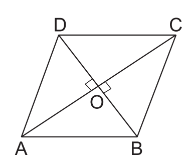
PROOF ar(trap. $ABCD$) = ar($\triangle ABC$) + ar($\triangle ACD$)

$$= \left(\frac{1}{2} \times AB \times CM\right) + \left(\frac{1}{2} \times DC \times AL\right)$$

$$= \frac{1}{2} \times (AB + DC) \times h \quad [\because CM = AL = h].$$

$\therefore$ area of a trapezium $= \frac{1}{2} \times$ (sum of the parallel sides) $\times$ (distance between them).

THEOREM 12 *Area of a rhombus* $= \frac{1}{2} \times$ *product of the diagonals.*



GIVEN A rhombus $ABCD$ whose diagonals AC and BD intersect at a point O.

TO PROVE ar(rhombus $ABCD$) $= \frac{1}{2} \times AC \times BD$.

PROOF Since the diagonals of a rhombus intersect each other at right angles, we have

$BO \perp AC$ and $DO \perp AC$.

$\therefore$ ar(rhombus $ABCD$) = ar($\triangle ABC$) + ar($\triangle ACD$)

$$= \left(\frac{1}{2} \times AC \times BO\right) + \left(\frac{1}{2} \times AC \times DO\right)$$

$$= \frac{1}{2} \times AC \times (BO + DO)$$

$$= \frac{1}{2} \times AC \times BD$$

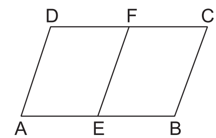
$$= \frac{1}{2} \times \text{(product of diagonals)}.$$

$\therefore$ area of a rhombus $= \frac{1}{2} \times$ (product of the diagonals).

SOLVED EXAMPLES

EXAMPLE 1 *Show that the line segment joining the midpoints of a pair of opposite sides of a parallelogram divides it into two equal parallelograms.*

GIVEN A ∥gm $ABCD$ in which E and F are the midpoints of AB and DC respectively. E and F are joined.



TO PROVE ar(□$AEFD$) = ar(□$EBCF$).

PROOF Since $ABCD$ is a ∥gm, we have $AB \parallel DC$ and $AB = DC$.

$\therefore$ $AE \parallel DF$ and $AE = DF$

$$\left[\because\ AB = DC \Rightarrow \frac{1}{2}AB = \frac{1}{2}DC \Rightarrow AE = DF\right].$$

$\therefore$ $AEFD$ is a ∥gm.

Again, $EB \parallel FC$ and $EB = FC$

$$\left[\because\ AB = DC \Rightarrow \frac{1}{2}AB = \frac{1}{2}DC \Rightarrow EB = FC\right]$$

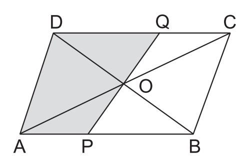
$\therefore$ $EBCF$ is a ∥gm.

Now, $AEFD$ and $EBCF$ being two parallelograms with equal bases and between the same parallels, they must have equal areas.

$\therefore$ ar(∥gm $AEFD$) = ar(∥gm $EBCF$).

EXAMPLE 2 *In the adjoining figure, ABCD is a ∥gm whose diagonals AC and BD intersect at O. A line segment through O meets AB at P and DC at Q. Prove that*

$$ar(\square APQD) = \frac{1}{2}ar(\|gm\ ABCD).$$



SOLUTION Diagonal AC of ∥gm $ABCD$ divides it into two triangles of equal area.

$$\therefore\quad ar(\triangle ACD) = \frac{1}{2}ar(\|gm\ ABCD). \qquad \dots (i)$$

In ⊿s OAP and OCQ, we have:

$OA = OC$ (diagonals of a ∥gm bisect each other)

$\angle AOP = \angle COQ$ (vert. opp. ∠s)

$\angle PAO = \angle QCO$ (alt. interior ∠s)

$\therefore$ $\triangle OAP \cong \triangle OCQ$.

$\therefore$ ar($\triangle OAP$) = ar($\triangle OCQ$)

$\Rightarrow$ ar($\triangle OAP$) + ar(quad. $AOQD$)

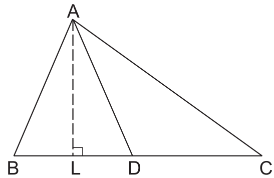
= ar($\triangle OCQ$) + ar(quad. $AOQD$)

$\Rightarrow$ ar(quad. $APQD$) = ar($\triangle ACD$)

$$= \frac{1}{2}\text{ar}(\|\text{gm } ABCD) \quad \text{[using (i)]}.$$

$\therefore$ ar($\square APQD$) $= \frac{1}{2}$ar($\|$gm $ABCD$).

EXAMPLE 3 *Show that a median of a triangle divides it into two triangles of equal area.*



GIVEN A $\triangle ABC$ in which AD is a median.

TO PROVE ar($\triangle ABD$) = ar($\triangle ADC$).

CONSTRUCTION Draw $AL \perp BC$.

PROOF Since D is the midpoint of BC, we have

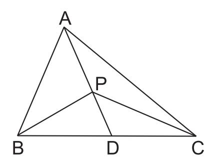
$BD = DC$.

Now, $BD = DC$

$$\Rightarrow \quad \frac{1}{2}BD \times AL = \frac{1}{2}DC \times AL \;\Rightarrow\; \text{ar}(\triangle ABD) = \text{ar}(\triangle ADC).$$

Hence, a median of a triangle divides it into two triangles of equal area.

EXAMPLE 4 *In the adjoining figure, AD is one of the medians of a $\triangle ABC$ and P is a point on AD. Prove that (i) ar($\triangle BDP$) = ar($\triangle CDP$), (ii) ar($\triangle ABP$) = ar($\triangle ACP$).*



GIVEN A $\triangle ABC$ in which AD is a median. P is any point on AD. BP and CP are joined.

TO PROVE (i) ar($\triangle BDP$) = ar($\triangle CDP$),

(ii) ar($\triangle ABP$) = ar($\triangle ACP$).

PROOF AD is a median of $\triangle ABC$

$\Rightarrow$ ar($\triangle ABD$) = ar($\triangle ACD$) ... (i)

[$\because$ a median divides a $\triangle$ into two $\triangle$s of equal area].

Now, D is the midpoints of BC

$\Rightarrow$ PD is a median of $\triangle PBC$

$\Rightarrow$ ar($\triangle BDP$) = ar($\triangle CDP$) ... (ii)

[$\because$ a median divides a $\triangle$ into two $\triangle$s of equal area].

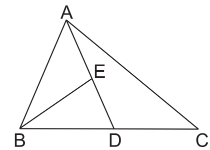
Subtracting (ii) from (i), we get

$$\text{ar}(\triangle ABD) - \text{ar}(\triangle BDP) = \text{ar}(\triangle ACD) - \text{ar}(\triangle CDP)$$

$\Rightarrow$ ar($\triangle ABP$) = ar($\triangle ACP$).

<u>EXAMPLE 5</u> *ABC is a triangle in which D is the midpoint of BC and E is the midpoint of AD.*

Prove that $ar(\triangle BED) = \frac{1}{4} ar(\triangle ABC)$.



GIVEN A $\triangle ABC$ in which D is the midpoint of BC and E is the midpoint of AD.

TO PROVE $\text{ar}(\triangle BED) = \frac{1}{4}\text{ar}(\triangle ABC)$.

PROOF D is the midpoint of $BC \Rightarrow AD$ is a median of $\triangle ABC$

$\Rightarrow \quad \text{ar}(\triangle ABD) = \text{ar}(\triangle ACD)$

[$\because$ median divides a $\triangle$ into two $\triangle$s of equal area]

$\Rightarrow \quad \text{ar}(\triangle ABD) = \frac{1}{2}\text{ar}(\triangle ABC)$. ... (i)

E is the midpoint of $AD \Rightarrow BE$ is a median of $\triangle ABD$

$\Rightarrow \quad \text{ar}(\triangle BED) = \text{ar}(\triangle BEA)$

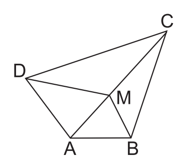
[$\because$ median divides a $\triangle$ into two $\triangle$s of equal area]

$\Rightarrow \quad \text{ar}(\triangle BED) = \frac{1}{2}\text{ar}(\triangle ABD) = \frac{1}{2}\left\{\frac{1}{2}\text{ar}(\triangle ABC)\right\}$ [using (i)]

$\Rightarrow \quad \text{ar}(\triangle BED) = \frac{1}{4}\text{ar}(\triangle ABC)$.

<u>EXAMPLE 6</u> *In a quadrilateral ABCD, it is being given that M is the midpoint of AC. Prove that*

ar(□ABMD)= ar(□DMBC).



SOLUTION Median BM divides $\triangle ABC$ into two $\triangle$s of equal area.

$\therefore \quad \text{ar}(\triangle ABM) = \text{ar}(\triangle BMC)$. ... (i)

Median DM divides $\triangle DAC$ into two $\triangle$s of equal area.

$\therefore \quad \text{ar}(\triangle AMD) = \text{ar}(\triangle DMC)$. ... (ii)

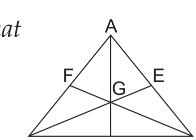
From (i) and (ii), we get

$$\text{ar}(\triangle ABM) + \text{ar}(\triangle AMD) = \text{ar}(\triangle BMC) + \text{ar}(\triangle DMC)$$

$\Rightarrow \quad \text{ar}(\square ABMD) = \text{ar}(\square DMBC)$.

<u>EXAMPLE 7</u> *If the medians of a $\triangle ABC$ intersect at G, show that* $ar(\triangle AGB) = ar(\triangle AGC) = ar(\triangle BGC)$

$$= \frac{1}{3} ar(\triangle ABC).$$



GIVEN A $\triangle ABC$. Its medians AD, BE and CF intersect at G.

TO PROVE $\text{ar}(\triangle AGB) = \text{ar}(\triangle AGC) = \text{ar}(\triangle BGC) = \frac{1}{3}\text{ar}(\triangle ABC)$.

PROOF We know that a median of a triangle divides it into two triangles of equal area.

In $\triangle ABC$, AD is the median.

$\therefore \quad \text{ar}(\triangle ABD) = \text{ar}(\triangle ACD)$. ... (i)

In $\triangle GBC$, GD is the median.

$\therefore \quad \text{ar}(\triangle GBD) = \text{ar}(\triangle GCD)$. ... (ii)

From (i) and (ii), we get

$$\text{ar}(\triangle ABD) - \text{ar}(\triangle GBD) = \text{ar}(\triangle ACD) - \text{ar}(\triangle GCD)$$

$\Rightarrow \quad \text{ar}(\triangle AGB) = \text{ar}(\triangle AGC)$.

Similarly, $\text{ar}(\triangle AGB) = \text{ar}(\triangle BGC)$.

$\therefore \quad \text{ar}(\triangle AGB) = \text{ar}(\triangle AGC) = \text{ar}(\triangle BGC)$. ... (iii)

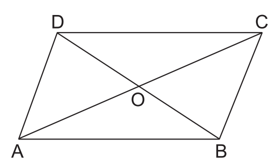
But, $\text{ar}(\triangle ABC) = \text{ar}(\triangle AGB) + \text{ar}(\triangle AGC) + \text{ar}(\triangle BGC)$

$= 3\text{ar}(\triangle AGB)$ [using (iii)].

$\therefore \quad \text{ar}(\triangle AGB) = \frac{1}{3}\text{ar}(\triangle ABC)$.

Hence, $\text{ar}(\triangle AGB) = \text{ar}(\triangle AGC) = \text{ar}(\triangle BGC) = \frac{1}{3}\text{ar}(\triangle ABC)$.

EXAMPLE 8 *Show that the diagonals of a parallelogram divides it into four triangles of equal area.*



GIVEN A $\|$gm $ABCD$. Its diagonals AC and BD intersect at O.

TO PROVE $\text{ar}(\triangle OAB) = \text{ar}(\triangle OBC)$

$= \text{ar}(\triangle OCD) = \text{ar}(\triangle OAD)$.

PROOF Since the diagonals of a $\|$gm bisect each other, we have

$OA = OC$ and $OB = OD$.

Also, a median of a triangle divides it into two $\triangle$s of equal area.

Now, in $\triangle ABC$, BO is the median.

$\therefore \quad \text{ar}(\triangle OAB) = \text{ar}(\triangle OBC)$. ... (i)

In $\triangle ABD$, AO is the median.

$\therefore \quad \text{ar}(\triangle OAB) = \text{ar}(\triangle OAD)$. ... (ii)

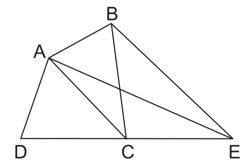
In $\triangle ACD$, DO is the median.

$\therefore \quad \text{ar}(\triangle OAD) = \text{ar}(\triangle OCD)$. ... (iii)

From (i), (ii) and (iii), we get

$$\text{ar}(\triangle OAB) = \text{ar}(\triangle OBC) = \text{ar}(\triangle OCD) = \text{ar}(\triangle OAD).$$

EXAMPLE 9 *In the adjoining figure, ABCD is a quadrilateral. BE is drawn parallel to AC and it meets DC produced in E. Show that the area of $\triangle ADE$ is equal to the area of the quadrilateral ABCD.*



SOLUTION We have

$$ar(\triangle ABC) = ar(\triangle AEC)$$

[$\because$ $\triangle$s on the same base and between the same parallels are equal in area].

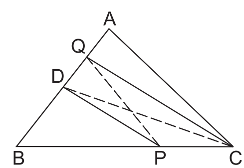
And so, $ar(\triangle ABC) + ar(\triangle ADC) = ar(\triangle AEC) + ar(\triangle ADC)$

[adding same areas on both sides]

$\Rightarrow$ $ar(\text{quad. } ABCD) = ar(\triangle ADE)$.

EXAMPLE 10 *In the adjoining figure, D is the midpoint of side AB of $\triangle ABC$ and P is any point on BC. If $CQ \parallel PD$ meets AB in Q, prove that*

$$ar(\triangle BPQ) = \frac{1}{2} ar(\triangle ABC).$$



GIVEN D is the midpoint of side AB of $\triangle ABC$ and P is any point on BC. $CQ \parallel PD$ meets AB in Q.

TO PROVE $ar(\triangle BPQ) = \frac{1}{2} ar(\triangle ABC)$.

CONSTRUCTION Join CD and PQ.

PROOF We know that a median of a triangle divides it into two triangles of equal area.

And, in $\triangle ABC$, CD is a median.

$\therefore$ $ar(\triangle BCD) = \frac{1}{2} ar(\triangle ABC)$

$\Rightarrow$ $ar(\triangle BPD) + ar(\triangle DPC) = \frac{1}{2} ar(\triangle ABC)$. ... (i)

But, $\triangle$s DPC and DPQ being on the same base DP and between the same parallels DP and CQ, we have

$ar(\triangle DPC) = ar(\triangle DPQ)$. ... (ii)

Using (ii) in (i), we get

$$ar(\triangle BPD) + ar(\triangle DPQ) = \frac{1}{2} ar(\triangle ABC).$$

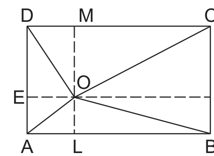
$\therefore$ $ar(\triangle BPQ) = \frac{1}{2} ar(\triangle ABC)$.

EXAMPLE 11 *A point O inside a rectangle ABCD is joined to the vertices. Prove that the sum of the areas of a pair of opposite triangles so formed is equal to the sum of the areas of the other pair of triangles.*

GIVEN A rect. $ABCD$ and O is a point inside it. OA, OB, OC and OD have been joined.

TO PROVE $\text{ar}(\triangle AOD)+\text{ar}(\triangle BOC)=\text{ar}(\triangle AOB)+\text{ar}(\triangle COD)$.

CONSTRUCTION Draw $EOF \parallel AB$ and $LOM \parallel AD$.



PROOF $EOF \parallel AB$ and DA cuts them.

$\therefore \quad \angle DEO = \angle EAB = 90^\circ$ (corres. $\angle$s).

$\therefore \quad OE \perp AD$.

Similarly, $OF \perp BC$; $OL \perp AB$ and $OM \perp DC$.

$\therefore \quad \text{ar}(\triangle AOD)+\text{ar}(\triangle BOC)$

$$=\left(\frac{1}{2}\times AD\times OE\right)+\left(\frac{1}{2}\times BC\times OF\right)$$

$$=\frac{1}{2}AD\times(OE+OF) \quad [\because\ BC=AD]$$

$$=\frac{1}{2}\times AD\times EF=\frac{1}{2}\times AD\times AB \quad [\because\ EF=AB]$$

$$=\frac{1}{2}\times \text{ar(rect. } ABCD).$$

Again, $\text{ar}(\triangle AOB)+\text{ar}(\triangle COD)$

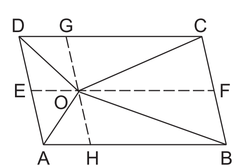
$$=\left(\frac{1}{2}\times AB\times OL\right)+\left(\frac{1}{2}\times DC\times OM\right)=\frac{1}{2}\times AB\times(OL+OM) \quad [\because\ DC=AB]$$

$$=\frac{1}{2}\times AB\times LM=\frac{1}{2}\times AB\times AD \quad [\because LM=AD]$$

$$=\frac{1}{2}\times \text{ar(rect. } ABCD).$$

$\therefore \quad \text{ar}(\triangle AOD)+\text{ar}(\triangle BOC)=\text{ar}(\triangle AOB)+\text{ar}(\triangle COD)$.

EXAMPLE 12 *ABCD is a parallelogram and O is a point in its interior. Prove that*



(i) $ar(\triangle AOB)+ar(\triangle COD)=\frac{1}{2}ar(\parallel gm\ ABCD)$.

(ii) $ar(\triangle AOB)+ar(\triangle COD)=ar(\triangle AOD)+ar(\triangle BOC)$.

GIVEN A $\parallel$gm $ABCD$ and O is a point in its interior.

TO PROVE (i) $\text{ar}(\triangle AOB)+\text{ar}(\triangle COD)=\frac{1}{2}\text{ar}(\parallel\text{gm } ABCD)$,

(ii) $\text{ar}(\triangle AOB)+\text{ar}(\triangle COD)=\text{ar}(\triangle AOD)+\text{ar}(\triangle BOC)$.

CONSTRUCTION Draw $EOF \parallel AB \parallel DC$ and $GOH \parallel AD \parallel BC$.

PROOF $\triangle AOB$ and ‖gm $EABF$ being on the same base AB and between the same parallels AB and EF, we have

$$\text{ar}(\triangle AOB) = \frac{1}{2}\text{ar}(\|\text{gm } EABF). \quad \dots \text{(i)}$$

Similarly, $\text{ar}(\triangle COD) = \frac{1}{2}\text{ar}(\|\text{gm } EFCD)$, ... (ii)

$$\text{ar}(\triangle AOD) = \frac{1}{2}\text{ar}(\|\text{gm } AHGD) \quad \dots \text{(iii)}$$

and $\text{ar}(\triangle BOC) = \frac{1}{2}\text{ar}(\|\text{gm } BCGH)$. ... (iv)

Adding (i) and (ii), we get

$$\text{ar}(\triangle AOB) + \text{ar}(\triangle COD) = \frac{1}{2}\text{ar}(\|\text{gm } EABF) + \frac{1}{2}\text{ar}(\|\text{gm } EFCD)$$
$$= \frac{1}{2}\text{ar}(\|\text{gm } ABCD). \quad \dots \text{(v)}$$

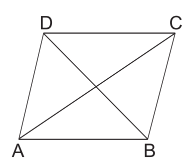
Adding (iii) and (iv), we get

$$\text{ar}(\triangle AOD) + \text{ar}(\triangle BOC) = \frac{1}{2}\text{ar}(\|\text{gm } AHGD) + \frac{1}{2}\text{ar}(\|\text{gm } BCGH)$$
$$= \frac{1}{2}\text{ar}(\|\text{gm } ABCD). \quad \dots \text{(vi)}$$

$$\therefore \quad \text{ar}(\triangle AOB) + \text{ar}(\triangle COD) = \text{ar}(\triangle AOD) + \text{ar}(\triangle BOC)$$

[from (v) and (vi)].

EXAMPLE 13 *If each diagonal of a quadrilateral separates it into two triangles of equal area then show that the quadrilateral is a parallelogram.*



GIVEN A quad. $ABCD$ in which diagonals AC and BD are such that

$$\text{ar}(\triangle ABD) = \text{ar}(\triangle CDB) \text{ and } \text{ar}(\triangle ABC) = \text{ar}(\triangle ACD).$$

TO PROVE $ABCD$ is a parallelogram.

PROOF $\text{ar}(\text{quad. } ABCD) = \text{ar}(\triangle ABC) + \text{ar}(\triangle ACD)$

$$= 2\text{ar}(\triangle ABC) \quad [\because \text{ar}(\triangle ACD) = \text{ar}(\triangle ABC)].$$

$$\therefore \quad \text{ar}(\triangle ABC) = \frac{1}{2}\text{ar}(\text{quad. } ABCD). \quad \dots \text{(i)}$$

Again, $\text{ar}(\text{quad. } ABCD) = \text{ar}(\triangle ABD) + \text{ar}(\triangle CDB) = 2\text{ar}(\triangle ABD)$

[$\because$ $\text{ar}(\triangle CDB) = \text{ar}(\triangle ABD)$].

$$\therefore \quad \text{ar}(\triangle ABD) = \frac{1}{2}\text{ar}(\text{quad. } ABCD). \quad \dots \text{(ii)}$$

From (i) and (ii), we get

$$\text{ar}(\triangle ABC) = \text{ar}(\triangle ABD).$$

Also, $\triangle ABC$ and $\triangle ABD$ have the same base AB.

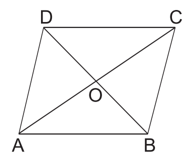
$\therefore$ they lie between the same parallels AB and DC [Theorem 9]

i.e., $AB \parallel DC$.

Similarly, $AD \parallel BC$.

Hence, $ABCD$ is a ‖gm.

EXAMPLE 14 *If the diagonals AC and BD of a quadrilateral ABCD intersect at O and separate the quadrilateral ABCD into four triangles of equal area, show that the quadrilateral is a parallelogram.*



GIVEN A quad. $ABCD$ whose diagonals AC and BD intersect at O in such a way that

$$\text{ar}(\triangle AOB) = \text{ar}(\triangle BOC) = \text{ar}(\triangle AOD) = \text{ar}(\triangle COD).$$

TO PROVE $ABCD$ is a ‖gm.

PROOF $\text{ar}(\triangle AOD) = \text{ar}(\triangle BOC)$

$\Rightarrow \quad \text{ar}(\triangle AOD) + \text{ar}(\triangle AOB) = \text{ar}(\triangle BOC) + \text{ar}(\triangle AOB)$

$\Rightarrow \quad \text{ar}(\triangle ABD) = \text{ar}(\triangle ABC)$.

Also, $\triangle ABD$ and $\triangle ABC$ have the same base.

$\therefore$ they lie between the same parallels AB and DC, i.e., $AB \parallel DC$ [Theorem 9].

Similarly, $AD \parallel BC$.

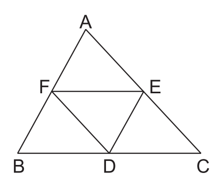
Hence, $ABCD$ is a parallelogram.

EXAMPLE 15 *D, E, F are the midpoints of the sides BC, CA and AB respectively of* $\triangle ABC$. *Prove that*

(i) BDEF is a ‖gm,

(ii) $ar(\triangle DEF) = \frac{1}{4} ar(\triangle ABC)$ *and*

(iii) $ar(\|gm\ BDEF) = \frac{1}{2} ar(\triangle ABC)$.



SOLUTION By midpoint theorem, $DE \parallel BA$ and $FE \parallel BC$.

$\therefore \quad DE \parallel BF$ and $FE \parallel BD$.

$\therefore \quad BDEF$ is a ‖gm.

Similarly, $AFDE$ is a ‖gm.

And, $DCEF$ is ‖gm.

Diagonal DF of ‖gm $BDEF$ divides it into two $\triangle$s of equal area.

$\therefore \quad \text{ar}(\triangle DEF) = \text{ar}(\triangle BDF)$.

Similarly, $\text{ar}(\triangle DEF) = \text{ar}(\triangle DCE)$.

And, $\text{ar}(\triangle DEF) = \text{ar}(\triangle AFE)$.

$\therefore \quad \text{ar}(\triangle DEF) = \text{ar}(\triangle BDF) = \text{ar}(\triangle DCE) = \text{ar}(\triangle AFE)$

$\therefore \quad \text{ar}(\triangle DEF) = \frac{1}{4}\text{ar}(\triangle ABC)$.

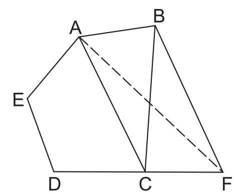
$\therefore \quad \text{ar}(\| \text{gm } BDEF) = 2 \times \text{ar}(\triangle DEF)$

$$= 2 \times \frac{1}{4} \times \text{ar}(\triangle ABC) = \frac{1}{2} \times \text{ar}(\triangle ABC).$$

EXAMPLE 16 *In the adjoining figure, ABCDE is a pentagon. A line through B parallel to AC meets DC produced at F. Show that*

(i) $ar(\triangle ACB) = ar(\triangle ACF)$,

(ii) $ar(AEDF) = ar(ABCDE)$.



SOLUTION Join AF.

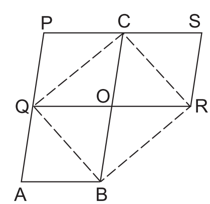
$\triangle ACB$ and $\triangle ACF$ have the same base AC and lie between the same parallels AC and BF.

$\therefore \quad \text{ar}(\triangle ACB) = \text{ar}(\triangle ACF)$.

And so, $\text{ar}(\triangle ACB) + \text{ar}(ACDE) = \text{ar}(\triangle ACF) + \text{ar}(ACDE)$

$\Rightarrow \quad \text{ar}(ABCDE) = \text{ar}(AEDF)$.

EXAMPLE 17 *In the adjoining figure, PQRS and PABC are two parallelograms of equal area. Prove that* $QC \| BR$.



SOLUTION Let BC and QR intersect at O.

Join BQ, QC, CR and RB.

Now, $\text{ar}(\| \text{gm } PQRS) = \text{ar}(\| \text{gm } PABC)$ (given)

$\Rightarrow \quad \text{ar}(\| \text{gm } PQRS) - \text{ar}(\| \text{gm } QOCP)$

$$= \text{ar}(\| \text{gm } PABC) - \text{ar}(\| \text{gm } QOCP)$$

$\Rightarrow \quad \text{ar}(\| \text{gm } ORSC) = \text{ar}(\| \text{gm } ABOQ)$

$\Rightarrow \quad 2 \times \text{ar}(\triangle ORC) = 2 \times \text{ar}(\triangle OBQ) \Rightarrow \text{ar}(\triangle ORC) = \text{ar}(\triangle OBQ)$

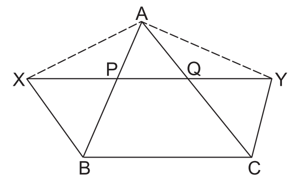
$\Rightarrow \quad \text{ar}(\triangle ORC) + \text{ar}(\triangle OBR) = \text{ar}(\triangle OBQ) + \text{ar}(\triangle OBR)$

$\Rightarrow \quad \text{ar}(\triangle BRC) = \text{ar}(\triangle BRQ)$.

Now, $\triangle BRC$ and $\triangle BRQ$ being on the same base BR and equal in area, they must be between the same parallels.

$\therefore \quad QC \| BR$.

EXAMPLE 18 *In the adjoining figure, PQ is a line parallel to side BC of* $\triangle ABC$. *If* $BX \parallel CA$ *and* $CY \parallel BA$ *meet the line PQ produced in X and Y respectively, show that* $ar(\triangle ABX) = ar(\triangle ACY)$.



GIVEN A $\triangle ABC$, $PQ \parallel BC$; $BX \parallel CA$ and $CY \parallel BA$.

TO PROVE $\text{ar}(\triangle ABX) = \text{ar}(\triangle ACY)$.

PROOF ‖gm $XBCQ$ and $\triangle ABX$ being on the same base XB and between the same parallels XB and CA, we have

$$\text{ar}(\triangle ABX) = \frac{1}{2}\text{ar}(\|\text{gm } XBCQ). \qquad \ldots \text{(i)}$$

Again, ‖gm $BCYP$ and $\triangle ACY$ being on the same base CY and between the same parallels CY and BA, we have

$$\text{ar}(\triangle ACY) = \frac{1}{2}\text{ar}(\|\text{gm } BCYP). \qquad \ldots \text{(ii)}$$

But, ‖gm $XBCQ$ and ‖gm $BCYP$ being on the same base BC and between the same parallels BC and XY, we have

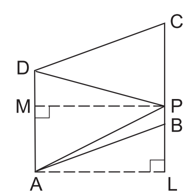
$$\text{ar}(\|\text{gm } XBCQ) = \text{ar}(\|\text{gm } BCYP). \qquad \ldots \text{(iii)}$$

$\therefore$ from (i), (ii) and (iii), we get

$$\text{ar}(\triangle ABX) = \text{ar}(\triangle ACY).$$

EXAMPLE 19 *In the adjoining figure, ABCD is a parallelogram and P is any point on BC. Prove that*

$$ar(\triangle ABP) + ar(\triangle DPC) = ar(\triangle PDA).$$



SOLUTION Draw $AL \perp BC$ and $PM \perp AD$.

Since $BC \parallel AD$, so distance between them remains the same.

$\therefore \quad AL = PM$.

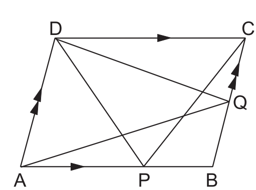
$\text{ar}(\triangle ABP) + \text{ar}(\triangle DPC)$

$= \frac{1}{2} \times BP \times AL + \frac{1}{2} \times PC \times AL = \frac{1}{2} \times AL \times (BP + PC)$

$= \frac{1}{2} \times AL \times BC = \frac{1}{2} \times PM \times AD \quad [\because\ AL = PM \text{ and } BC = AD]$

$= \text{ar}(\triangle PDA)$.

EXAMPLE 20 *In the adjoining figure, ABCD is a parallelogram. If P and Q are any two points on the sides AB and BC respectively, prove that* $\text{ar}(\triangle CPD) = \text{ar}(\triangle AQD)$.



SOLUTION $\triangle CPD$ and $\|$gm $ABCD$ are on the same base DC and between the same parallels CD and AB.

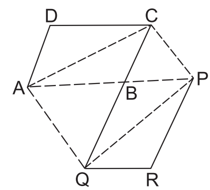
$\therefore \quad \text{ar}(\triangle CPD) = \frac{1}{2}\text{ar}(\|\text{gm } ABCD)$.

Similarly, $\text{ar}(\triangle AQD) = \frac{1}{2}\text{ar}(\|\text{gm } ABCD)$.

$\therefore \quad \text{ar}(\triangle CPD) = \text{ar}(\triangle AQD)$.

EXAMPLE 21 *In the adjoining figure, the side AB of $\|$gm ABCD is produced to a point P and a line through A, parallel to CP, meets CB produced in Q and the $\|$gm BQRP is completed. Prove that*

$$ar(\|gm\ ABCD) = ar(\|gm\ BQRP).$$



SOLUTION Join AC, CP, PQ and QA.

Now, $\triangle AQC$ and $\triangle AQP$ being on the same base AQ and between the same parallels AQ and CP, we have

$\text{ar}(\triangle AQC) = \text{ar}(\triangle AQP)$

$\Rightarrow \quad \text{ar}(\triangle AQC) - \text{ar}(\triangle AQB) = \text{ar}(\triangle AQP) - \text{ar}(\triangle AQB)$

$\Rightarrow \quad \text{ar}(\triangle ABC) = \text{ar}(\triangle BQP)$

$\Rightarrow \quad \frac{1}{2}\text{ar}(\|\text{gm } ABCD) = \frac{1}{2}\text{ar}(\|\text{gm } BQRP)$

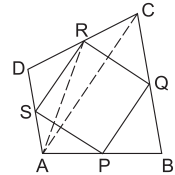
[$\because$ AC divides $\|$gm $ABCD$ into two $\triangle$s of equal area and QP divides $\|$gm $BQRP$ into two $\triangle$s of equal area]

$\Rightarrow \quad \text{ar}(\|\text{gm } ABCD) = \text{ar}(\|\text{gm } BQRP)$.

Hence, $\text{ar}(\|\text{gm } ABCD) = \text{ar}(\|\text{gm } BQRP)$.

EXAMPLE 22 *Let P, Q, R, S be respectively the midpoints of the sides AB, BC, CD and DA of quad. ABCD. Show that PQRS is a parallelogram such that*

$$ar(\|gm\ PQRS) = \frac{1}{2}ar(quad.\ ABCD).$$



SOLUTION Join AC and AR.

In $\triangle ABC$, P and Q are midpoints of AB and BC respectively.

$\therefore \quad PQ \parallel AC$ and $PQ = \frac{1}{2}AC$.

In $\triangle DAC$, S and R are midpoints of AD and DC respectively.

$\therefore \quad SR \parallel AC$ and $SR = \frac{1}{2}AC$.

Thus, $PQ \parallel SR$ and $PQ = SR$.

$\therefore$ $PQRS$ is a $\parallel$gm.

Now, median AR divides $\triangle ACD$ into two $\triangle$s of equal area.

$\therefore$ $\text{ar}(\triangle ARD) = \frac{1}{2}\text{ar}(\triangle ACD)$. ... (i)

Median RS divides $\triangle ARD$ into two $\triangle$s of equal area.

$\therefore$ $\text{ar}(\triangle DSR) = \frac{1}{2}\text{ar}(\triangle ARD)$. ... (ii)

From (i) and (ii), we get $\text{ar}(\triangle DSR) = \frac{1}{4}\text{ar}(\triangle ACD)$.

Similarly, $\text{ar}(\triangle BQP) = \frac{1}{4}\text{ar}(\triangle ABC)$.

$\therefore$ $\text{ar}(\triangle DSR) + \text{ar}(\triangle BQP) = \frac{1}{4}[\text{ar}(\triangle ACD) + \text{ar}(\triangle ABC)]$

$\Rightarrow$ $\text{ar}(\triangle DSR) + \text{ar}(\triangle BQP) = \frac{1}{4}\text{ar}[\text{quad. } ABCD]$. ... (iii)

Similarly, $\text{ar}(\triangle CRQ) + \text{ar}(\triangle ASP) = \frac{1}{4}\text{ar}(\text{quad. } ABCD)$. ... (iv)

Adding (iii) and (iv), we get

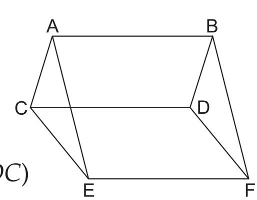
$$\text{ar}(\triangle DSR) + \text{ar}(\triangle BQP) + \text{ar}(\triangle CRQ) + \text{ar}(\triangle ASP) = \frac{1}{2}\text{ar}(\text{quad. } ABCD). \quad \text{... (v)}$$

But, $\text{ar}(\triangle DSR) + \text{ar}(\triangle BQP) + \text{ar}(\triangle CRQ) + \text{ar}(\triangle ASP)$
$+ \text{ar}(\parallel\text{gm } PQRS) = \text{ar}(\text{quad. } ABCD)$. ... (vi)

Subtracting (v) from (vi), we get

$$\text{ar}(\parallel\text{gm } PQRS) = \frac{1}{2}\text{ar}(\text{quad. } ABCD).$$

EXAMPLE 23 *In the adjoining figure, ABDC, CDFE and ABFE are parallelograms. Show that* $ar(\triangle ACE) = ar(\triangle BDF)$.



SOLUTION We have:

$AC = BD$ (opp. sides of $\parallel$gm $ABDC$)

$CE = DF$ (opp. sides of $\parallel$gm $CDFE$)

$AE = BF$ (opp. sides of $\parallel$gm $ABFE$)

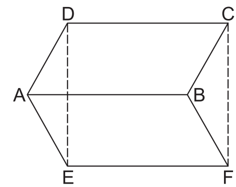
$\therefore$ $\triangle ACE \cong \triangle BDF$ [SSS-criterion].

And so, $\text{ar}(\triangle ACE) = \text{ar}(\triangle BDF)$

[$\because$ congruent figures have equal areas].

EXAMPLE 24 *In the adjoining figure, two parallelograms ABCD and AEFB are drawn on opposite sides of AB. Prove that*

$$ar(\|gm\ ABCD) + ar(\|gm\ AEFB) = ar(\|gm\ EFCD).$$



SOLUTION In $\triangle AED$ and $\triangle BFC$, we have

$AD = BC$, $AE = BF$ and $DE = CF$ (opp. sides of a ‖gm).

$\therefore$ $\triangle AED \cong \triangle BFC$.

So, $\text{ar}(\triangle AED) = \text{ar}(\triangle BFC)$. … (i)

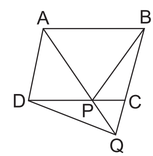
$\text{ar}(\|\text{gm } ABCD) + \text{ar}(\|\text{gm } AEFB)$

$= \text{ar}(\|\text{gm } DEFC) + \text{ar}(\triangle AED) - \text{ar}(\triangle BFC)$

$= \text{ar}(\|\text{gm } EFCD)$ [using (i)].

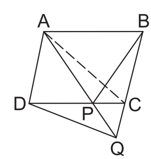
EXAMPLE 25 *In the adjoining figure, ABCD is a parallelogram and a line through A cuts DC at P and BC produced at Q. Prove that*

$$ar(\triangle BPC) = ar(\triangle DPQ).$$



SOLUTION Join AC.

Now, $\triangle ADC$ and $\triangle ADQ$ being on the same base AD and between the same parallels AD and BC, are equal in area.



$\therefore$ $\text{ar}(\triangle ADC) = \text{ar}(\triangle ADQ)$

$\Rightarrow$ $\text{ar}(\triangle ADC) - \text{ar}(\triangle ADP) = \text{ar}(\triangle ADQ) - \text{ar}(\triangle ADP)$

$\Rightarrow$ $\text{ar}(\triangle APC) = \text{ar}(\triangle DPQ)$. … (i)

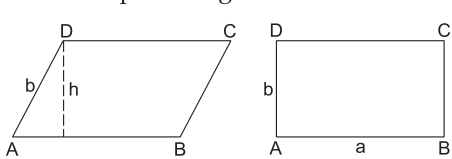
Also, $\triangle APC$ and $\triangle BPC$ being on the same base PC and between the same parallels PC and AB, are equal in area.

$\therefore$ $\text{ar}(\triangle APC) = \text{ar}(\triangle BPC)$. … (ii)

From (i) and (ii), we get $\text{ar}(\triangle BPC) = \text{ar}(\triangle DPQ)$.

EXAMPLE 26 *Prove that of all the parallelograms of given sides, the parallelogram which is a rectangle has the greatest area.*

SOLUTION Let us consider all parallelograms of sides a and b.



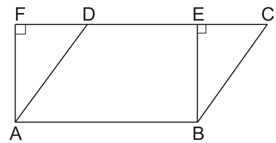
In ∥gm $ABCD$ of sides a and b, let h be the height corresponding to the base a.

Then, $h \leq b$.

Now, $h \leq b \Rightarrow ah \leq ab$.

$\therefore$ ar(∥gm $ABCD$) $\leq$ ar(rect. $ABCD$).

EXAMPLE 27 *∥gm ABCD and rect. ABEF have the same base AB and equal area. Show that the perimeter of the ∥gm is greater than that of the rectangle.*



SOLUTION We have

$$AB = EF \text{ and } CD = AB \Rightarrow AB + CD = AB + EF. \quad \ldots \text{(i)}$$

We know that, of all the line segments to a given line from a point, outside it, the perpendicular is the least.

$\therefore$ $BE < BC$ and $AF < AD$.

$\therefore$ $BC + AD > BE + AF$. … (ii)

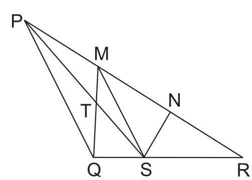
From (i) and (ii), we get

$$AB + BC + CD + AD > AB + BE + EF + AF.$$

Hence, the perimeter of the ∥gm is greater than the perimeter of the rectangle.

EXAMPLE 28 *M and N are points on the side PR of the triangle PQR such that PM = MN = NR. Through M, a line is drawn parallel to PQ to meet QR at S, as shown in the adjoining figure. Prove that*

$$ar(\triangle PSN) = ar(QSNM).$$



SOLUTION $\triangle PSM$ and $\triangle QSM$ have the same base SM and lie between the same parallels PQ and MS.

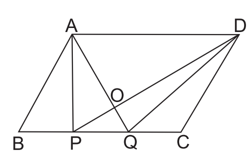
$\therefore$ ar($\triangle PSM$) = ar($\triangle QSM$)

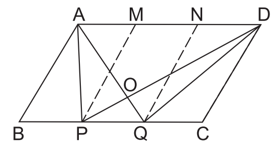
$\Rightarrow$ ar($\triangle PSM$) + ar($\triangle MSN$) = ar($\triangle QSM$) + ar($\triangle MSN$)

$\Rightarrow$ ar($\triangle PSN$) = ar($QSNM$).

EXAMPLE 29 *In the adjoining figure, ABCD is a parallelogram. Points P and Q on BC trisect BC. Prove that*

$$ar(\triangle APQ) = ar(\triangle DPQ) = \frac{1}{6} ar(ABCD).$$





SOLUTION Through P and Q, we draw PM and QN parallel to AB. Parallelograms $ABPM$, $MPQN$ and $NQCD$ have equal areas since they have equal bases and lie between the same parallels [Theorem 3].

$\therefore$ $\text{ar}(\|\text{gm } ABPM) = \text{ar}(\|\text{gm } MPQN)$

$$= \text{ar}(\|\text{gm } NQCD) = \frac{1}{3}\text{ar}(\|\text{gm } ABCD). \quad \ldots \text{(i)}$$

Now, $\text{ar}(\triangle APQ) = \text{ar}(\triangle DPQ)$

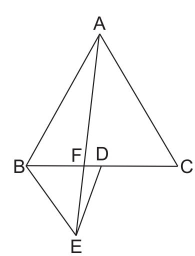
$$= \frac{1}{2}\text{ar}(\|\text{gm } MPQN)$$

[$\because$ $\triangle APQ$, $\triangle DPQ$ and $\|$gm $MPQN$ have the same base PQ and lie between the same parallels AD and BC]

$\Rightarrow$ $\text{ar}(\triangle APQ) = \text{ar}(\triangle DPQ) = \frac{1}{2}\left\{\frac{1}{3}\text{ar}(\|\text{gm } ABCD)\right\}$ [using (i)].

Hence, $\text{ar}(\triangle APQ) = \text{ar}(\triangle DPQ) = \frac{1}{6}\text{ar}(\|\text{gm } ABCD)$.

EXAMPLE 30 *In the adjoining figure, ABC and BDE are two equilateral triangles such that D is the midpoint of BC. If AE intersects BC at F, show that*



(i) $ar(\triangle BDE) = \frac{1}{4}ar(\triangle ABC)$

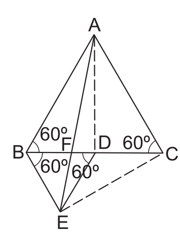
(ii) $ar(\triangle BDE) = \frac{1}{2}ar(\triangle BAE)$

(iii) $ar(\triangle ABC) = 2ar(\triangle BEC)$

(iv) $ar(\triangle BFE) = ar(\triangle AFD)$

(v) $ar(\triangle BFE) = 2ar(\triangle FED)$

(vi) $ar(\triangle FED) = \frac{1}{8}ar(\triangle AFC)$.



SOLUTION Join AD and EC.

We have $\angle ABC = \angle DBE = 60^\circ \Rightarrow DE \parallel AB$

(alt. interior $\angle$s)

and $\angle ACB = \angle DBE = 60^\circ \Rightarrow BE \parallel AC$

[$\because$ $\triangle ABC$ and $\triangle BDE$ are eq. $\triangle$s].

Now, let each side of $\triangle BDE = a$.

Then, $\text{ar}(\triangle BDE) = \frac{\sqrt{3}}{4}a^2$. ... (i)

Each side of $\triangle ABC = 2a$ [$\because BC = 2BD$].

$\therefore \quad \text{ar}(\triangle ABC) = \frac{\sqrt{3}}{4}(2a)^2 = 4\left(\frac{\sqrt{3}}{4}a^2\right)$. ... (ii)

From (i) and (ii), we get

$\text{ar}(\triangle BDE) = \frac{1}{4}\text{ar}(\triangle ABC)$. ... (iii)

Now, AD is a median of $\triangle ABC$

$\Rightarrow \quad \text{ar}(\triangle ABD) = \frac{1}{2}\text{ar}(\triangle ABC)$. ... (iv)

But, $\text{ar}(\triangle BAE) = \text{ar}(\triangle ABD)$ ... (v)

[$\because$ same base AB and same parallels AB and DE].

From (iv) and (v), we get

$\text{ar}(\triangle BAE) = \frac{1}{2}\text{ar}(\triangle ABC)$. ... (vi)

From (iii) and (vi), we get

$\text{ar}(\triangle BDE) = \frac{1}{2}\text{ar}(\triangle BAE)$. ... (vii)

Now, $\text{ar}(\triangle BDE) = \frac{1}{2}\text{ar}(\triangle BEC)$ ... (viii)

[$\because ED$ is a median of $\triangle BEC$].

From (iii) and (viii), we get

$\text{ar}(\triangle ABC) = 2\text{ar}(\triangle BEC)$. ... (ix)

Now, $\text{ar}(\triangle AED) = \text{ar}(\triangle BDE)$

[$\because$ same base DE and same parallels AB and DE]

$\Rightarrow \quad \text{ar}(\triangle AED) - \text{ar}(\triangle FED) = \text{ar}(\triangle BDE) - \text{ar}(\triangle FED)$

$\Rightarrow \quad \text{ar}(\triangle AFD) = \text{ar}(\triangle BFE)$. ... (x)

Now, height of $\triangle AFD$ = height of $\triangle ABC = H = \frac{\sqrt{3}}{2}(2a) = \sqrt{3}a$

and height of $\triangle FED$ = height of $\triangle BDE = h = \frac{\sqrt{3}}{2}(a) = \frac{\sqrt{3}a}{2} = \frac{H}{2}$.

$\therefore \quad \text{ar}(\triangle AFD) = \frac{1}{2} \times FD \times H$ and $\text{ar}(\triangle FED) = \frac{1}{2} \times FD \times \frac{H}{2}$

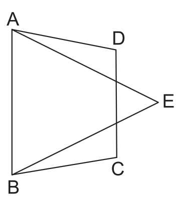
$\Rightarrow \quad \text{ar}(\triangle AFD) = 2\text{ar}(\triangle FED)$ ... (xi)

$\Rightarrow \quad \text{ar}(\triangle BFE) = 2\text{ar}(\triangle FED)$ ... (xii) [using (x)].

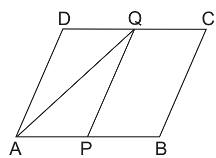
$$\text{Now, } ar(\triangle AFC) = ar(\triangle AFD) + ar(\triangle ADC)$$
$$= 2ar(\triangle FED) + \frac{1}{2}\{ar(\triangle ABC)\} \quad \text{[using (xi)]}$$
$$= 2ar(\triangle FED) + \frac{1}{2}\{4ar(\triangle BDE)\} \quad \text{[using (iii)]}$$
$$= 2ar(\triangle FED) + 2\{ar(\triangle BFE) + ar(\triangle FED)\}$$
$$= 2ar(\triangle FED) + 2\{ar(\triangle FED) + ar(\triangle FED)\} \quad \text{[using (xii)]}$$
$$= 8ar(\triangle FED)$$
$$\Rightarrow \quad ar(\triangle FED) = \frac{1}{8}ar(\triangle AFC).$$

EXERCISE 11

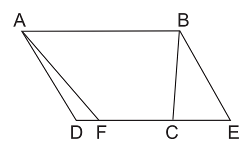
1. Which of the following figures lie on the same base and between the same parallels. In such a case, write the common base and the two parallels.



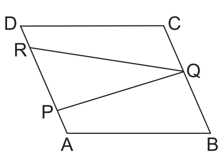
(i)



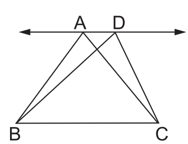
(ii)



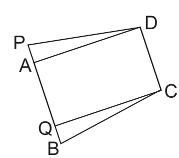
(iii)



(iv)

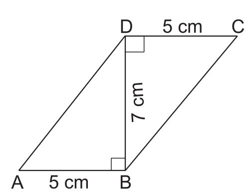


(v)

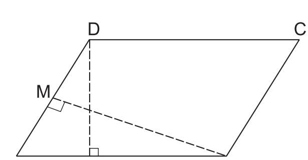


(vi)

2. In the adjoining figure, show that $ABCD$ is a parallelogram.
Calculate the area of $\parallel$gm $ABCD$.



3. In a parallelogram $ABCD$, it is being given that $AB = 10$ cm and the altitudes corresponding to the sides AB and AD are $DL = 6$ cm and $BM = 8$ cm, respectively. Find AD.

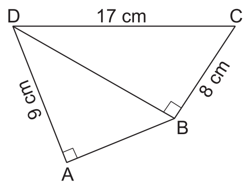


4. Find the area of a figure formed by joining the midpoints of the adjacent sides of a rhombus with diagonals 12 cm and 16 cm.

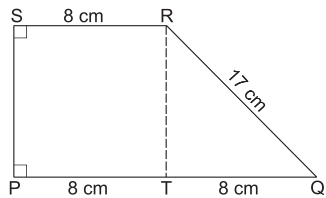
5. Find the area of a trapezium whose parallel sides are 9 cm and 6 cm respectively and the distance between these sides is 8 cm.

6. (i) Calculate the area of quad. $ABCD$, given in Fig. (i).

(ii) Calculate the area of trap. $PQRS$, given in Fig. (ii).

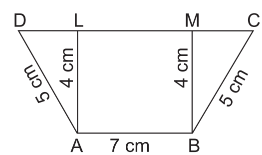


(i)



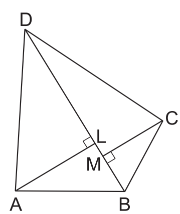
(ii)

7. In the adjoining figure, $ABCD$ is a trapezium in which $AB \parallel DC$; $AB = 7$ cm; $AD = BC = 5$ cm and the distance between AB and DC is 4 cm. Find the length of DC and hence, find the area of trap. $ABCD$.



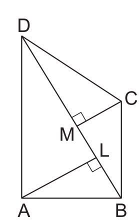
8. BD is one of the diagonals of a quad. $ABCD$. If $AL \perp BD$ and $CM \perp BD$, show that

$$\text{ar(quad. } ABCD) = \frac{1}{2} \times BD \times (AL + CM).$$



9. M is the midpoint of the side AB of a parallelogram $ABCD$. If ar($AMCD$) $= 24 \text{ cm}^2$, find ar($\triangle ABC$).

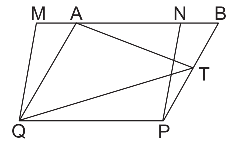
10. In the adjoining figure, $ABCD$ is a quadrilateral in which diag. $BD = 14$ cm. If $AL \perp BD$ and $CM \perp BD$ such that $AL = 8$ cm and $CM = 6$ cm, find the area of quad. $ABCD$.



11. If P and Q are any two points lying respectively on the sides DC and AD of a parallelogram $ABCD$ then show that

$$\text{ar}(\triangle APB) = \text{ar}(\triangle BQC).$$

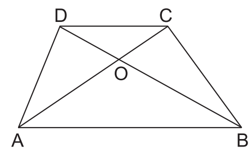
12. In the adjoining figure, $MNPQ$ and $ABPQ$ are parallelograms and T is any point on the side BP. Prove that



(i) ar($MNPQ$) = ar($ABPQ$)

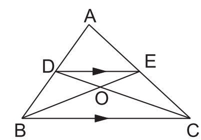
(ii) ar($\triangle ATQ$) $= \frac{1}{2}$ar($MNPQ$).

13. In the adjoining figure, $ABCD$ is a trapezium in which $AB \parallel DC$ and its diagonals AC and BD intersect at O.



Prove that ar($\triangle AOD$) = ar($\triangle BOC$).

14. In the adjoining figure, $DE \parallel BC$.



Prove that

(i) ar($\triangle ACD$) = ar($\triangle ABE$)

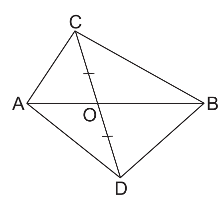
(ii) ar($\triangle OCE$) = ar($\triangle OBD$).

15. Prove that a median divides a triangle into two triangles of equal area.

16. Show that a diagonal divides a parallelogram into two triangles of equal area.

17. In the adjoining figure, ABC and ABD are two triangles on the same base AB. If line segment CD is bisected by AB at O, show that

ar($\triangle ABC$) = ar($\triangle ABD$).

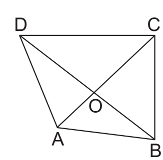


18. D and E are points on sides AB and AC respectively of $\triangle ABC$ such that ar($\triangle BCD$) = ar($\triangle BCE$). Prove that $DE \parallel BC$.

19. P is any point on the diagonal AC of a parallelogram $ABCD$. Prove that

ar($\triangle ADP$) = ar($\triangle ABP$).

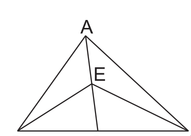
20. In the adjoining figure, the diagonals AC and BD of a quadrilateral $ABCD$ intersect at O.



If $BO = OD$, prove that

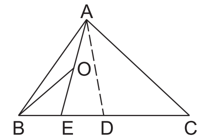
ar($\triangle ABC$) = ar($\triangle ADC$).

21. The vertex A of $\triangle ABC$ is joined to a point D on the side BC. The midpoint of AD is E.

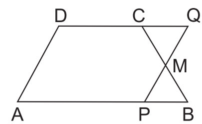


Prove that ar($\triangle BEC$) $= \frac{1}{2}$ar($\triangle ABC$).

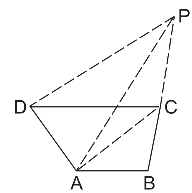
22. D is the midpoint of side BC of $\triangle ABC$ and E is the midpoint of BD. If O is the midpoint of AE, prove that $\text{ar}(\triangle BOE) = \frac{1}{8}\text{ar}(\triangle ABC)$.



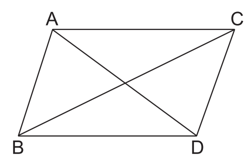
23. In a trapezium $ABCD$, $AB \parallel DC$ and M is the midpoint of BC. Through M, a line $PQ \parallel AD$ has been drawn which meets AB in P and DC produced in Q, as shown in the adjoining figure. Prove that $\text{ar}(ABCD) = \text{ar}(APQD)$.



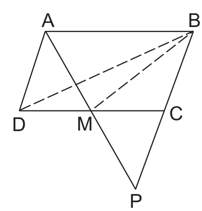
24. In the adjoining figure, $ABCD$ is a quadrilateral. A line through D, parallel to AC, meets BC produced in P.
Prove that $\text{ar}(\triangle ABP) = \text{ar}(\text{quad. } ABCD)$.



25. In the adjoining figure, $\triangle ABC$ and $\triangle DBC$ are on the same base BC with A and D on opposite sides of BC such that $\text{ar}(\triangle ABC) = \text{ar}(\triangle DBC)$. Show that BC bisects AD.



26. $ABCD$ is a parallelogram in which BC is produced to P such that $CP = BC$, as shown in the adjoining figure. AP intersects CD at M. If $\text{ar}(DMB) = 7\text{ cm}^2$, find the area of parallelogram $ABCD$.

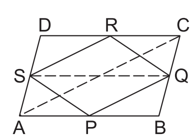


27. In a parallelogram $ABCD$, any point E is taken on the side BC. AE and DC when produced meet at a point M. Prove that

$$\text{ar}(\triangle ADM) = \text{ar}(ABMC)$$

28. P, Q, R, S are respectively the midpoints of the sides AB, BC, CD and DA of $\parallel$gm $ABCD$. Show that $PQRS$ is a parallelogram and also show that

$$\text{ar}(\parallel\text{gm } PQRS) = \frac{1}{2} \times \text{ar}(\parallel\text{gm } ABCD).$$



29. In a triangle ABC, the medians BE and CF intersect at G. Prove that

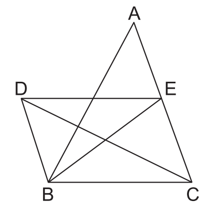
$$\text{ar}(\triangle BCG) = \text{ar}(AFGE).$$

30. The base BC of $\triangle ABC$ is divided at D such that $BD = \frac{1}{2}DC$. Prove that

$$\text{ar}(\triangle ABD) = \frac{1}{3} \times \text{ar}(\triangle ABC).$$

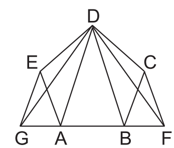
31. In the adjoining figure, $BD \parallel CA$, E is the midpoint of CA and $BD = \frac{1}{2}CA$. Prove that

$$\text{ar}(\triangle ABC) = 2\text{ar}(\triangle DBC).$$

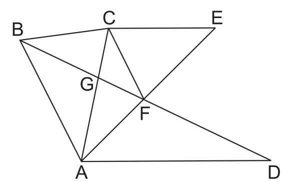


32. The given figure shows a pentagon $ABCDE$. EG, drawn parallel to DA, meets BA produced at G, and CF, drawn parallel to DB, meets AB produced at F. Show that

$$\text{ar(pentagon } ABCDE) = \text{ar}(\triangle DGF).$$

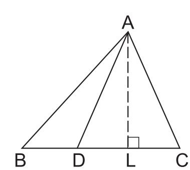


33. In the adjoining figure, $CE \parallel AD$ and $CF \parallel BA$. Prove that $\text{ar}(\triangle CBG) = \text{ar}(\triangle AFG)$.

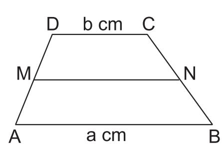


34. In the adjoining figure, the point D divides the side BC of $\triangle ABC$ in the ratio $m:n$. Prove that

$$\text{ar}(\triangle ABD) : \text{ar}(\triangle ADC) = m : n.$$



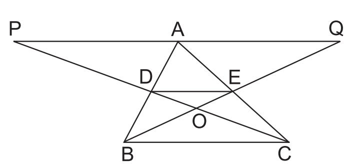
35. In a trapezium $ABCD$, $AB \parallel DC$, $AB = a$ cm, and $DC = b$ cm. If M and N are the midpoints of the nonparallel sides, AD and BC respectively then find the ratio of ar($DCNM$) and ar($MNBA$).



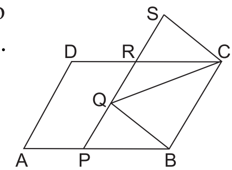
36. $ABCD$ is a trapezium in which $AB \parallel DC$, $AB = 16$ cm and $DC = 24$ cm. If E and F are respectively the midpoints of AD and BC, prove that

$$\text{ar}(ABFE) = \frac{9}{11}\text{ar}(EFCD).$$

37. In the adjoining figure, D and E are respectively the midpoints of sides AB and AC of $\triangle ABC$. If $PQ \parallel BC$ and CDP and BEQ are straight lines then prove that $\text{ar}(\triangle ABQ) = \text{ar}(\triangle ACP)$.



38. In the adjoining figure, $ABCD$ and $BQSC$ are two parallelograms. Prove that $ar(\triangle RSC) = ar(\triangle PQB)$.



ANSWERS (EXERCISE 11)

1. (i) No (ii) No (iii) Yes; AB; AB and DE (iv) No
(v) Yes; BC; BC and AD (vi) Yes; CD; CD and BP

2. 35 cm^2 **3.** 7.5 cm **4.** 48 cm^2 **5.** 60 cm^2

6. (i) 114 cm^2 (ii) 180 cm^2 **7.** $DC = 13$ cm, area = 40 cm^2

9. 36 cm^2 **10.** 98 cm^2 **26.** 28 cm^2 **35.** $(a+3b):(3a+b)$

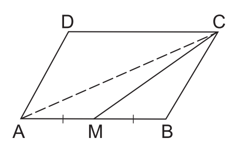
HINTS TO SOME SELECTED QUESTIONS

2. $\angle ABD = \angle BDC = 90^\circ$ (corresponding $\angle$s).
$\therefore$ $AB \parallel DC$. Also, $AB = DC$.

3. ar(||gm $ABCD$) $= AB \times DL = AD \times BM$.

4. Area of rhombus $= \frac{1}{2} \times$ product of diagonals $= \left(\frac{1}{2} \times 12 \times 16\right) \text{cm}^2 = 96 \text{ cm}^2$.

Area of figure formed by joining the midpoints $= \frac{1}{2} \times$ area of rhombus $= 48 \text{ cm}^2$.



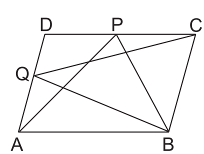
9. $ar(\triangle ACD) = ar(\triangle ABC) = \frac{1}{2} ar$(||gm $ABCD$),

$ar(\triangle AMC) = ar(\triangle MBC) = \frac{1}{2} ar(\triangle ABC)$.

$\therefore$ $ar(AMCD) = 3ar(\triangle AMC) \Rightarrow 24 = 3ar(\triangle AMC)$

$\Rightarrow ar(\triangle AMC) = 8 \text{ cm}^2$.

And so, $ar(\triangle ABC) = 2ar(\triangle AMC) = 16 \text{ cm}^2$.



11. $ar(\triangle APB) = \frac{1}{2} ar(ABCD)$ and $ar(\triangle BQC) = \frac{1}{2} ar(ABCD)$

[$\because$ $\triangle$ and ||gm have same base and lie between same parallels].

$\therefore$ $ar(\triangle APB) = ar(\triangle BQC)$.

12. $ar(MNPQ) = ar(ABPQ)$ (they have same base and lie between same parallels).

$ar(\triangle ATQ) = \frac{1}{2} ar(ABPQ)$ (they have same base and lie between same parallels).

13. $ar(\triangle CDA) = ar(\triangle CDB)$ ($\triangle$s on the same base and between the same parallels)

$\Rightarrow$ $ar(\triangle CDA) - ar(\triangle OCD) = ar(\triangle CDB) - ar(\triangle OCD)$.

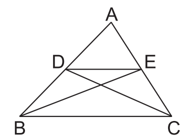
14. $ar(\triangle DEC) = ar(\triangle DEB)$ ($\triangle$s on the same base and between the same parallels).

(i) Add ar($\triangle ADE$) on both sides.

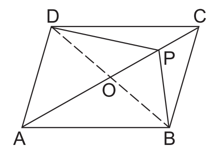
(ii) Subtract ar($\triangle ODE$) from both sides.

17. AO is the median of $\triangle CAD \Rightarrow$ ar($\triangle COA$) = ar($\triangle DOA$).
BO is the median of $\triangle CBD \Rightarrow$ ar($\triangle COB$) = ar($\triangle DOB$).
$\therefore$ ar($\triangle COA$) + ar($\triangle COB$) = ar($\triangle DOA$) + ar($\triangle DOB$) $\Rightarrow$ ar($\triangle ABC$) = ar($\triangle ABD$).

18. $\triangle BCE$ and $\triangle BCD$ are on the same base BC and have equal area. So, their corresponding altitudes must be equal.
$\therefore$ $DE \parallel BC$.



19. Draw diagonal BD. Let AC and BD intersect at O. Then, O is the midpoint of BD.



ar($\triangle DPO$) = ar($\triangle BPO$) [$\because$ PO is a median of $\triangle BPD$].
ar($\triangle ADO$) = ar($\triangle ABO$) [$\because$ AO is a median of $\triangle ABD$].
$\therefore$ ar($\triangle DPO$) + ar($\triangle ADO$) = ar($\triangle BPO$) + ar($\triangle ABO$)
$\Rightarrow$ ar($\triangle ADP$) = ar($\triangle ABP$).

20. CO is a median of $\triangle BCD \Rightarrow$ ar($\triangle COD$) = ar($\triangle COB$).
AO is a median of $\triangle ABD \Rightarrow$ ar($\triangle AOD$) = ar($\triangle AOB$).
$\therefore$ ar($\triangle COD$) + ar($\triangle AOD$) = ar($\triangle COB$) + ar($\triangle AOB$).

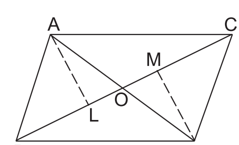
21. ar($\triangle DBE$) = $\frac{1}{2}$ar($\triangle ABD$) and ar($\triangle CDE$) = $\frac{1}{2}$ar($\triangle ADC$).

22. ar($\triangle ABD$) = $\frac{1}{2}$ar($\triangle ABC$); ar($\triangle ABE$) = $\frac{1}{2}$ar($\triangle ABD$) = $\frac{1}{4}$ar($\triangle ABC$).
ar($\triangle BOE$) = $\frac{1}{2}$ar($\triangle ABE$) = $\frac{1}{8}$ar($\triangle ABC$).

23. $\triangle MQC \cong \triangle MPB$ [$\because$ $CM = BM$, $\angle CMQ = \angle BMP$, $\angle MCQ = \angle MBP$].
$\therefore$ ar($\triangle MQC$) = ar($\triangle MPB$)
$\Rightarrow$ ar($\triangle MQC$) + ar($APCD$) = ar($\triangle MPB$) + ar($APCD$)
$\Rightarrow$ ar($APQD$) = ar($ABCD$).

24. $\triangle ACD$ and $\triangle ACP$ are on the same base and between the same parallels.
$\therefore$ ar($\triangle ACP$) = ar($\triangle ACD$)
$\Rightarrow$ ar($\triangle ACP$) + ar($\triangle ABC$) = ar($\triangle ACD$) + ar($\triangle ABC$).

25. Draw $AL \perp BC$ and $DM \perp BC$.
Since $\triangle ABC$ and $\triangle DBC$ have the same base BC and are equal in area, their altitudes must be equal. So, $AL = DM$.
Let AD and BC intersect at O.
Now, $\triangle ALO \cong \triangle DMO$. Therefore, $OA = OD$.

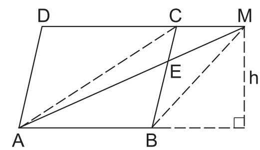


26. $\triangle MDA \cong \triangle MCP$ [$\because$ $\angle DMA = \angle CMP$; $\angle MDA = \angle MCP$;
$AD = CP$ since $AD = BC$ and $CP = BC$]
$\therefore$ $DM = MC$ (c.p.c.t.) and so BM is a median of $\triangle BDC$.
Thus, ar(DMB) = $\frac{1}{2}$ar(BDC).

But, $\text{ar}(BDC) = \frac{1}{2}\text{ar}(ABCD)$ [$\because$ BD is a diagonal of $\|$gm $ABCD$].

$\therefore$ $\text{ar}(DMB) = \frac{1}{4}\text{ar}(ABCD)$ and so $\text{ar}(ABCD) = 28 \text{ cm}^2$.

27. Join BM and AC. We have



$$\text{ar}(\triangle ADC) = \frac{1}{2} \times DC \times h$$

and $\text{ar}(\triangle ABM) = \frac{1}{2} \times AB \times h = \frac{1}{2} \times DC \times h$

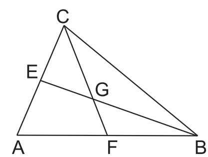
[$\because$ $AB = DC$ in $\|$gm $ABCD$].

$\therefore$ $\text{ar}(\triangle ADC) = \text{ar}(\triangle ABM)$

$\Rightarrow$ $\text{ar}(\triangle ADC) + \text{ar}(\triangle AMC) = \text{ar}(\triangle ABM) + \text{ar}(\triangle AMC)$

$\Rightarrow$ $\text{ar}(\triangle ADM) = \text{ar}(ABMC)$.

29. CF is a median of $\triangle ABC \Rightarrow \text{ar}(\triangle BCF) = \frac{1}{2}\text{ar}(\triangle ABC)$.



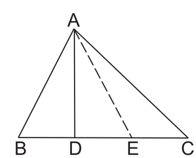
BE is a median of $\triangle ABC \Rightarrow \text{ar}(\triangle ABE) = \frac{1}{2}\text{ar}(\triangle ABC)$.

$\therefore$ $\text{ar}(\triangle BCF) = \text{ar}(\triangle ABE)$

$\Rightarrow$ $\text{ar}(\triangle BCF) - \text{ar}(\triangle BFG) = \text{ar}(\triangle ABE) - \text{ar}(\triangle BFG)$

$\Rightarrow$ $\text{ar}(\triangle BCG) = \text{ar}(AFGE)$.

30. Let E be the midpoint of DC. Then,



$$BD = DE = EC = \frac{1}{3}BC.$$

Join AD and AE.

In $\triangle ABE$, AD is a median and so $\text{ar}(\triangle ABD) = \text{ar}(\triangle ADE)$.

In $\triangle ADC$, AE is a median and so $\text{ar}(\triangle ADE) = \text{ar}(\triangle AEC)$.

$\therefore$ $\text{ar}(\triangle ABD) = \text{ar}(\triangle ADE) = \text{ar}(\triangle AEC) = \frac{1}{3}\text{ar}(\triangle ABC)$.

31. $BD = \frac{1}{2}CA$ and E is the midpoint of $CA \Rightarrow BD = CE$.

Now, $BD \| CE$ and $BD = CE \Rightarrow BCED$ is a $\|$gm.

$\text{ar}(\triangle DBC) = \text{ar}(\triangle BCE) = \frac{1}{2}\text{ar}(\triangle ABC)$ [$\because$ BE is a median of $\triangle ABC$]

$\Rightarrow$ $\text{ar}(\triangle ABC) = 2\text{ar}(\triangle DBC)$.

32. $\text{ar}(\triangle DBC) = \text{ar}(\triangle DBF)$

$\text{ar}(\triangle ADE) = \text{ar}(\triangle ADG)$

$\text{ar}(\triangle DBC) + \text{ar}(\triangle ADE) + \text{ar}(\triangle ABD) = \text{ar}(\triangle DBF) + \text{ar}(\triangle ADG) + \text{ar}(\triangle ABD)$

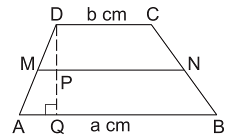
$\therefore$ ar(pentagon $ABCDE$) $= \text{ar}(\triangle DGF)$.

33. $\text{ar}(\triangle CFA) = \text{ar}(\triangle CFB)$ [$\because$ same base CF and same parallels BA and CF]

$\Rightarrow$ $\text{ar}(\triangle CFA) - \text{ar}(\triangle CFG) = \text{ar}(\triangle CFB) - \text{ar}(\triangle CFG)$

$\Rightarrow$ $\text{ar}(\triangle AFG) = \text{ar}(\triangle CBG)$.

34. $\dfrac{\text{ar}(\triangle ABD)}{\text{ar}(\triangle ADC)} = \dfrac{\frac{1}{2} \times BD \times AL}{\frac{1}{2} \times DC \times AL} = \dfrac{BD}{DC} = \dfrac{m}{n}$.



35. M and N are midpoints of AD and BC respectively

$\Rightarrow \quad MN \parallel AB \parallel DC$ and $MN = \frac{1}{2}(AB + DC) = \left(\frac{a+b}{2}\right)$.

Draw $DQ \perp AB$. Let DQ cut MN at P.

Then, P is the midpoint of DQ,

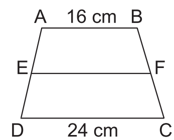
i.e., $DP = PQ = h$ (say).

Now, $\text{ar}(DCNM) = \frac{1}{2} \times (DC + MN) \times DP = \frac{1}{2} \times \left(b + \frac{a+b}{2}\right) \times h = \frac{h}{4} \times (a + 3b)$

and $\text{ar}(MNBA) = \frac{1}{2} \times (MN + AB) \times PQ = \frac{1}{2} \times \left(\frac{a+b}{2} + a\right) \times h = \frac{h}{4} \times (3a + b)$.

$\therefore \quad \text{ar}(DCNM) : \text{ar}(MNBA) = (a + 3b) : (3a + b)$.

36. $\dfrac{\text{ar}(ABFE)}{\text{ar}(EFCD)} = \dfrac{24 + 3 \times 16}{3 \times 24 + 16} = \dfrac{72}{88} = \dfrac{9}{11}$.



37. In $\triangle PAC$, we have $PA \parallel DE$ and E is the midpoint of AC.

$\therefore \quad D$ is the midpoint of PC (by converse of midpoint theorem).

And so, $DE = \frac{1}{2}PA$, i.e., $PA = 2(DE)$.

Similarly, $AQ = 2(DE)$ and so, $PA = AQ$.

Now, $\text{ar}(\triangle ABQ) = \text{ar}(\triangle ACP)$

[$\because$ $\triangle ABQ$ and $\triangle ACP$ have equal bases and lie between the same parallels].

38. In $\triangle RSC$ and $\triangle PQB$, we have:

$\angle CRS = \angle BPQ$ (corres. $\angle$s)

$\angle CSR = \angle BQP$ (corres. $\angle$s)

$SC = QB$ (opp. sides of a $\parallel$gm)

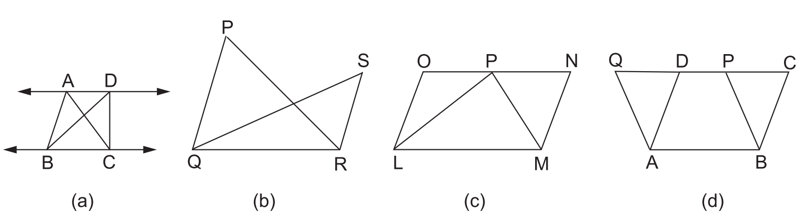
$\therefore \quad \triangle RSC \cong \triangle PQB$ (AAS-criterion).

And so, $\text{ar}(\triangle RSC) = \text{ar}(\triangle PQB)$.

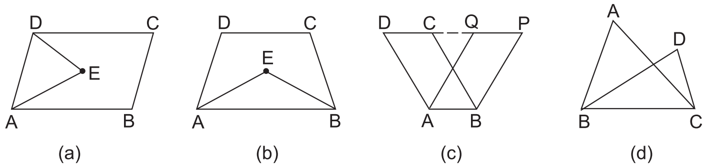
MULTIPLE-CHOICE QUESTIONS (MCQ)

Choose the correct answer in each of the following questions:

1. Out of the following given figures, which are on the same base but not between the same parallels?



2. In which of the following figures, you find polygons on the same base and between the same parallels?

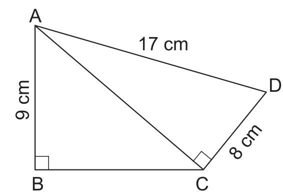


3. The median of a triangle divides it into two

(a) triangles of equal area (b) congruent triangles

(c) isosceles triangles (d) right triangles

4. The area of quadrilateral *ABCD* in the given figure is



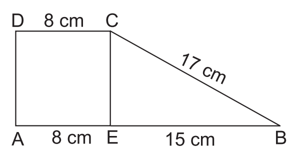
(a) 57 cm^2

(b) 108 cm^2

(c) 114 cm^2

(d) 195 cm^2

5. The area of trapezium *ABCD* in the given figure is



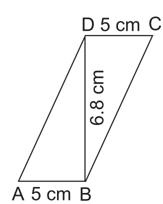
(a) 62 cm^2

(b) 93 cm^2

(c) 124 cm^2

(d) 155 cm^2

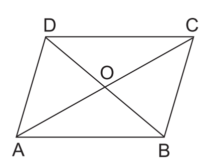
6. In the given figure, *ABCD* is a ||gm in which $AB = CD = 5$ cm and $BD \perp DC$ such that $BD = 6.8$ cm. Then, the area of $ABCD = ?$



(a) 17 cm^2 (b) 25 cm^2

(c) 34 cm^2 (d) 68 cm^2

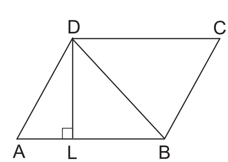
7. In the given figure, *ABCD* is a ||gm in which diagonals *AC* and *BD* intersect at *O*. If ar(||gm *ABCD*) is 52 cm^2 then the ar($\triangle AOB$) = ?



(a) 26 cm^2 (b) 18.5 cm^2

(c) 39 cm^2 (d) 13 cm^2

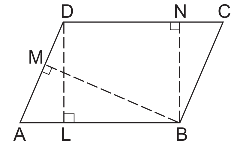
8. In the given figure, *ABCD* is a ||gm in which $DL \perp AB$. If $AB = 10$ cm and $DL = 4$ cm then the ar(||gm *ABCD*) = ?



(a) 40 cm^2 (b) 80 cm^2

(c) 20 cm^2 (d) 196 cm^2

9. The area of $\|$gm $ABCD$ is



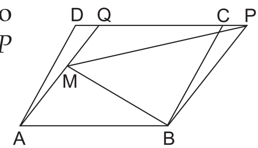
(a) $AB \times BM$ (b) $BC \times BN$

(c) $DC \times DL$ (d) $AD \times DL$

10. Two parallelograms are on equal bases and between the same parallels. The ratio of their areas is

(a) 1 : 2 (b) 1 : 1 (c) 2 : 1 (d) 3 : 1

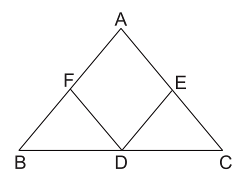
11. In the given figure, $ABCD$ and $ABPQ$ are two parallelograms and M is a point on AQ and BMP is a triangle.



Then, ar($\triangle BMP$) $= \frac{1}{2}$ar($\|$gm $ABCD$) is

(a) true (b) false

12. The midpoints of the sides of a triangle along with any of the vertices as the fourth point makes a parallelogram of area equal to



(a) $\frac{1}{2}$(ar $\triangle ABC$)

(b) $\frac{1}{3}$(ar $\triangle ABC$)

(c) $\frac{1}{4}$(ar $\triangle ABC$)

(d) ar($\triangle ABC$)

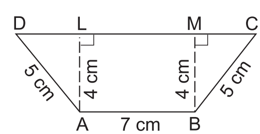
13. The lengths of the diagonals of a rhombus are 12 cm and 16 cm. The area of the rhombus is

(a) 192 cm^2 (b) 96 cm^2 (c) 64 cm^2 (d) 80 cm^2

14. Two parallel sides of a trapezium are 12 cm and 8 cm long and the distance between them is 6.5 cm. The area of the trapezium is

(a) 74 cm^2 (b) 32.5 cm^2 (c) 65 cm^2 (d) 130 cm^2

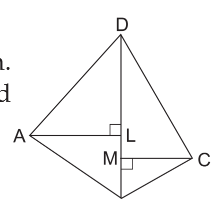
15. In the given figure, $ABCD$ is a trapezium such that $AL \perp DC$ and $BM \perp DC$. If $AB = 7$ cm, $BC = AD = 5$ cm and $AL = BM = 4$ cm then ar(trap. $ABCD$) = ?



(a) 24 cm^2 (b) 40 cm^2

(c) 55 cm^2 (d) 27.5 cm^2

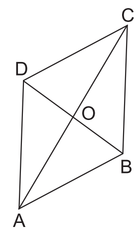
16. In a quadrilateral $ABCD$, it is given that $BD = 16$ cm. If $AL \perp BD$ and $CM \perp BD$ such that $AL = 9$ cm and $CM = 7$ cm then ar(quad. $ABCD$) = ?



(a) 256 cm^2 (b) 128 cm^2

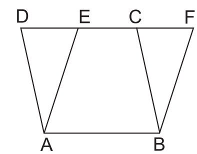
(c) 64 cm^2 (d) 96 cm^2

17. $ABCD$ is a rhombus in which $\angle C = 60°$. Then, $AC : BD = ?$



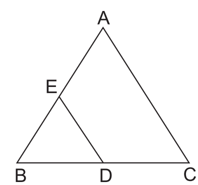
(a) $\sqrt{3} : 1$ (b) $\sqrt{3} : \sqrt{2}$
(c) $3 : 1$ (d) $3 : 2$

18. In the given figure, $ABCD$ and $ABFE$ are parallelograms such that ar(quad. $EABC$) = 17 cm^2 and ar(‖gm $ABCD$) = 25 cm^2. Then, ar($\triangle BCF$) = ?



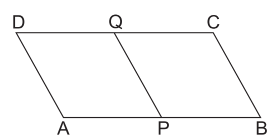
(a) 4 cm^2 (b) 4.8 cm^2
(c) 6 cm^2 (d) 8 cm^2

19. $\triangle ABC$ and $\triangle BDE$ are two equilateral triangles such that D is the midpoint of BC. Then, ar($\triangle BDE$) : ar($\triangle ABC$) = ?



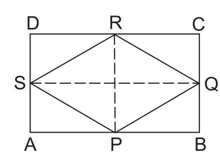
(a) $1 : 2$ (b) $1 : 4$
(c) $\sqrt{3} : 2$ (d) $3 : 4$

20. In a ‖gm $ABCD$, if P and Q are midpoints of AB and CD respectively and ar(‖gm $ABCD$) = 16 cm^2 then ar(‖gm $APQD$) = ?



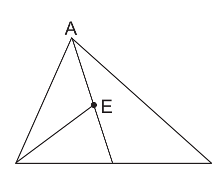
(a) 8 cm^2 (b) 12 cm^2
(c) 6 cm^2 (d) 9 cm^2

21. The figure formed by joining the midpoints of the adjacent sides of a rectangle of sides 8 cm and 6 cm is a



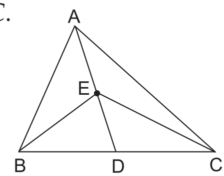
(a) rectangle of area 24 cm^2
(b) square of area 24 cm^2
(c) trapezium of area 24 cm^2
(d) rhombus of area 24 cm^2

22. In $\triangle ABC$, if D is the midpoint of BC and E is the midpoint of AD then ar($\triangle BED$) = ?



(a) $\frac{1}{2}$ar($\triangle ABC$) (b) $\frac{1}{3}$ar($\triangle ABC$)
(c) $\frac{1}{4}$ar($\triangle ABC$) (d) $\frac{2}{3}$ar($\triangle ABC$)

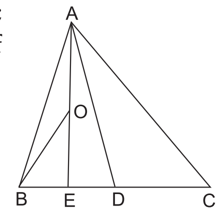
23. The vertex A of $\triangle ABC$ is joined to a point D on BC. If E is the midpoint of AD then $\text{ar}(\triangle BEC) = ?$



(a) $\frac{1}{2}\text{ar}(\triangle ABC)$ (b) $\frac{1}{3}\text{ar}(\triangle ABC)$

(c) $\frac{1}{4}\text{ar}(\triangle ABC)$ (d) $\frac{1}{6}\text{ar}(\triangle ABC)$

24. In $\triangle ABC$, it is given that D is the midpoint of BC; E is the midpoint of BD and O is the midpoint of AE. Then, $\text{ar}(\triangle BOE) = ?$



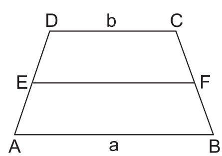
(a) $\frac{1}{3}\text{ar}(\triangle ABC)$ (b) $\frac{1}{4}\text{ar}(\triangle ABC)$

(c) $\frac{1}{6}\text{ar}(\triangle ABC)$ (d) $\frac{1}{8}\text{ar}(\triangle ABC)$

25. If a triangle and a parallelogram are on the same base and between the same parallels then the ratio of the area of the triangle to the area of the parallelogram is

(a) 1 : 2 (b) 1 : 3 (c) 1 : 4 (d) 3 : 4

26. In the given figure, $ABCD$ is a trapezium in which $AB \parallel DC$ such that $AB = a$ cm and $DC = b$ cm. If E and F are the midpoints of AD and BC respectively then $\text{ar}(ABFE) : \text{ar}(EFCD) = ?$



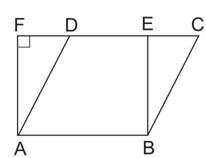
(a) $a : b$ (b) $(a + 3b) : (3a + b)$

(c) $(3a + b) : (a + 3b)$ (d) $(2a + b) : (3a + b)$

27. $ABCD$ is a quadrilateral whose diagonal AC divides it into two parts, equal in area. Then, $ABCD$ is

(a) a rectangle (b) a $\parallel$gm (c) a rhombus (d) all of these

28. In the given figure, a $\parallel$gm $ABCD$ and a rectangle $ABEF$ are of equal area. Then,



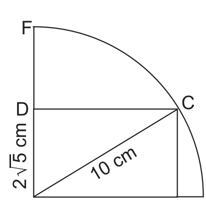
(a) perimeter of $ABCD$ = perimeter of $ABEF$

(b) perimeter of $ABCD$ < perimeter of $ABEF$

(c) perimeter of $ABCD$ > perimeter of $ABEF$

(d) perimeter of $ABCD = \frac{1}{2}$(perimeter of $ABEF$)

29. In the given figure, $ABCD$ is a rectangle inscribed in a quadrant of a circle of radius 10 cm. If $AD = 2\sqrt{5}$ cm then area of the rectangle is



(a) 32 cm^2 (b) 40 cm^2

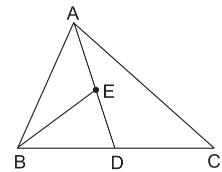
(c) 44 cm^2 (d) 48 cm^2

30. Which of the following is a false statement?

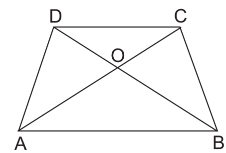
(a) A median of a triangle divides it into two triangles of equal area.

(b) The diagonals of a ||gm divide it into four triangles of equal area.

(c) In a $\triangle ABC$, if E is the midpoint of median AD then $\text{ar}(\triangle BED) = \frac{1}{4}\text{ar}(\triangle ABC)$.



(d) In a trap. $ABCD$, it is given that $AB \parallel DC$ and the diagonals AC and BD intersect at O. Then, $\text{ar}(\triangle AOB) = \text{ar}(\triangle COD)$.



31. Which of the following is a false statement?

(a) If the diagonals of a rhombus are 18 cm and 14 cm then its area is 126 cm^2.

(b) Area of a ||gm $= \frac{1}{2} \times$ base $\times$ corresponding height.

(c) A parallelogram and a rectangle on the same base and between the same parallels are equal in area.

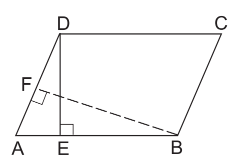
(d) If the area of a ||gm with one side 24 cm and corresponding height h cm is 192 cm^2 then $h = 8$ cm.

Based on Synthesis

32. Look at the statements given below:

I. A parallelogram and a rectangle on the same base and between the same parallels are equal in area.

II. In a ||gm $ABCD$, it is given that $AB = 10$ cm. The altitudes DE on AB and BF on AD being 6 cm and 8 cm respectively, then $AD = 7.5$ cm.



III. Area of a ||gm $= \frac{1}{2} \times$ base $\times$ altitude.

Which is true?

(a) I only (b) II only (c) I and II (d) II and III

Assertion-and-Reason Type

Each question consists of two statements, namely, Assertion (A) and Reason (R). For selecting the correct answer, use the following code:

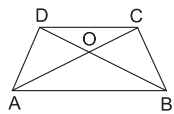
(a) Both Assertion (A) and Reason (R) are true and Reason (R) is a correct explanation of Assertion (A).

(b) Both Assertion (A) and Reason (R) are true but Reason (R) is not a correct explanation of Assertion (A).

(c) Assertion (A) is true and Reason (R) is false.

(d) Assertion (A) is false and Reason (R) is true.

33.

Assertion (A)	**Reason (R)**
In a trapezium $ABCD$, we have $AB \parallel DC$ and the diagonals AC and BD intersect at O. Then, $\text{ar}(\triangle AOD) = \text{ar}(\triangle BOC)$. 	Triangles on the same base and between the same parallels are equal in area.

The correct answer is: (a)/(b)/(c)/(d).

34.

Assertion (A)	**Reason (R)**
If $ABCD$ is a rhombus whose one angle is $60°$ then the ratio of the lengths of its diagonals is $\sqrt{3} : 1$.	Median of a triangle divides it into two triangles of equal area.

The correct answer is: (a)/(b)/(c)/(d).

35.

Assertion (A)	**Reason (R)**
The diagonals of a $\parallel$gm divide it into four triangles of equal area.	A diagonal of a $\parallel$gm divides it into two triangles of equal area.

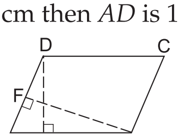
The correct answer is: (a)/(b)/(c)/(d).

36.

Assertion (A)	**Reason (R)**
The area of a trapezium whose parallel sides measure 25 cm and 15 cm respectively and the distance between them is 6 cm, is 120 cm^2.	The area of an equilateral triangle of side 8 cm is $16\sqrt{3}\ \text{cm}^2$.

The correct answer is: (a)/(b)/(c)/(d).

37.

Assertion (A)	**Reason (R)**
In the given figure, $ABCD$ is a $\parallel$gm in which $DE \perp AB$ and $BF \perp AD$. If $AB = 16$ cm, $DE = 8$ cm and $BF = 10$ cm then AD is 12 cm. 	Area of a $\parallel$gm = base $\times$ height.

The correct answer is: (a)/(b)/(c)/(d).

ANSWERS (MCQ)

1. (b)	**2.** (c)	**3.** (a)	**4.** (c)	**5.** (c)	**6.** (c)	**7.** (d)	**8.** (a)
9. (c)	**10.** (b)	**11.** (a)	**12.** (a)	**13.** (b)	**14.** (c)	**15.** (b)	**16.** (b)
17. (a)	**18.** (d)	**19.** (b)	**20.** (a)	**21.** (d)	**22.** (c)	**23.** (a)	**24.** (d)
25. (a)	**26.** (c)	**27.** (d)	**28.** (c)	**29.** (b)	**30.** (d)	**31.** (b)	**32.** (c)
33. (a)	**34.** (b)	**35.** (a)	**36.** (b)	**37.** (d)			

HINTS TO SOME SELECTED QUESTIONS

4. In right $\triangle ACD$, $AC^2 = (AD^2 - CD^2) = (17)^2 - (8)^2 = (289 - 64) = 225$.

$\therefore \quad AC = \sqrt{225} = 15$ cm.

In right $\triangle ABC$, $BC^2 = AC^2 - AB^2 = (15)^2 - (9)^2 = (225 - 81) = 144$.

$\therefore \quad BC = \sqrt{144} = 12$ cm.

ar(quad. $ABCD$) = ar($\triangle ACD$) + ar($\triangle ABC$)

$$= \left[\left(\frac{1}{2} \times 8 \times 15\right) + \left(\frac{1}{2} \times 12 \times 9\right)\right] \text{cm}^2 = (60 + 54) \text{ cm}^2 = 114 \text{ cm}^2.$$

5. ar(trap. $ABCD$) = ar(sq. $AECD$) + ar($\triangle CEB$)

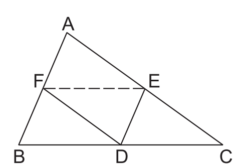
$$= \left[(8 \times 8) + \left(\frac{1}{2} \times 15 \times 8\right)\right] \text{cm}^2 = (64 + 60) \text{ cm}^2 = 124 \text{ cm}^2.$$

7. ar($\triangle OAB$) $= \frac{1}{4}$ar(||gm $ABCD$) $= \left(\frac{1}{4} \times 52\right)$ cm^2 = 13 cm^2.

8. ar(||gm $ABCD$) = (base $\times$ height) $= AB \times DL = (10 \times 4)$ cm^2 = 40 cm^2.

11. ar($\triangle BMP$) $= \frac{1}{2}$ar(||gm $ABPQ$) $= \frac{1}{2}$ar(||gm $ABCD$).

12. $\triangle ABC$ has been divided into 4 triangles of equal area.



$\therefore \quad$ ar(||gm $AFDE$) = ar($\triangle AFE$) + ar($\triangle FED$)

$$= 2\text{ar}(\triangle AFE) = 2 \times \frac{1}{4}\text{ar}(\triangle ABC)$$

$$= \frac{1}{2} \times \text{ar}(\triangle ABC).$$

13. Area of the rhombus $= \left(\frac{1}{2} \times d_1 \times d_2\right) = \left(\frac{1}{2} \times 12 \times 16\right)$ cm^2 = 96 cm^2.

14. Area of the trapezium $= \frac{1}{2} \times$ sum of parallel sides $\times$ distance between them

$$= \left\{\frac{1}{2} \times (12 + 8) \times \frac{13}{2}\right\} \text{cm}^2 = 65 \text{ cm}^2.$$

15. $CM^2 = BC^2 - BM^2 = (5)^2 - (4)^2 = 9 \Rightarrow CM = 3$ cm.

Similarly, $DL = 3$ cm.

$\therefore \quad CD = DL + LM + CM = (3 + 7 + 3)$ cm = 13 cm.

ar(trap. $ABCD$) $= \left[\frac{1}{2} \times (7 + 13) \times 4\right]$ cm^2 = 40 cm^2.

16. ar(quad. $ABCD$) = ar($\triangle ABD$) + ar($\triangle BCD$)

$$= \left(\frac{1}{2} \times BD \times AL\right) + \left(\frac{1}{2} \times BD \times CM\right)$$

$$= \left(\frac{1}{2} \times 16 \times 9\right) + \left(\frac{1}{2} \times 16 \times 7\right)$$

$$= (72 + 56) = 128 \text{ cm}^2.$$

17. $ABCD$ is a rhombus. So, its all sides are equal.

Now, $BC = DC \Rightarrow \angle BDC = \angle DBC = x^\circ$ (say).

Also, $\angle BCD = 60^\circ$ (given).

$\therefore \quad x^\circ + x^\circ + 60^\circ = 180^\circ \Rightarrow 2x = 120 \Rightarrow x = 60.$

$\therefore \quad \angle BDC = \angle DBC = \angle BCD = 60^\circ.$

So, $\triangle BCD$ is an equilateral triangle.

$\therefore \quad BD = BC = a$ (say).

$$AB^2 = OA^2 + OB^2 \Rightarrow OA^2 = AB^2 - OB^2 = a^2 - \left(\frac{a}{2}\right)^2 = a^2 - \frac{a^2}{4} = \frac{3a^2}{4}$$

$$\Rightarrow OA = \frac{\sqrt{3}a}{2} \Rightarrow AC = \left(2 \times \frac{\sqrt{3}a}{2}\right) = \sqrt{3}a.$$

$\therefore \quad AC : BD = \sqrt{3}a : a = \sqrt{3} : 1.$

18. ||gm $ABCD$ and ||gm $ABFE$ being on the same base and between the same parallels, we have:

ar(||gm $ABFE$) = ar(||gm $ABCD$) = 25 cm^2.

ar($\triangle BCF$) = ar(||gm $ABFE$) − ar(quad. $EABC$) = (25 − 17) cm^2 = 8 cm^2.

19. Let $BC = a$. Then, $BD = \frac{a}{2} \cdot$

$$\therefore \quad \frac{\text{ar}(\triangle BDE)}{\text{ar}(\triangle ABC)} = \frac{\frac{\sqrt{3}}{4} \cdot \left(\frac{a}{2}\right)^2}{\frac{\sqrt{3}}{4} \cdot a^2} = \frac{1}{4} \cdot$$

So, the required ratio is 1 : 4.

20. Let the distance between AB and CD be h cm.

Then, ar(||gm $APQD$) = $AP \times h = \frac{1}{2} \times AB \times h = \frac{1}{2}$ar(||gm $ABCD$)

$$= \left(\frac{1}{2} \times 16\right) \text{cm}^2 = 8 \text{ cm}^2.$$

21. Clearly, $PQRS$ is a rhombus whose area is

$$\frac{1}{2} \times PR \times SQ = \left(\frac{1}{2} \times 6 \times 8\right) \text{cm}^2 = 24 \text{ cm}^2.$$

22. Median AD divides $\triangle ABC$ into two triangles of equal area.

$\therefore \quad$ ar($\triangle ABD$) = ar($\triangle ADC$) = $\frac{1}{2}$ar($\triangle ABC$).

In $\triangle ABD$, median BE divides it into two $\triangle$s of equal area.

$\therefore$ $\text{ar}(\triangle BED) = \text{ar}(\triangle ABE) = \frac{1}{2}\text{ar}(\triangle ABD) = \frac{1}{4}\text{ar}(\triangle ABC)$.

23. Median BE in $\triangle ABD$ divides it into two $\triangle$s of equal area.

$\therefore$ $\text{ar}(\triangle BED) = \frac{1}{2}\text{ar}(\triangle ABD)$.

Median CE in $\triangle ACD$ divides it into two $\triangle$s of equal area.

$\therefore$ $\text{ar}(\triangle CED) = \frac{1}{2}\text{ar}(\triangle ACD)$.

$\therefore$ $\text{ar}(\triangle BED) + \text{ar}(\triangle CED) = \frac{1}{2}\{\text{ar}(\triangle ABD) + \text{ar}(\triangle ACD)\} = \frac{1}{2}\text{ar}(\triangle ABC)$

$\Rightarrow$ $\text{ar}(\triangle BEC) = \frac{1}{2}\text{ar}(\triangle ABC)$.

24. Median AD divides $\triangle ABC$ into two $\triangle$s of equal area.

$\therefore$ $\text{ar}(\triangle ABD) = \frac{1}{2}\text{ar}(\triangle ABC)$.

Median AE divides $\triangle ABD$ into two $\triangle$s of equal area.

$\therefore$ $\text{ar}(\triangle ABE) = \frac{1}{2}\text{ar}(\triangle ABD) = \frac{1}{4}\text{ar}(\triangle ABC)$.

Median OB divides $\triangle ABE$ into two $\triangle$s of equal area.

$\therefore$ $\text{ar}(\triangle BOE) = \frac{1}{2}\text{ar}(\triangle ABE) = \frac{1}{8}\text{ar}(\triangle ABC)$.

25. Area of triangle $= \frac{1}{2}$(area of $\parallel$gm)

$\Rightarrow$ (area of triangle) : (area of $\parallel$gm) $= 1 : 2$.

27. Since in all the quadrilaterals mentioned a diagonal divides them into two triangles of equal area, the answer is (d).

29. $AB^2 = AC^2 - BC^2 = (AC^2 - AD^2) = \{(10)^2 - 2(\sqrt{5})^2\} = (100 - 20) = 80$

$\Rightarrow$ $AB = \sqrt{80} = \sqrt{16 \times 5} = 4\sqrt{5}$ cm.

$\therefore$ ar(rect. $ABCD$) $= (AB \times AD) = (4\sqrt{5} \times 2\sqrt{5})\ \text{cm}^2 = 40\ \text{cm}^2$.

32. Clearly I is true.

$AB \times DE = (10 \times 6) = 60$ and $(AD \times BF) = (7.5 \times 8) = 60$.

$\therefore$ II is true.

But, III is clearly false, as we know that

area of a $\parallel$gm = base $\times$ altitude.

33. Clearly, Reason (R) is true.

In trapezium $ABCD$, we find that $\triangle ABC$ and $\triangle ABD$ are on the same base and between the same parallels.

$\therefore$ $\text{ar}(\triangle ABC) = \text{ar}(\triangle ABD)$

$\Rightarrow$ $\text{ar}(\triangle ABC) - \text{ar}(\triangle AOB) = \text{ar}(\triangle ABD) - \text{ar}(\triangle AOB)$

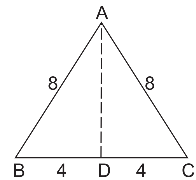
$\Rightarrow$ $\text{ar}(\triangle BOC) = \text{ar}(\triangle AOD)$.

$\therefore$ Assertion (A) is true. And, clearly Reason (R) gives Assertion (A).

Hence, the correct answer is (a).

36. Reason (R): Let $\triangle ABC$ be an equilateral triangle having each side 8 cm.

Let $AD \perp BC$. Then, D is the midpoint of BC.



$AD^2 = AB^2 - BD^2 = (8^2 - 4^2) = (64 - 16) = 48$

$\Rightarrow \quad AD = \sqrt{48} = \sqrt{16 \times 3} = 4\sqrt{3}$ cm.

$\therefore \quad \text{ar}(\triangle ABC) = \left(\frac{1}{2} \times 8 \times 4\sqrt{3}\right) \text{cm}^2 = 16\sqrt{3} \text{ cm}^2.$

Thus, Reason (R) is true.

Assertion (A): Area of the trapezium $= \left\{\frac{1}{2}(25 + 15) \times 6\right\} \text{cm}^2 = 120 \text{ cm}^2.$

$\therefore$ Assertion (A) is true.

But, Reason (R) does not give Assertion (A).

So, the correct answer is (b).

37. Reason (R) is clearly true.

Area of $\|$gm $= AB \times DE = AD \times BF$.

Let $AD = x$ cm. Then,

$$16 \times 8 = x \times 10 \Rightarrow x = \frac{16 \times 8}{10} = 12.8 \text{ cm.}$$

But, AD is 12 cm.

$\therefore$ Assertion (A) is false.

Thus, Assertion (A) is false and Reason (R) is true.

So, the correct answer is (d).

REVIEW OF FACTS AND FORMULAE

1. (i) Area of a $\|$gm = (base $\times$ height).

(ii) Area of a triangle $= \left(\frac{1}{2} \times \text{base} \times \text{height}\right)$.

(iii) Area of a trapezium

$$= \frac{1}{2} \times (\text{sum of parallel sides}) \times (\text{distance between them}).$$

(iv) Area of a rhombus $= \frac{1}{2} \times$ product of diagonals.

2. (i) Parallelograms on the same base and between the same parallels are equal in area.

(ii) A parallelogram and a rectangle on the same base and between the same parallels are equal in area.

3. (i) Triangles on the same base and between the same parallels are equal in area.

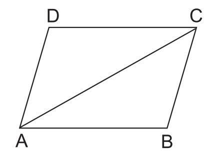
(ii) If a triangle and a $\|$gm are on the same base and between the same parallels then

$$\textit{(area of triangle)} = \frac{\mathbf{1}}{\mathbf{2}} \times \textit{(area of the } \|\textit{gm).}$$

4. (i) A diagonal of a ∥gm divides it into two triangles of equal area.

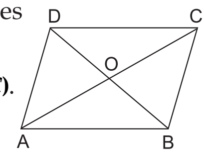
In ∥gm *ABCD*, we have

$$\boldsymbol{ar(\triangle ABC) = ar(\triangle ACD).}$$



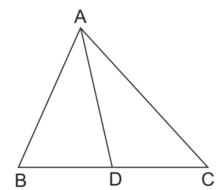
(ii) The diagonals of a ∥gm divide it into four triangles of equal area. Therefore,

$$\boldsymbol{ar(\triangle AOB) = ar(\triangle COD) = ar(\triangle AOD) = ar(\triangle BOC).}$$



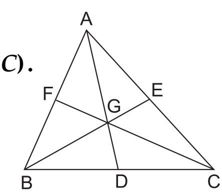
5. (i) A median *AD* of a △*ABC* divides it into two triangles of equal area. Therefore,

$$\boldsymbol{ar(\triangle ABD) = ar(\triangle ACD).}$$



(ii) If the medians of a △*ABC* intersect at *G* then

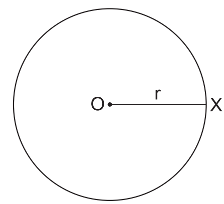
$$\boldsymbol{ar(\triangle AGB) = ar(\triangle AGC) = ar(\triangle BGC) = \frac{1}{3} ar(\triangle ABC).}$$



✿

12 Circles

CIRCLE *A circle is the locus of a point which moves in a plane in such a way that its distance from a given fixed point is always constant.*

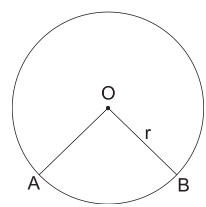


The fixed point is called the *centre* and the constant distance is called the *radius* of the circle.

A circle with centre O and radius r is denoted by $C(O, r)$.

TERMS RELATED TO A CIRCLE

RADIUS *A line segment joining the centre and a point on the circle is called its radius.*



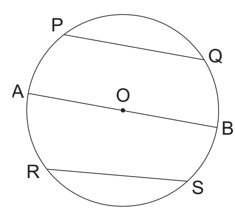
The plural of radius is *radii*.

In the given figure, OA and OB are radii of circle $C(O, r)$.

CIRCUMFERENCE *The perimeter of a circle is called its circumference.*

$$\text{Circumference} = 2\pi r.$$

CHORD *A chord of a circle is a line segment joining any two points on the circle.*



In the given figure, PQ, RS and AOB are the chords of a circle with centre O.

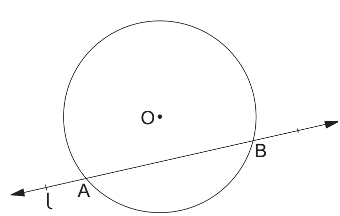
DIAMETER *A diameter is a chord of a circle passing through the centre of the circle.*

Thus, AOB is a diameter of a circle with centre O.

A diameter is the longest chord of a circle.

$$\text{Diameter} = 2 \times \text{radius}.$$

SECANT *A line which intersects a circle in two distinct points is called a secant of the circle.*

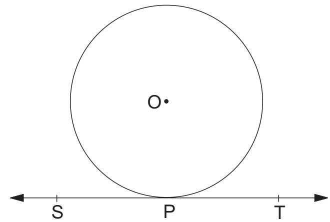


In the given figure, the line l cuts the circle in two points A and B. So, l is a secant of the circle.

TANGENT *A line that intersects the circle in exactly one point is called a tangent to the circle.*

The point at which the tangent meets the circle is called its *point of contact*.

In the given figure, SPT is the tangent to the circle at the point P.



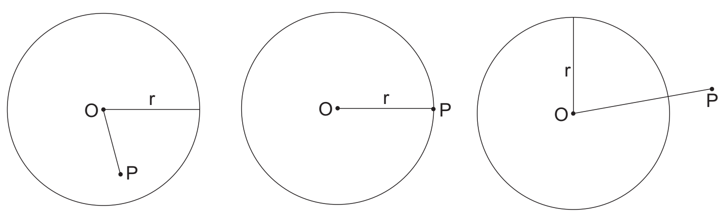
POSITION OF A POINT WITH RESPECT TO A CIRCLE

A point P is said to lie

(i) inside the circle $C(O, r)$ if $OP < r$,

(ii) on the circle $C(O, r)$ if $OP = r$,

(iii) outside the circle $C(O, r)$ if $OP > r$.



(i) (ii) (iii)

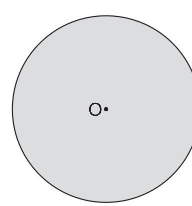
INTERIOR OF A CIRCLE *The region consisting of all points lying on the circumference of a circle and inside it is called the interior of the circle.*

Thus, a point P lies in the interior of a circle $C(O, r)$ if $OP \leq r$.

EXTERIOR OF A CIRCLE *The region consisting of all points lying outside a circle is called the exterior of the circle.*

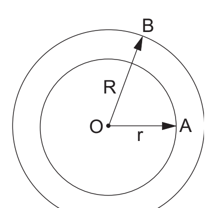
Thus, a point P lies in the exterior of a circle $C(O, r)$ if $OP > r$.

CIRCULAR REGION *The region consisting of all points which are either on the circle or lie inside the circle is called the circular region or circular disc.*

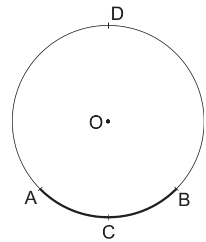


CONCENTRIC CIRCLES *Circles which have the same centre and different radii are called concentric circles.*

In the given figure, $C(O, r)$ and $C(O, R)$ are concentric circles having the same centre O but different radii r and R respectively.



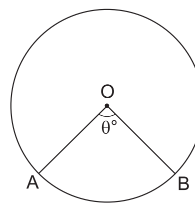
ARC OF A CIRCLE *A continuous piece of a circle is called an arc of the circle.*



Let A, B be two points on a circle $C(O, r)$. Here, the whole circle has been divided into two pieces, namely, arc ACB and arc BDA.

We may denote them by $\overparen{ACB}$ and $\overparen{BDA}$ respectively or simply by $\overparen{BA}$ and $\overparen{BA}$ respectively.

DEGREE MEASURE OF AN ARC



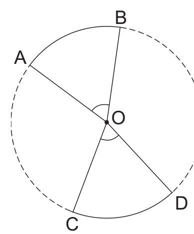
Let $\overparen{AB}$ be an arc of a circle with centre O. If $\angle AOB = \theta°$ then *degree measure of* $\overparen{AB} = \theta°$. And, we write, $m(\overparen{AB}) = \theta°$.

Thus, $m(\overparen{AB}) = \theta° \Rightarrow \angle AOB = \theta°$.

If $m(\overparen{AB}) = \theta°$ then $m(\overparen{BA}) = (360 - \theta)°$.

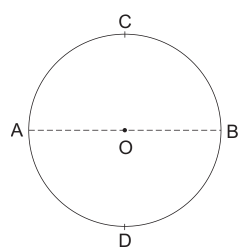
Degree measure of a circle is 360°.

CONGRUENT ARCS *Two arcs $\overparen{AB}$ and $\overparen{CD}$ of a circle are said to be congruent if they have the same degree measures.*



$$\overparen{AB} \cong \overparen{CD} \Rightarrow m(\overparen{AB}) = m(\overparen{CD})$$
$$\Rightarrow \angle ABO = \angle COD.$$

SEMICIRCLE *A diameter of a circle divides it into two equal arcs. Each of these two arcs is called a semicircle.*



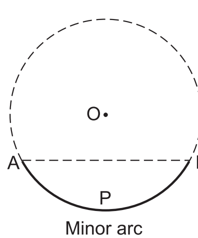
In the given figure, $\overparen{BCA}$ and $\overparen{BDA}$ are two semicircles.

The degree measure of a semicircle is 180°.

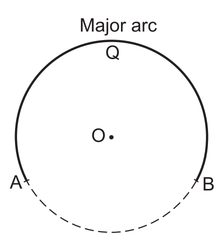
MINOR AND MAJOR ARCS OF A CIRCLE

If the length of an arc is less than the length of the arc of the semicircle then it is called a minor arc. Otherwise, it is a *major arc.*

In Fig. (i), $\overparen{APB}$ is a minor arc, since $l(\overparen{APB})$ is less than the length of the semicircle. In Fig. (ii), $\overparen{AQB}$ is a major arc, its length being more than the length of the semicircle.



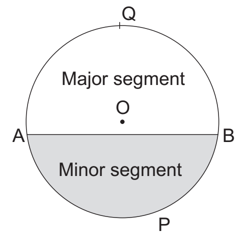
(i)



(ii)

SEGMENT OF A CIRCLE *The part of the circular region bounded by an arc and a chord, including the arc and the chord, is called a segment of the circle.*

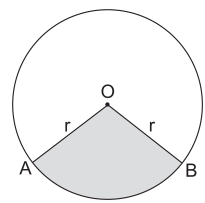
The segment containing the minor arc is called the minor segment. Thus, *APBA* is the *minor segment* of the circle *C*(*O, r*). The segment containing the major arc is called the *major segment*.



ALTERNATE SEGMENTS OF A CIRCLE *The minor and major segments of a circle are called the alternate segments of the circle.*

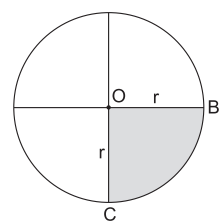
SECTOR OF A CIRCLE *The region enclosed by an arc of a circle and its two bounding radii is called a sector of the circle.*

Thus, in the adjoining figure, *OABO* is the sector of the circle *C*(*O, r*).



QUADRANT OF A CIRCLE *One fourth of a circle is called a quadrant.*

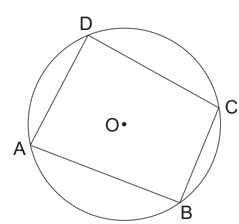
Thus, in the adjoining figure, *OBCO* is a quadrant of the circle *C*(*O, r*).



CONGRUENT CIRCLES *Two circles* *C*(*O, r*) *and* *C*(*O′, s*) *are said to be congruent only when* $r = s$.

CYCLIC QUADRILATERAL *A quadrilateral ABCD is said to be cyclic if all its vertices lie on a circle. Points lying on a circle are said to be concyclic.*

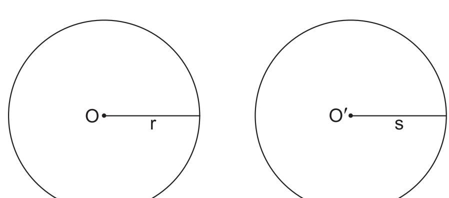
In the given figure, *ABCD* is a *cyclic quadrilateral* and hence the points *A, B, C, D* are concyclic.



CONGRUENCY OF CIRCLES

THEOREM 1 *Two circles are congruent if and only if they have equal radii.*

PROOF Let *C*(*O, r*) and *C*(*O′, s*) be two circles.



Let the circle $C(O', s)$ be placed on the circle $C(O, r)$ so that O' coincides with O.

Then, clearly $C(O', s)$ will cover $C(O, r)$ completely only when $r = s$.

Hence, the two circles will be congruent if and only if they have equal radii.

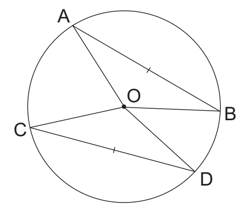
CHORD PROPERTIES OF CIRCLES

THEOREM 2 *Equal chords of a circle subtend equal angles at the centre.*

GIVEN A circle $C(O, r)$ in which chord AB = chord CD.

TO PROVE $\angle AOB = \angle COD$.

PROOF In $\triangle AOB$ and $\triangle COD$, we have



$OA = OC$ [each equal to r]

$OB = OD$ [each equal to r]

$AB = CD$ [given]

$\therefore$ $\triangle AOB \cong \triangle COD$ [by SSS-congruence].

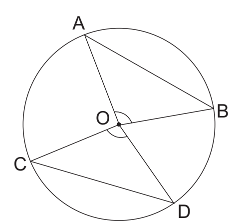
Hence, $\angle AOB = \angle COD$.

THEOREM 3 (Converse of Theorem 2) *If the angles subtended by two chords at the centre of a circle are equal then the chords are equal.*

GIVEN A circle $C(O, r)$ in which AB and CD are the chords such that $\angle AOB = \angle COD$.

TO PROVE $AB = CD$.

PROOF In $\triangle AOB$ and $\triangle COD$, we have



$OA = OC$ [each equal to r]

$OB = OD$ [each equal to r]

$\angle AOB = \angle COD$ [given]

$\therefore$ $\triangle AOB \cong \triangle COD$ [by SAS-congruence].

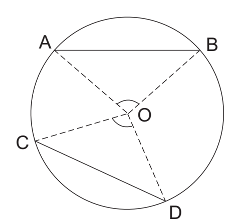
Hence, $AB = CD$ [c.p.c.t.].

THEOREM 4 *If two arcs of a circle are congruent then the corresponding chords are equal.*

GIVEN A circle $C(O, r)$ in which $\overset{\frown}{AB} \cong \overset{\frown}{CD}$.

TO PROVE Chord AB = chord CD.

PROOF **Case I** *When* $\overset{\frown}{AB}$ *and* $\overset{\frown}{CD}$ *are minor arcs*



Join OA, OB, OC and OD.

In $\triangle AOB$ and $\triangle COD$, we have

$OA = OC$ [each equal to r]

$OB = OD$ [each equal to r]

$\angle AOB = \angle COD$

$[\because \overparen{AB} \cong \overparen{CD} \Rightarrow m(\overparen{AB}) = m(\overparen{CD}) \Rightarrow \angle AOB = \angle COD]$

$\therefore \quad \triangle AOB \cong \triangle COD$ [by SAS-congruence].

Hence, $AB = CD$ [c.p.c.t.].

Case II *When $\overparen{AB}$ and $\overparen{CD}$ are major arcs*

In this case, $\overparen{BA}$ and $\overparen{DC}$ are minor arcs.

$\therefore \quad \overparen{AB} \cong \overparen{CD} \Rightarrow \overparen{BA} \cong \overparen{DC} \Rightarrow BA = DC$ [Case I]

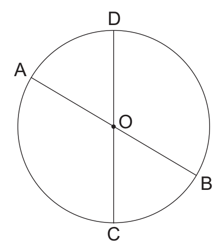
$\Rightarrow AB = CD.$

Hence, in both the cases, we have $AB = CD$.

THEOREM 5 (Converse of Theorem 4) *If two chords of a circle are equal then their corresponding arcs (semicircular, minor or major) are congruent.*

GIVEN A circle $C(O, r)$ in which chord AB = chord CD.

TO PROVE $\overparen{AB} \cong \overparen{CD}$, where both $\overparen{AB}$ and $\overparen{CD}$ are either semicircular, minor or major arcs.



PROOF **Case I** *When AB and CD are diameters*

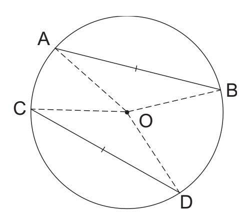
In this case, $\overparen{AB}$ and $\overparen{CD}$ are semicircles with the same radii. So, $\overparen{AB} \cong \overparen{CD}$.

Thus, $AB = CD \Rightarrow \overparen{AB} \cong \overparen{CD}$.

Case II *When chord AB = chord CD, where $\overparen{AB}$ and $\overparen{CD}$ are minor arcs*

Join OA, OB, OC and OD.

In $\triangle AOB$ and $\triangle COD$, we have



$AB = CD$ [given]

$OA = OC$ [each equal to r]

$OB = OD$ [each equal to r]

$\therefore \quad \triangle AOB \cong \triangle COD$.

So, $\angle AOB = \angle COD$.

Now, $\angle AOB = \angle COD \Rightarrow m(\overparen{AB}) = m(\overparen{CD})$

$\Rightarrow \overparen{AB} \cong \overparen{CD}.$

Case III *When chord AB = chord CD, where $\overparen{AB}$ and $\overparen{CD}$ are major arcs*

In this case, $\overparen{BA}$ and $\overparen{DC}$ are minor arcs.

$$\therefore \quad AB = CD \Rightarrow BA = DC$$
$$\Rightarrow \overset{\frown}{BA} \cong \overset{\frown}{DC}$$
$$\Rightarrow m(\overset{\frown}{BA}) = m(\overset{\frown}{DC})$$
$$\Rightarrow 360^\circ - m(\overset{\frown}{AB}) = 360^\circ - m(\overset{\frown}{DC})$$
$$\Rightarrow m(\overset{\frown}{AB}) = m(\overset{\frown}{CD})$$
$$\Rightarrow \overset{\frown}{AB} \cong \overset{\frown}{CD}.$$

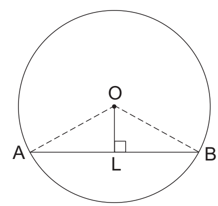
Hence, in all the cases, $AB = CD \Rightarrow \overset{\frown}{AB} \cong \overset{\frown}{CD}$.

THEOREM 6 *The perpendicular from the centre of a circle to a chord bisects the chord.*

GIVEN A chord AB of a circle $C(O, r)$ and $OL \perp AB$.

TO PROVE $LA = LB$.

CONSTRUCTION Join OA and OB.



PROOF In the right $\triangle$s OLA and OLB, we have

$OA = OB$ [each equal to r]

$OL = OL$ [common]

$\therefore$ $\triangle OLA \cong \triangle OLB$ [by RHS-congruence].

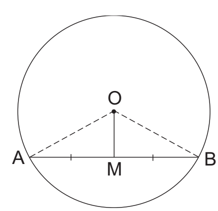
Hence, $LA = LB$.

THEOREM 7 (Converse of Theorem 6) *The line drawn through the centre of a circle to bisect a chord is perpendicular to the chord.*

GIVEN M is the midpoint of the chord AB of a circle $C(O, r)$.

TO PROVE $OM \perp AB$.

CONSTRUCTION Join OA and OB.



PROOF In $\triangle OMA$ and $\triangle OMB$, we have

$OA = OB$ [each equal to r]

$OM = OM$ [common]

$MA = MB$ [given]

$\therefore$ $\triangle OMA \cong \triangle OMB$ [by SSS-congruence].

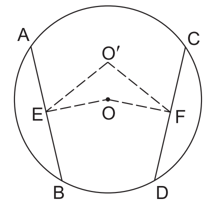
$\therefore$ $\angle OMA = \angle OMB$ [c.p.c.t.].

But, $\angle OMA + \angle OMB = 180^\circ$ [linear-pair axiom].

$\therefore$ $\angle OMA = \angle OMB = 90^\circ$.

Hence, $OM \perp AB$.

COROLLARY *The perpendicular bisectors of two chords of a circle intersect at its centre.*



GIVEN AB and CD are two chords of a circle $C(O, r)$ and let the perpendicular bisectors $O'E$ and $O'F$ of AB and CD respectively meet at O'.

TO PROVE O' coincides with O.

CONSTRUCTION Join OE and OF.

PROOF E is the midpoint of chord AB

$\Rightarrow$ $OE \perp AB$

$\Rightarrow$ OE is the perpendicular bisector of AB

$\Rightarrow$ OE as well as $O'E$ is the perpendicular bisector of AB

$\Rightarrow$ $O'E$ lies along OE.

Similarly, F is the midpoint of chord CD

$\Rightarrow$ $OF \perp CD$

$\Rightarrow$ OF is the perpendicular bisector of CD

$\Rightarrow$ OF as well as $O'F$ is the perpendicular bisector of CD

$\Rightarrow$ $O'F$ lies along OF.

Thus, $O'E$ lies along OE and $O'F$ lies along OF

$\Rightarrow$ the point of intersection of $O'E$ and $O'F$ coincides with the point of intersection of OE and OF

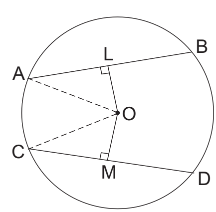
$\Rightarrow$ O' coincides with O.

Hence, the perpendicular bisectors of AB and CD intersect at O.

THEOREM 8 *Equal chords of a circle are equidistant from the centre.*

GIVEN A circle $C(O, r)$ in which chord AB = chord CD, $OL \perp AB$ and $OM \perp CD$.

TO PROVE $OL = OM$.



CONSTRUCTION Join OA and OC.

PROOF We know that the perpendicular from the centre of a circle to a chord bisects the chord.

$\therefore \quad AL = \frac{1}{2}AB$ and $CM = \frac{1}{2}CD$.

$\therefore \quad AB = CD \Rightarrow \frac{1}{2}AB = \frac{1}{2}CD \Rightarrow AL = CM.$ ... (i)

Now, in the right $\triangle$s OLA and OMC, we have

$AL = CM$ [from (i)]

$OA = OC$ [each equal to r]

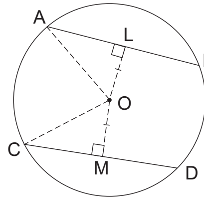
$\therefore \quad \triangle OLA \cong \triangle OMC \quad$ [by RHS-congruence].

So, $\quad OL = OM$.

Hence, AB and CD are equidistant from O.

THEOREM 9 *The chords of a circle which are equidistant from the centre are equal.*

GIVEN AB and CD are two chords of a circle $C(O, r)$; $OL \perp AB$ and $OM \perp CD$ such that $OL = OM$.



TO PROVE $AB = CD$.

CONSTRUCTION Join OA and OC.

PROOF We know that the perpendicular from the centre of a circle to a chord bisects the chord.

$$\therefore \quad AL = \frac{1}{2}AB \quad \text{and} \quad CM = \frac{1}{2}CD.$$

Now, in the right $\triangle$s OLA and OMC, we have

$OA = OC \quad$ [each equal to r]

$OL = OM \quad$ [given]

$\therefore \quad \triangle OLA \cong \triangle OMC \quad$ [by RHS-congruence].

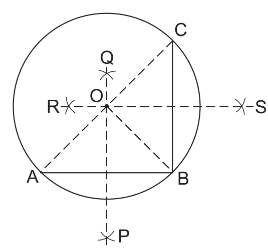
$\therefore \quad AL = CM \Rightarrow 2AL = 2CM$

$$\Rightarrow AB = CD \quad \left[\because AL = \frac{1}{2}AB, CM = \frac{1}{2}CD\right].$$

Hence, $AB = CD$.

THEOREM 10 *There is one and only one circle passing through three given noncollinear points.*

GIVEN Three noncollinear points A, B, C.



TO PROVE There is one and only one circle passing through A, B, C.

CONSTRUCTION Join AB and BC. Draw the perpendicular bisectors PQ and RS of AB and BC respectively. Let PQ and RS intersect at O.

Join OA, OB and OC.

PROOF Since O lies on the perpendicular bisector of AB, we have

$OA = OB$. … (i)

Again, O lies on the perpendicular bisector of BC.

$\therefore \quad OB = OC$. … (ii)

Thus, $OA = OB = OC = r$ (say) $\quad$ [from (i) and (ii)].

With O as centre and radius r draw a circle $C(O, r)$.

Clearly, $C(O, r)$ passes through A, B and C.

We shall show that this is the only circle passing through A, B, C.

If possible, let there be another circle $C(O', s)$, passing through the points A, B, C.

Then, O' will lie on the perpendicular bisectors PQ and RS of AB and BC respectively. Also, PQ and RS intersect at O.

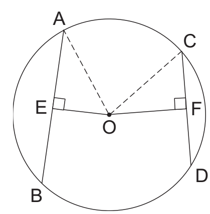
Since two lines cannot intersect at more than one point, so O' must coincide with O.

Since $OA = r$, $O'A = s$ and O coincides with O', we must have $r = s$.

$\therefore \quad C(O, r) \cong C(O', s)$.

Hence, there is one and only one circle, passing through three noncollinear points A, B, C.

THEOREM 11 *Of any two chords of a circle, show that the one which is larger is nearer to the centre.*



GIVEN AB and CD are two chords of a circle $C(O, r)$ such that $AB > CD$. Also, $OE \perp AB$ and $OF \perp CD$.

TO PROVE $OE < OF$.

CONSTRUCTION Join OA and OC.

PROOF We know that the perpendicular from the centre of a circle to a chord bisects the chord.

$$\left.\begin{aligned} \therefore \quad & AE = \frac{1}{2}AB \\ \text{and} \quad & CF = \frac{1}{2}CD. \end{aligned}\right\} \qquad \ldots \text{(i)}$$

Also, $OA = OC = r$. … (ii)

And, $AB > CD \Rightarrow CD < AB \Rightarrow \frac{1}{2}CD < \frac{1}{2}AB$

$\Rightarrow CF < AE$ … (iii) [from (i)].

Now, from the right $\triangle$s OEA and OFC, we have

$$OA^2 = OE^2 + AE^2 \quad \text{and} \quad OC^2 = OF^2 + CF^2$$

$\Rightarrow \quad OE^2 + AE^2 = OF^2 + CF^2$ $\quad [\because OA = OC \Rightarrow OA^2 = OC^2]$

$\Rightarrow \quad OE^2 + AE^2 < OF^2 + AE^2$ [using (iii)]

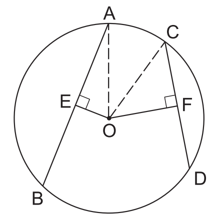
$\Rightarrow \quad OE^2 < OF^2$

$\Rightarrow \quad OE < OF$

$\Rightarrow \quad$ (distance of AB from O) < (distance of CD from O)

$\Rightarrow \quad AB$ is nearer to the centre.

THEOREM 12 *Of any two chords of a circle, show that the one which is nearer to the centre is longer.*



GIVEN Two chords AB and CD of a circle $C(O, r)$, $OE \perp AB$ and $OF \perp CD$ such that $OE < OF$.

TO PROVE $AB > CD$.

CONSTRUCTION Join OA and OC.

PROOF We know that the perpendicular from the centre of a circle to a chord bisects the chord.

$$\left.\begin{aligned} \therefore \quad AE &= \frac{1}{2}AB \\ \text{and} \quad CF &= \frac{1}{2}CD. \end{aligned}\right\} \qquad \ldots \text{(i)}$$

Also, $OA = OC = r$. $\quad \ldots$ (ii)

And, $OE < OF \;\Rightarrow\; OF > OE \;\Rightarrow\; OF^2 > OE^2$. $\quad \ldots$ (iii)

Now, from the right $\triangle$s OEA and OFC, we have

$$OA^2 = OE^2 + AE^2 \quad \text{and} \quad OC^2 = OF^2 + CF^2$$

$\Rightarrow \quad OE^2 + AE^2 = OF^2 + CF^2 \qquad [\because OA = OC = r]$

$\Rightarrow \quad OE^2 + AE^2 = OF^2 + CF^2 > OE^2 + CF^2 \qquad$ [using (iii)]

$\Rightarrow \quad AE^2 > CF^2$

$\Rightarrow \quad \left(\frac{1}{2}AB\right)^2 + \left(\frac{1}{2}CD\right)^2 \quad$ [from (i)]

$\Rightarrow \quad AB^2 > CD^2 \;\Rightarrow\; AB > CD$.

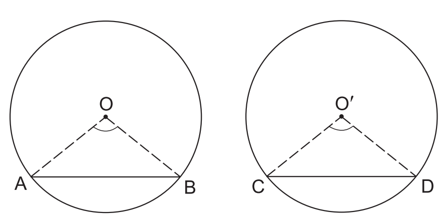
Hence, $AB > CD$.

SOME RESULTS ON CONGRUENT CIRCLES

THEOREM 13 *If two arcs of congruent circles are congruent then the corresponding chords are equal.*

GIVEN Two congruent circles $C(O, r)$ and $C(O', r)$ in which $\overset{\frown}{AB} \cong \overset{\frown}{CD}$.

TO PROVE $AB = CD$.



CONSTRUCTION Draw line segments OA, OB, $O'C$ and $O'D$.

PROOF **Case I** *When $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$ are minor arcs*

In $\triangle AOB$ and $\triangle CO'D$, we have

$OA = O'C = r$

$OB = O'D = r$

$\angle AOB = \angle CO'D$ [$\because \overset{\frown}{AB} \cong \overset{\frown}{CD} \Rightarrow \angle AOB = \angle CO'D$]

$\therefore$ $\triangle AOB \cong \triangle CO'D$ [by SAS-congruence].

Hence, $AB = CD$.

Case II *When $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$ are major arcs*

In this case, $\overset{\frown}{BA}$ and $\overset{\frown}{DC}$ are minor arcs.

$\therefore$ $\overset{\frown}{AB} \cong \overset{\frown}{CD} \Rightarrow \overset{\frown}{BA} \cong \overset{\frown}{DC} \Rightarrow BA = DC$ [from Case I]

$\Rightarrow AB = CD$.

Hence, $AB = CD$.

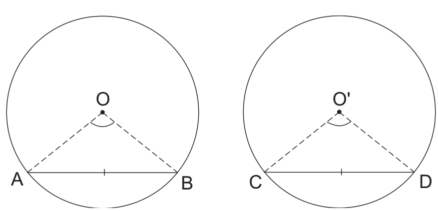
THEOREM 14 *If two chords of congruent circles are equal then the corresponding arcs (minor, major or semicircular) are congruent.*

GIVEN Two congruent circles $C(O, r)$ and $C(O', r)$ in which chord AB = chord CD.

TO PROVE $\overset{\frown}{AB} = \overset{\frown}{CD}$, where both $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$ are either minor arcs, major arcs or semicircles.

CONSTRUCTION If AB and CD are not diameters, draw line segments OA, OB, $O'C$ and $O'D$.

PROOF **Case I** *When $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$ are minor arcs*



In $\triangle AOB$ and $\triangle CO'D$, we have

$OA = O'C$ [each equal to r]

$OB = O'D$ [each equal to r]

$AB = CD$ [given]

$\therefore$ $\triangle AOB \cong \triangle CO'D$ [by SSS-congruence].

$\therefore$ $\angle AOB = \angle CO'D$ [c.p.c.t.]

$\Rightarrow$ $m(\overset{\frown}{AB}) \cong m(\overset{\frown}{CD})$

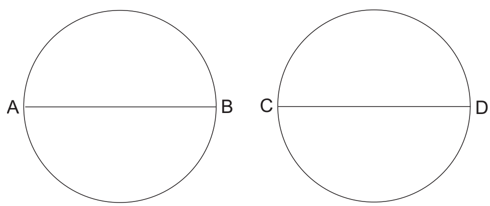
$\Rightarrow$ $\overset{\frown}{AB} \cong \overset{\frown}{CD}$.

Case II *When $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$ are major arcs*

In this case, $\overset{\frown}{BA}$ and $\overset{\frown}{DC}$ will be minor arcs.

$$\therefore \quad AB = CD \Rightarrow BA = DC$$
$$\Rightarrow m(\overset{\frown}{BA}) \cong m(\overset{\frown}{DC})$$
$$\Rightarrow 360^\circ - m(\overset{\frown}{BA}) = 360^\circ - m(\overset{\frown}{DC})$$
$$\Rightarrow m(\overset{\frown}{AB}) \cong m(\overset{\frown}{DC})$$
$$\Rightarrow \overset{\frown}{AB} \cong \overset{\frown}{CD}.$$

Case III *When $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$ are semicircles*



In this case, $\overset{\frown}{AB}$ and $\overset{\frown}{CD}$ are semicircles of equal radii and so they are congruent.

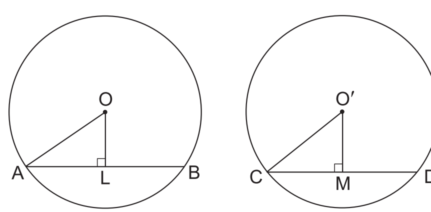
Hence, in all the cases, we have $\overset{\frown}{AB} \cong \overset{\frown}{CD}$.

THEOREM 15 *Equal chords of congruent circles are equidistant from the corresponding centres.*

GIVEN Two congruent circles $C(O, r)$ and $C(O', r)$ in which chord AB = chord CD; $OL \perp AB$ and $O'M \perp CD$.

TO PROVE $OL = O'M$.

CONSTRUCTION Join OA and $O'C$.



PROOF We know that the perpendicular from the centre of a circle to a chord bisects the chord.

$$\therefore \quad AL = \frac{1}{2}AB, \quad \text{and}$$
$$CM = \frac{1}{2}CD.$$

Now, $AB = CD \Rightarrow \frac{1}{2}AB = \frac{1}{2}CD$

$$\Rightarrow AL = CM. \quad \ldots \text{(i)}$$

From the right $\triangle$s OLA and $O'MC$, we have

$OA = O'C$ [each equal to r]

$AL = CM$ [from (i)]

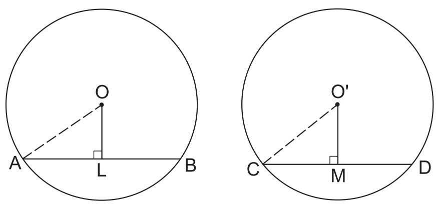
$\therefore$ $\triangle OLA \cong \triangle O'MC$ [by RHS-congruence].

Hence, $OL = O'M$ [c.p.c.t.].

Thus, AB and CD are equidistant from O and O' respectively.

THEOREM 16 *Chords of congruent circles which are equidistant from the corresponding centres are equal.*

GIVEN Two congruent circles $C(O, r)$ and $C(O', r)$, having chords AB and CD respectively; $OL \perp AB$ and $O'M \perp CD$ such that $OL = O'M$.



TO PROVE $AB = CD$.

CONSTRUCTION Join OA and $O'C$.

PROOF From $\triangle OLA$ and $\triangle O'MC$, we have

$OL = O'M$ [given]

$OA = O'C$ [each equal to r]

$\angle OLA \cong \angle O'MC = 90°$

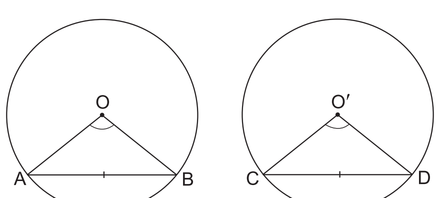
$\therefore$ $\triangle OLA \cong \triangle O'MC$ [by RHS-congruence].

$\therefore$ $AL = CM$ [c.p.c.t.]

$\Rightarrow$ $2AL = 2CM \Rightarrow AB = CD$ $\left[\because AL = \frac{1}{2}AB, CM = \frac{1}{2}CD\right]$.

THEOREM 17 *Equal chords of congruent circles subtend equal angles at the centre.*

GIVEN Two congruent circles $C(O, r)$ and $C(O', r)$ which have chords AB and CD respectively such that $AB = CD$.



TO PROVE $\angle AOB = \angle CO'D$.

PROOF From $\triangle AOB$ and $\triangle CO'D$, we have

$AB = CD$ [given]

$OA = O'C$ [each equal to r]

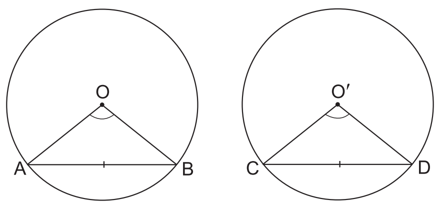
$OB = O'D$ [each equal to r]

$\therefore$ $\triangle AOB \cong \triangle CO'D$. [by SSS-congruence]

$\Rightarrow$ $\angle AOB = \angle CO'D$ [c.p.c.t.].

THEOREM 18 *If the angles subtended by the two chords of congruent circles at the corresponding centres are equal then the chords are equal.*

GIVEN Two congruent circles $C(O, r)$ and $C(O', r)$ which have chords AB and CD respectively such that $\angle AOB = \angle CO'D$.



TO PROVE $AB = CD$.

PROOF In $\triangle AOB$ and $\triangle CO'D$, we have

$OA = O'C$ [each equal to r]

$OB = O'D$ [each equal to r]

$\angle AOB = \angle CO'D$ [given]

$\therefore$ $\triangle AOB \cong \triangle CO'D$ [by SAS-congruence].

Hence, $AB = CD$ [c.p.c.t.].

SOLVED EXAMPLES

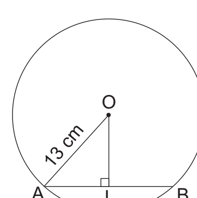
EXAMPLE 1 *The radius of a circle is* 13 cm *and the length of one of its chords is* 10 cm. *Find the distance of the chord from the centre.*

SOLUTION Let AB be a chord of the given circle with centre O and radius 13 cm.

Then, $OA = 13$ cm and $AB = 10$ cm.

From O, draw $OL \perp AB$.

We know that the perpendicular from the centre of a circle to a chord bisects the chord.



$$\therefore \quad AL = \frac{1}{2}AB = \left(\frac{1}{2} \times 10\right) \text{cm} = 5 \text{ cm}.$$

From the right $\triangle OLA$, we have

$$OA^2 = OL^2 + AL^2$$

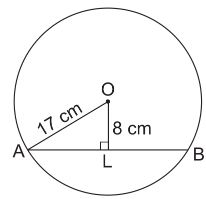
$$\Rightarrow \quad OL^2 = OA^2 - AL^2 = [(13)^2 - (5)^2]\text{ cm}^2 = 144\text{ cm}^2.$$

$$\Rightarrow \quad OL = \sqrt{144}\text{ cm} = 12\text{ cm}.$$

Hence, the distance of the chord from the centre is 12 cm.

EXAMPLE 2 *Find the length of a chord which is at a distance of* 8 cm *from the centre of a circle of radius* 17 cm.

SOLUTION Let AB be a chord of a circle with centre O and radius 17 cm.



Draw $OL \perp AB$. Join OA.

Then, $OL = 8$ cm and $OA = 17$ cm.

From the right $\triangle OLA$, we have

$$OA^2 = OL^2 + AL^2$$

$$\Rightarrow \quad AL^2 = OA^2 - OL^2 = [(17)^2 - (8)^2]\text{ cm}^2$$

$$= (17+8)(17-8)\text{ cm}^2 = 225\text{ cm}^2$$

$$\Rightarrow \quad AL = \sqrt{225}\text{ cm} = 15\text{ cm}.$$

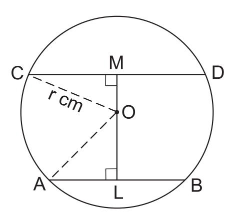
Since the perpendicular from the centre of a circle to a chord bisects the chord, we have

$$AB = 2 \times AL = (2 \times 15)\text{ cm} = 30\text{ cm}.$$

EXAMPLE 3 *AB and CD are two parallel chords of a circle which are on opposite sides of the centre such that* $AB = 10$ cm, $CD = 24$ cm *and the distance between AB and CD is* 17 cm. *Find the radius of the circle.*

SOLUTION Let AB and CD be two chords of a circle $C(O, r)$ such that $AB \parallel CD$. Also, $AB = 10$ cm and $CD = 24$ cm.

Draw $OL \perp AB$ and $OM \perp CD$.



Join OA and OC.

Then, $OA = OC = r$ cm.

Since $OL \perp AB$ and $OM \perp CD$ and $AB \parallel CD$, the points L, O, M are collinear.

$\therefore \quad LM = 17$ cm.

Let $OL = x$ cm. Then, $OM = (17 - x)$ cm.

Since the perpendicular from the centre of a circle to a chord bisects the chord, we have

$$AL = \frac{1}{2}AB = \left(\frac{1}{2} \times 10\right)\text{ cm} = 5\text{ cm, and}$$

$$CM = \frac{1}{2}CD = \left(\frac{1}{2} \times 24\right) \text{cm} = 12 \text{ cm}.$$

From the right $\triangle OLA$, we have

$$OA^2 = OL^2 + AL^2 \Rightarrow r^2 = x^2 + (5)^2. \qquad \ldots \text{(i)}$$

From the right $\triangle OMC$, we have

$$OC^2 = OM^2 + CM^2 \Rightarrow r^2 = (17 - x)^2 + (12)^2. \qquad \ldots \text{(ii)}$$

From (i) and (ii), we get

$$x^2 + (5)^2 = (17 - x)^2 + (12)^2$$

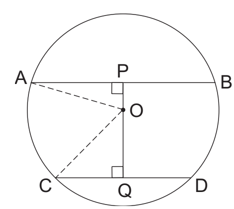
$\Leftrightarrow \quad x^2 + 25 = x^2 - 34x + 433$

$\Leftrightarrow \quad 34x = 408 \Leftrightarrow x = 12.$

Putting $x = 12$ in (i), we get $r^2 = 169 \Rightarrow r = 13.$

Hence, the radius of the circle is 13 cm.

EXAMPLE 4 *In the given figure, AB and CD are two parallel chords of a circle with centre O and radius* 5 cm *such that* $AB = 8$ cm *and* $CD = 6$ cm. *If* $OP \perp AB$ *and* $OQ \perp CD$, *determine the length PQ.*



SOLUTION We know that the perpendicular from the centre of a circle to a chord bisects the chord.

$$\therefore \quad AP = \frac{1}{2}AB = \left(\frac{1}{2} \times 8\right) \text{cm} = 4 \text{ cm},$$

$$CQ = \frac{1}{2}CD = \left(\frac{1}{2} \times 6\right) \text{cm} = 3 \text{ cm}.$$

Join OA and OC.

Then, $OA = OC = 5$ cm.

From the right-angled $\triangle OPA$, we have

$$OP^2 = OA^2 - AP^2 = [(5)^2 - (4)^2] \text{ cm}^2 = 9 \text{ cm}^2$$

$\Rightarrow \quad OP = 3$ cm.

From the right-angled $\triangle OQC$, we have

$$OQ^2 = OC^2 - CQ^2 = [(5)^2 - (3)^2] \text{ cm}^2 = 16 \text{ cm}^2$$

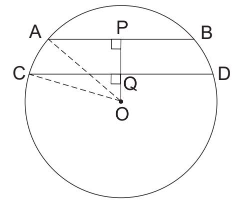
$\Rightarrow \quad OQ = 4$ cm.

Since $OP \perp AB$, $OQ \perp CD$ and $AB \parallel CD$, the points P, O, Q are collinear.

$\therefore \quad PQ = OP + OQ = (3 + 4) \text{ cm} = 7$ cm.

EXAMPLE 5 *In the given figure, AB and CD are two parallel chords of a circle with centre O and radius* 5 cm *such that* $AB = 6$ cm *and* $CD = 8$ cm.

If $OP \perp AB$ *and* $OQ \perp CD$, *determine the length of PQ.*



SOLUTION Since $OP \perp AB$, $OQ \perp CD$ and $AB \parallel CD$, the points O, Q, P are collinear.

We know that the perpendicular from the centre of a circle to a chord bisects the chord.

$$\therefore \quad AP = \frac{1}{2}AB = \left(\frac{1}{2} \times 6\right) \text{cm} = 3 \text{ cm},$$

$$CQ = \frac{1}{2}CD = \left(\frac{1}{2} \times 8\right) \text{cm} = 4 \text{ cm}.$$

Join OA and OC.

Then, $OA = OC = 5$ cm.

From the right $\triangle OPA$, we have

$$OP^2 = OA^2 - AP^2 = [(5)^2 - (3)^2] \text{ cm}^2 = 16 \text{ cm}^2$$

$\Rightarrow \quad OP = 4$ cm.

From the right $\triangle OQC$, we have

$$OQ^2 = OC^2 - CQ^2 = [(5)^2 - (4)^2] \text{ cm}^2 = 9 \text{ cm}^2$$

$\Rightarrow \quad OQ = 3$ cm.

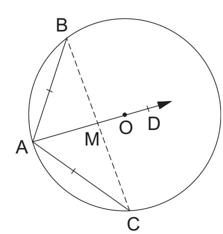
$\therefore \quad PQ = OP - OQ = (4 - 3)$ cm $= 1$ cm.

EXAMPLE 6 *Two chords AB and AC of a circle are equal. Prove that the centre of the circle lies on the angle bisector of* $\angle BAC$.

SOLUTION GIVEN AB and AC are two equal chords of a circle $C(O, r)$ and AD is the bisector of $\angle BAC$.

TO PROVE O lies on AD.

CONSTRUCTION Join BC, meeting AD at M.



PROOF In $\triangle BAM$ and $\triangle CAM$, we have

$AB = AC$ [given]

$\angle BAM = \angle CAM$ [given]

$AM = AM$ [common]

$\therefore \quad \triangle BAM \cong \triangle CAM$ [by SAS-congruence]

$\Rightarrow \quad BM = CM$ and $\angle BMA = \angle CMA$ [c.p.c.t.]

$\Rightarrow \quad BM = CM$ and $\angle BMA = \angle CMA = 90^\circ$

[$\because \angle BMA + \angle CMA = 180^\circ$ and $\angle BMA = \angle CMA$
$\Rightarrow \angle BMA = \angle CMA = 90^\circ$]

$\Rightarrow \quad AM$ is the perpendicular bisector of chord BC

$\Rightarrow$ *AD* is the perpendicular bisector of the chord *BC*.

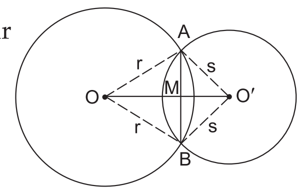
But, the perpendicular bisector of a chord always passes through the centre of the circle.

$\therefore$ *AD* passes through the centre *O* of the circle

$\Rightarrow$ *O* lies on *AD*.

EXAMPLE 7 *If two circles intersect in two points, prove that the line through their centres is the perpendicular bisector of the common chord.*

SOLUTION GIVEN Two circles $C(O, r)$ and $C(O', s)$ intersecting at points *A* and *B*.



TO PROVE OO' is the perpendicular bisector of *AB*.

CONSTRUCTION Draw line segments *OA*, *OB*, $O'A$ and $O'B$.

Let OO' and *AB* intersect at *M*.

PROOF In $\triangle OAO'$ and $\triangle OBO'$, we have

$OA = OB$ [each equal to *r*]

$O'A = O'B$ [each equal to *s*]

$OO' = OO'$ [common]

$\therefore$ $\triangle OAO' \cong \triangle OBO'$ [by SSS-congruence]

$\Rightarrow$ $\angle AOO' = \angle BOO'$

$\Rightarrow$ $\angle AOM = \angle BOM$ ... (i)

[$\because$ $\angle AOO' = \angle AOM$ and $\angle BOO' = \angle BOM$].

In $\triangle AOM$ and $\triangle BOM$, we have

$OA = OB$ [each equal to *r*]

$\angle AOM = \angle BOM$ [from (i)]

$OM = OM$ [common]

$\therefore$ $\triangle AOM \cong \triangle BOM$ [by SAS-congruence]

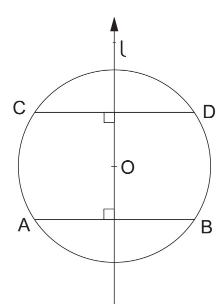
$\Rightarrow$ $AM = BM$ and $\angle AMO = \angle BMO$

$\Rightarrow$ $AM = BM$ and $\angle AMO = \angle BMO = 90^\circ$

[$\because$ $\angle AMO + \angle BMO = 180^\circ$ and $\angle AMO = \angle BMO$]

$\Rightarrow$ OO' is the perpendicular bisector of *AB*.

EXAMPLE 8 *Two chords AB and CD of a circle are parallel and a line l is the perpendicular bisector of AB. Show that l bisects CD.*



SOLUTION We know that the perpendicular bisector of any chord of a circle always passes through its centre.

Since *l* is the perpendicular bisector of *AB*, *l* passes through the centre *O* of the circle.

Now, $l \perp AB$ and $AB \parallel CD \Rightarrow l \perp CD$.

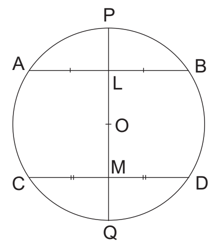
Thus, $l \perp CD$ and passes through the centre of the circle.

But, the perpendicular from the centre of a circle to a chord bisects the chord.

$\therefore$ l must bisect the chord CD.

EXAMPLE 9 *If a diameter of a circle bisects each of the two chords of the circle, prove that the chords are parallel.*

SOLUTION Let AB and CD be two chords of a circle $C(O, r)$.



Let PQ be a diameter, bisecting chords AB and CD at L and M respectively.

Since PQ is a diameter of the circle, it passes through the centre O.

Now, L is the midpoint of AB.

We know that the line joining the centre of a circle to the midpoint of a chord is perpendicular to the chord.

$\therefore$ $OL \perp AB$

$\Rightarrow$ $\angle ALO = 90°$.

Similarly, $\angle DMO = 90°$.

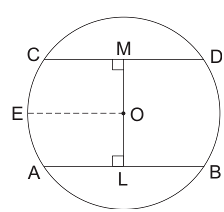
$\therefore$ $\angle ALO = \angle DMO$.

But, these are alternate angles.

$\therefore$ $AB \parallel CD$.

EXAMPLE 10 *Prove that the line joining the midpoints of two parallel chords of a circle passes through the centre of the circle.*

SOLUTION GIVEN L and M are the midpoints of two parallel chords AB and CD respectively of a circle $C(O, r)$.



TO PROVE LOM is a straight line.

CONSTRUCTION Join OL, OM and draw $OE \parallel AB \parallel CD$.

PROOF We know that the line joining the centre of a circle to the midpoint of a chord is perpendicular to the chord.

$\therefore$ $OL \perp AB$ and $OM \perp CD$.

Now, $OL \perp AB$ and $AB \parallel OE \Rightarrow OL \perp OE$

$\Rightarrow \angle EOL = 90°$.

$OM \perp CD$ and $CD \parallel OE \Rightarrow OM \perp OE$

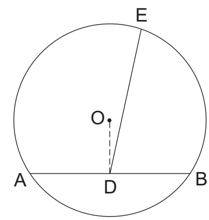
$\Rightarrow \angle EOM = 90°$.

$\therefore \quad \angle EOL + \angle EOM = 180°.$

Hence, LOM is a straight line.

EXAMPLE 11 *Prove that the perpendicular bisector of the chord of a circle always passes through its centre.*

SOLUTION Let DE be the perpendicular bisector of the given chord AB of a circle $C(O, r)$.



Then, $AD = DB$ and $\angle ADE = 90°$. ... (i)

Now, we have to show that DE passes through O.

If possible, suppose DE does not pass through O. Join OD.

We know that the line joining the centre of a circle to the midpoint of a chord is always perpendicular to the chord.

$\therefore \quad OD \perp AB \Rightarrow \angle ADO = 90°.$ ... (ii)

From (i) and (ii), we get $\angle ADE = \angle ADO$.

This is a contradiction, since $\angle ADO$ is a part of $\angle ADE$.

The contradiction arises by assuming that DE does not pass through O.

Hence, DE must pass through O.

EXAMPLE 12 *Prove that the right bisector of a chord of a circle bisects the corresponding minor arc of the circle.*

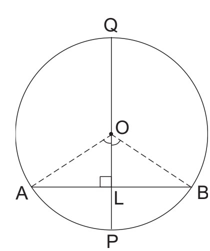
SOLUTION Let AB be a chord of a circle $C(O, r)$.

Let PQ be the right bisector of a chord AB, intersecting it at L and the circle at P and Q.

Since the right bisector of a chord always passes through the centre of the circle, PQ must pass through O.

Join OA and OB.

In $\triangle OLA$ and $\triangle OLB$, we have



$OA = OB$ [each equal to r]

$\angle ALO = \angle BLO$ [each equal to 90°]

$OL = OL$ [common]

$\therefore \quad \triangle OLA \cong \triangle OLB$

$\Rightarrow \quad \angle AOL = \angle BOL$

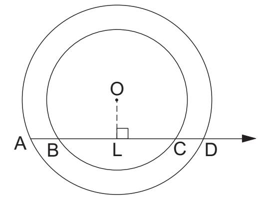
$\Rightarrow \quad \angle AOP = \angle BOP$

$\Rightarrow \quad m(\overset{\frown}{AP}) = m(\overset{\frown}{BP})$

$\Rightarrow \quad \overset{\frown}{AP} \cong \overset{\frown}{BP}.$

NOTE The right bisector of a chord also bisects the corresponding major arc.

EXAMPLE 13 *The given figure shows two concentric circles whose common centre is O. l is a line intersecting these circles at the points A, B, C and D. Show that AB = CD.*



SOLUTION Draw $OL \perp l$.

We know that the perpendicular from the centre of a circle to a chord bisects the chord.

Now, BC is a chord of the smaller circle and $OL \perp BC$.

$\therefore \quad BL = LC.$ … (i)

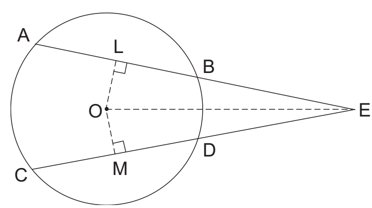
Again, AD is a chord of the bigger circle and $OL \perp AD$.

$\therefore \quad AL = LD.$ … (ii)

Subtracting (i) from (ii), we get

$$(AL - BL) = (LD - LC) \Rightarrow AB = CD.$$

EXAMPLE 14 *In the given figure, two equal chords AB and CD of a circle C(O, r) when produced meet at a point E.*
Prove that (i) BE = DE
(ii) AE = CE.



SOLUTION Join OE.

Draw $OL \perp AB$ and $OM \perp CD$.

We know that equal chords of a circle are equidistant from the centre.

$\therefore \quad OL = OM.$

In the right ▵ OLE and OME, we have

$OL = OM$

$OE = OE$ [common]

$\therefore \quad \triangle OLE \cong \triangle OME$ [by RHS-congruence]

$\Rightarrow \quad LE = ME$ … (i) [c.p.c.t.].

Also, $AB = CD \Rightarrow \frac{1}{2}AB = \frac{1}{2}CD$

$\Rightarrow BL = DM.$ … (ii)

On subtracting (ii) from (i), we get

$LE - BL = ME - DM \Rightarrow BE = DE.$ [Part (i) proved]

Now, $AB = CD$ and $BE = DE$

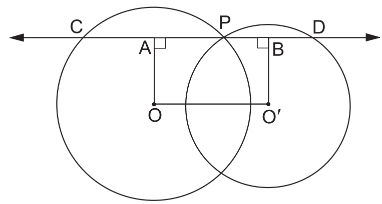
$\Rightarrow \quad AB + BE = CD + DE$

$\Rightarrow \quad AE = CE.$

Hence, $BE = DE$ and $AE = CE$.

EXAMPLE 15 *Two circles whose centres are O and O′ intersect at P. Through P, a line l parallel to OO′, intersecting the circles at C and D, is drawn. Prove that* $CD = 2OO'$.

SOLUTION Draw $OA \perp l$ and $O'B \perp l$.



$OA \perp l$

$\Rightarrow \quad OA \perp CP$

$\Rightarrow \quad CA = AP$

$\Rightarrow \quad CP = 2AP. \quad \ldots \text{(i)}$

Again, $O'B \perp l$

$\Rightarrow \quad O'B \perp PD$

$\Rightarrow \quad PB = BD$

$\Rightarrow \quad PD = 2PB. \quad \ldots \text{(ii)}$

$\therefore \quad CD = CP + PD = 2(AP + PB) = 2AB = 2OQ'$

[from (i) and (ii)].

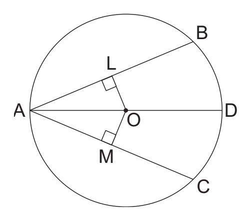
EXAMPLE 16 *If two chords of a circle are equally inclined to the diameter through their point of intersection, prove that the chords are equal.*

SOLUTION GIVEN Two chords AB and AC of a circle $C(O, r)$ and AOD is a diameter such that $\angle OAB = \angle OAC$.

TO PROVE $AB = AC$.

CONSTRUCTION Draw $OL \perp AB$ and $OM \perp AC$.

PROOF In $\triangle OLA$ and $\triangle OMA$, we have



$\angle OLA = \angle OMA$ [each equal to 90°]

$OA = OA$ [common]

$\angle OAL = \angle OAM$ [given].

$\therefore \quad \triangle OLA \cong \triangle OMA$

$\Rightarrow \quad OL = OM$

$\Rightarrow$ chords AB and AC are equidistant from O

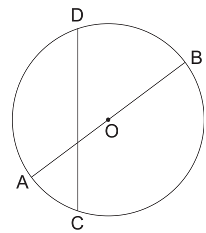
$\Rightarrow \quad AB = AC.$

EXAMPLE 17 *Prove that a diameter is the longest chord in a circle.*

SOLUTION GIVEN A circle $C(O, r)$ in which AB is a diameter and CD is any other chord.

TO PROVE $AB > CD$.

PROOF Clearly, the diameter AB is nearer to the centre than any other chord CD.



But, of any two chords of a circle, the one which is nearer to the centre is longer.

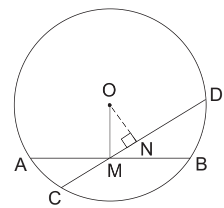
$\therefore$ $AB > CD$.

Thus, AB is longer than every other chord.

Hence, a diameter is the longest chord in a circle.

EXAMPLE 18 *Prove that of all chords of a circle through a given point within it, the shortest is the one which is bisected at that point.*

SOLUTION GIVEN A circle $C(O, r)$ and a point M within it; AB is a chord with midpoint M and CD is another chord through M.



TO PROVE $AB < CD$.

CONSTRUCTION Join OM and draw $ON \perp CD$.

PROOF In the right $\triangle ONM$, OM is the hypotenuse.

$\therefore$ $ON < OM$

$\Rightarrow$ chord CD is nearer to O than chord AB.

We know that, of any two chords of a circle, the one which is nearer to the centre is longer.

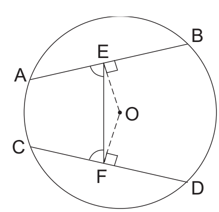
$\therefore$ $CD > AB$.

Hence, $AB < CD$.

Thus, of all chords through M, the shortest is the one which is bisected at M.

EXAMPLE 19 *Prove that the line joining the midpoints of two equal chords of a circle subtends equal angles with the chords.*

SOLUTION GIVEN Two equal chords AB and CD of a circle $C(O, r)$ which have E and F as their midpoints respectively.



TO PROVE $\angle AEF = \angle CFE$, and $\angle BEF = \angle DFE$.

CONSTRUCTION Join OE and OF.

PROOF We know that the line joining the centre of a circle to the midpoint of a chord is perpendicular to the chord.

$\therefore$ $OE \perp AB$ and $OF \perp CD$.

Now, since AB and CD are equal chords, they must be equidistant from the centre.

$\therefore$ $OE = OF$.

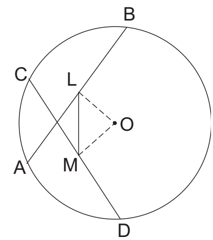
Now, in $\triangle OEF$, we have

$$OE = OF \Rightarrow \angle OEF = \angle OFE$$
$$\Rightarrow 90° - \angle OEF = 90° - \angle OFE$$
$$\text{and} \quad 90° + \angle OEF = 90° + \angle OFE$$
$$\Rightarrow \angle AEF = \angle CFE \quad \text{and} \quad \angle BEF = \angle DFE.$$

EXAMPLE 20 *In the given figure, L and M are the midpoints of two equal chords AB and CD of a circle C(O, r).*

Prove that (i) $\angle OLM = \angle OML$

(ii) $\angle ALM = \angle CML$.



SOLUTION PROOF Since L is the midpoint of chord AB, we have $OL \perp AB$.

Again, M is the midpoint of chord CD.

$\therefore \quad OM \perp CD$.

But, equal chords of a circle are equidistant from the centre.

$\therefore \quad AB = CD \Rightarrow OL = OM$

$\Rightarrow \angle OLM = \angle OML.$ … (i)

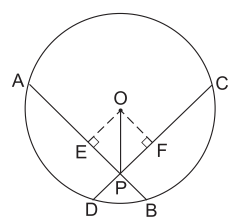
Now, $\angle ALM = \angle OLA - \angle OLM = 90° - \angle OLM$

$\angle CML = \angle OMC - \angle OML = 90° - \angle OML.$

But, $\angle OLM = \angle OML$ [from (i)].

$\therefore \quad \angle ALM = \angle CML.$

EXAMPLE 21 *In the given figure, O is the centre of a circle C(O, r) and PO bisects $\angle APC$. Prove that $AB = CD$.*



SOLUTION Draw $OE \perp AB$ and $OF \perp CD$.

Now, in $\triangle OEP$ and $\triangle OFP$, we have

$\angle OEP = \angle OFP$ [each equal to 90°]

$\angle OPE = \angle OPF$ [given]

$OP = OP$ [common]

$\therefore \quad \triangle OEP \cong \triangle OFP$ [AAS-congruence]

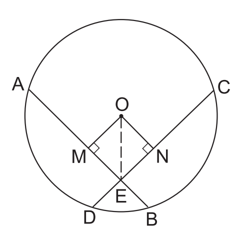
$\Rightarrow \quad OE = OF$ [c.p.c.t.]

$\Rightarrow$ chords AB and CD are equidistant from the centre O

$\Rightarrow \quad AB = CD$ [chords equidistant from the centre are equal].

Hence, $AB = CD$.

EXAMPLE 22 *In the given figure, equal chords AB and CD of a circle C(O, r) cut at right angles at E. If M and N are the midpoints of AB and CD respectively, prove that OMEN is a square.*



SOLUTION GIVEN A circle $C(O, r)$ in which chord AB = chord CD, and M, N are the midpoints of AB and CD respectively.

TO PROVE $OMEN$ is a square.

CONSTRUCTION Join OE.

PROOF Since the line joining the centre of a circle to the midpoint of a chord is perpendicular to the chord, we have

$OM \perp AB$ and $ON \perp CD$

$\Rightarrow \angle OMB = 90°$ and $\angle OND = 90°$

$\Rightarrow \angle OME = 90°$ and $\angle ONE = 90°$.

Also, equal chords of a circle are equidistant from the centre.

$\therefore OM = ON$. ... (i)

Now, in $\triangle OME$ and $\triangle ONE$, we have

$OM = ON$ [from (i)]

$\angle OME = \angle ONE$ [each equal to 90°]

$OE = OE$ [common]

$\therefore \triangle OME \cong \triangle ONE$ [by SAS-congruence]

$\Rightarrow ME = NE$ [c.p.c.t.].

Thus, in quad. $OMEN$, we have

$OM = ON$, $ME = NE$ and $\angle OME = \triangle ONE = 90°$.

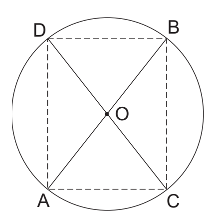
Hence, $OMEN$ is a square.

EXAMPLE 23 *Show that if two chords of a circle bisect each other, they must be diameters of the circle.*

SOLUTION GIVEN AB and CD are two chords of a circle, intersecting at O such that $OA = OB$ and $OC = OD$.

TO PROVE AB and CD are diameters of the circle.

CONSTRUCTION Join AC, AD, and BC, BD.



PROOF In $\triangle AOC$ and $\triangle BOD$, we have

$OA = OB$ [given]

$OC = OD$ [given]

$\angle AOC = \angle BOD$ [vert. opp. $\angle$s]

$\therefore \triangle AOC \cong \triangle BOD$

$\Rightarrow AC = BD$

$\Rightarrow \overparen{AC} = \overparen{BD}$. ... (i)

In $\triangle AOD$ and $\triangle BOC$, we have

$OA = OB$ [given]

$OD = OC$ [given]

$\angle AOD = \angle BOC$ [vert. opp. $\angle$s]

$\therefore \quad \triangle AOD \cong \triangle BOC$

$\Rightarrow \quad AD = BC$

$\Rightarrow \quad \overset{\frown}{AD} = \overset{\frown}{BC}$. … (ii)

From (i) and (ii), we get

$\overset{\frown}{AC} + \overset{\frown}{AD} = \overset{\frown}{BD} + \overset{\frown}{BC}$

$\Rightarrow \quad \overset{\frown}{CAD} = \overset{\frown}{CBD}$

$\Rightarrow$ CD divides the circle into two semicircles

$\Rightarrow$ CD is a diameter.

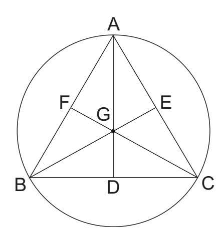
Similarly, AB is a diameter.

EXAMPLE 24 *In an equilateral triangle, prove that the centroid and the circumcentre coincide.*

SOLUTION The point of intersection of the medians of a triangle is called its centroid. The centroid of a triangle is the point located at $\frac{2}{3}$ of the distance from a vertex along a median. The centre of the circumcircle of this triangle is called the circumcentre.

Let $\triangle ABC$ be the given equilateral triangle and let its medians AD, BE and CF intersect at G.

Then G is the centroid of $\triangle ABC$.



In $\triangle BCE$ and $\triangle CBF$ we have

$BC = CB$ [common]

$\angle B = \angle C$ [each equal to 60°]

$CE = BF$ $\left[AC = AB \Rightarrow \frac{1}{2}AC = \frac{1}{2}AB\right]$

$\therefore \quad \triangle BCE = \triangle CBF$ [by SAS-congruence]

$\Rightarrow \quad BE = CF$.

Similarly, $AD = BE$.

Thus, $AD = BE = CF \Rightarrow \frac{2}{3}AD = \frac{2}{3}BE = \frac{2}{3}CF$

$\Rightarrow GA = GB = GC$.

This shows that G is the circumcentre of $\triangle ABC$.

Hence, G is the centroid as well as circumcentre of $\triangle ABC$.

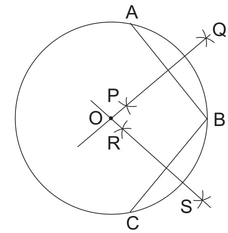
EXAMPLE 25 *Give a method to find the centre of a given circle.*

SOLUTION Take any three distinct points A, B, C on the given circle.

Join AB and BC.

Draw the perpendicular bisectors PQ and RS of AB and BC respectively to meet at a point O.

Then, O is the required centre of the circle.



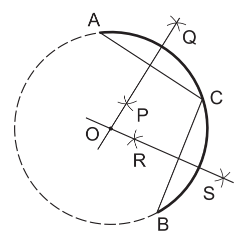
EXAMPLE 26 *Show how to complete a circle if an arc of the circle is given.*

SOLUTION Let $\overparen{AB}$ be the given arc.

Take a point C on $\overparen{AB}$.

Join AC and CB.

Draw the perpendicular bisectors PQ and RS of AC and CB respectively, meeting each other at a point O.



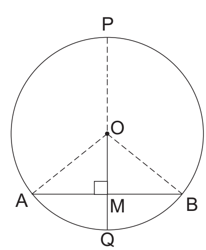
With O as centre and OA as radius, draw the required circle.

EXAMPLE 27 *Prove that the line joining the midpoint of a chord to the centre of a circle passes through the midpoint of the corresponding minor arc.*

SOLUTION GIVEN M is the midpoint of a chord AB of a circle $C(O, r)$, and OM is produced to meet the circle at Q.

TO PROVE $\overparen{AQ} \cong \overparen{BQ}$.

CONSTRUCTION Join OA and OB.



PROOF In $\triangle OMA$ and $\triangle OMB$, we have

$OA = OB$ [each equal to r]

$OM = OM$ [common]

$AM = BM$ [$\because$ M is the midpoint of AB]

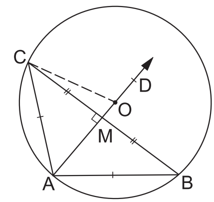
$\therefore$ $\triangle OMA \cong \triangle OMB$ [by SSS-congruence]

$\Rightarrow$ $\angle AOM = \angle BOM$ [c.p.c.t.]

$\Rightarrow$ $m(\overparen{AQ}) = m(\overparen{BQ}) \Rightarrow \overparen{AQ} \cong \overparen{BQ}$.

NOTE MO when produced to P also passes through the midpoint of the corresponding major arc.

EXAMPLE 28 *In a circle of radius* 5 cm, *AB and AC are two equal chords such that* $AB = AC = 6$ cm. *Find the length of the chord BC.*



SOLUTION Let AD be the bisector of $\angle BAC$.

Then, AD is the perpendicular bisector of BC and passes through the centre O. Join CO.

Let AD meet BC in M. Then, $BM = CM$.

From the right $\triangle AMC$, we have

$$MC^2 = AC^2 - AM^2 = 36 \text{ cm}^2 - AM^2. \qquad \ldots \text{(i)}$$

From the right $\triangle OMC$, we have

$$MC^2 = OC^2 - OM^2 = 25 \text{ cm}^2 - (OA - AM)^2$$

$$= 25 \text{ cm}^2 - (5 \text{ cm} - AM)^2. \qquad \ldots \text{(ii)}$$

$\therefore$ $36 \text{ cm}^2 - AM^2 = 25 \text{ cm}^2 - (5 \text{ cm} - AM)^2$ [from (i) and (ii)]

$\Rightarrow$ $36 \text{ cm}^2 - AM^2 = 25 \text{ cm}^2 - (25 \text{ cm}^2 + AM^2 - 10 \text{ cm} \times AM)$

$\Rightarrow$ $36 \text{ cm}^2 - AM^2 = 25 \text{ cm}^2 - 25 \text{ cm}^2 - AM^2 + 10 \text{ cm} \times AM$

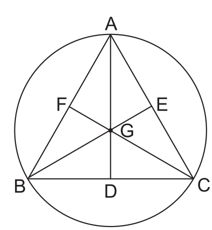
$\Rightarrow$ $36 \text{ cm}^2 = 10 \text{ cm} \times AM$

$\Rightarrow$ $AM = 3.6$ cm.

$\therefore$ $MC = \sqrt{36 - (3.6)^2}$ cm $= \sqrt{23.04}$ cm $= 4.8$ cm.

$\therefore$ $BC = 2 \times MC = 9.6$ cm.

EXAMPLE 29 *A circular park of radius* 20 m *is situated in a colony. Three boys Ankur, Brijesh and Chirag are sitting at equal distances on its boundary, each having a toy telephone in his hands to talk to each other. Find the length of the string of each phone.*



SOLUTION Let A, B and C be the positions of Ankur, Brijesh and Chirag respectively.

Now, $\overset{\frown}{AB} = \overset{\frown}{BC} = \overset{\frown}{CA}$

$\Rightarrow$ chord AB = chord BC = chord CA

$\Rightarrow$ $\triangle ABC$ is an equilateral triangle.

Let AD, BE and CF be the medians of $\triangle ABC$ and let G be its centroid.

Since, in an equilateral triangle, the centroid coincides with its circumcentre, we have

$$GA = GB = GC = 20 \text{ m}.$$

Since the centroid of a triangle divides a median in the ratio 2 : 1, we have

$$\frac{GA}{GD} = \frac{2}{1} \Rightarrow \frac{20 \text{ m}}{GD} = \frac{2}{1} \Rightarrow GD = 10 \text{ m}.$$

In right $\triangle BDG$, we have

$$BG^2 = BD^2 + GD^2$$

$\Rightarrow$ $(20\text{ m})^2 = BD^2 + (10\text{ m})^2$

$\Rightarrow$ $BD^2 = (400 - 100)\text{ m}^2 = 300\text{ m}^2$

$\Rightarrow$ $BD = \sqrt{300}\text{ m} = 10\sqrt{3}\text{ m}.$

$\therefore$ $BC = 2 \times BD = (2 \times 10\sqrt{3})\text{ m} = 20\sqrt{3}\text{ m}.$

Hence, the length of each telephone string is $20\sqrt{3}$ m.

EXERCISE 12A

1. A chord of length 16 cm is drawn in a circle of radius 10 cm. Find the distance of the chord from the centre of the circle.

2. Find the length of a chord which is at a distance of 3 cm from the centre of a circle of radius 5 cm.

3. A chord of length 30 cm is drawn at a distance of 8 cm from the centre of a circle. Find out the radius of the circle.

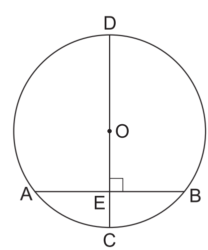
4. In a circle of radius 5 cm, AB and CD are two parallel chords of lengths 8 cm and 6 cm respectively. Calculate the distance between the chords if they are

(i) on the same side of the centre,

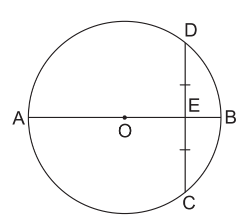
(ii) on the opposite sides of the centre.

5. Two parallel chords of lengths 30 cm and 16 cm are drawn on the opposite sides of the centre of a circle of radius 17 cm. Find the distance between the chords.

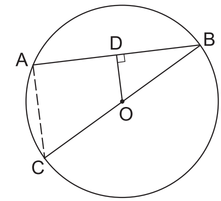
6. In the given figure, the diameter CD of a circle with centre O is perpendicular to chord AB. If $AB = 12$ cm and $CE = 3$ cm, calculate the radius of the circle.



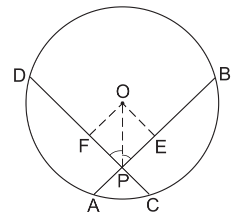
7. In the given figure, a circle with centre O is given in which a diameter AB bisects the chord CD at a point E such that $CE = ED = 8$ cm and $EB = 4$ cm. Find the radius of the circle.



8. In the adjoining figure, OD is perpendicular to the chord AB of a circle with centre O. If BC is a diameter, show that $AC \parallel DO$ and $AC = 2 \times OD$.



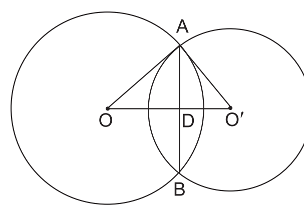
9. In the given figure, O is the centre of a circle in which chords AB and CD intersect at P such that PO bisects $\angle BPD$. Prove that $AB = CD$.



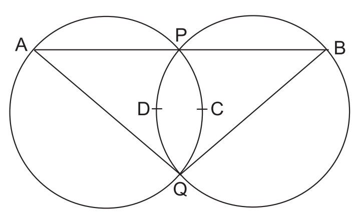
10. Prove that the diameter of a circle perpendicular to one of the two parallel chords of a circle is perpendicular to the other and bisects it.

11. Prove that two different circles cannot intersect each other at more than two points.

12. Two circles of radii 10 cm and 8 cm intersect each other, and the length of the common chord is 12 cm. Find the distance between their centres.

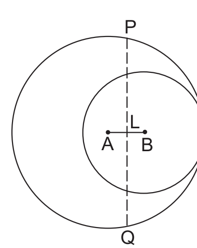


13. Two equal circles intersect in P and Q. A straight line through P meets the circles in A and B. Prove that $QA = QB$.

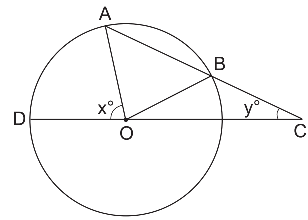


14. If a diameter of a circle bisects each of the two chords of a circle then prove that the chords are parallel.

15. In the adjoining figure, two circles with centres at A and B, and of radii 5 cm and 3 cm touch each other internally. If the perpendicular bisector of AB meets the bigger circle in P and Q, find the length of PQ.

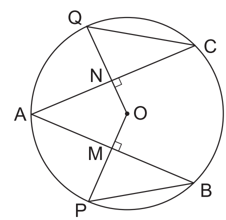


16. In the given figure, AB is a chord of a circle with centre O and AB is produced to C such that $BC = OB$. Also, CO is joined and produced to meet the circle in D. If $\angle ACD = y°$ and $\angle AOD = x°$, prove that $x = 3y$.

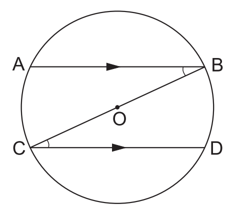


17. AB and AC are two chords of a circle of radius r such that $AB = 2AC$. If p and q are the distances of AB and AC from the centre then prove that $4q^2 = p^2 + 3r^2$.

18. In the adjoining figure, O is the centre of a circle. If AB and AC are chords of the circle such that $AB = AC$, $OP \perp AB$ and $OQ \perp AC$, prove that $PB = QC$.

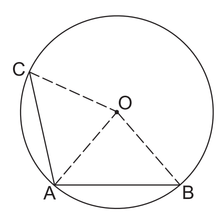


19. In the adjoining figure, BC is a diameter of a circle with centre O. If AB and CD are two chords such that $AB \parallel CD$, prove that $AB = CD$.

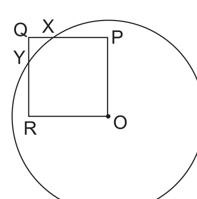


20. An equilateral triangle of side 9 cm is inscribed in a circle. Find the radius of the circle.

21. In the adjoining figure, AB and AC are two equal chords of a circle with centre O. Show that O lies on the bisector of $\angle BAC$.



22. In the adjoining figure, $OPQR$ is a square. A circle drawn with centre O cuts the square in X and Y. Prove that $QX = QY$.



23. Two circles with centres O and O' intersect at two points A and B. A line PQ is drawn parallel to OO' through A or B, intersecting the circles at P and Q. Prove that $PQ = 2OO'$.

ANSWERS (EXERCISE 12A)

1. 6 cm	**2.** 8 cm	**3.** 17 cm	**4.** (i) 1 cm (ii) 7 cm
5. 23 cm	**6.** 7.5 cm	**7.** 10 cm	**12.** $(8+2\sqrt{7})$ cm
15. $4\sqrt{6}$ cm	**20.** $3\sqrt{3}$ cm		

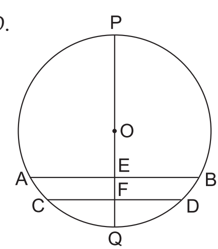
HINTS TO SOME SELECTED QUESTIONS

6. Let $OA = OC = r$ cm. Then, $OE = (r-3)$ cm and $AE = \frac{1}{2}AB = 6$ cm.
   Use $OA^2 = OE^2 + AE^2$.

8. $OD \perp AB \Rightarrow D$ is the midpoint of AB.
   Also, O is the midpoint of BC. Join AC.
   Now, in $\triangle ABC$, D is the midpoint of AB and O is the midpoint of BC.
   $\therefore \quad DO \parallel AC$ and $DO = \frac{1}{2}AC$.

9. Draw $OE \perp AB$ and $OF \perp CD$.
   Then, $\angle OEP = \angle OFP = 90°$, $OP = OP$, $\angle OPE = \angle OPF$.
   $\therefore \quad \triangle OEP \cong \triangle OFP$, and hence $OE = OF$.
   And, chords which are equidistant from the centre are equal.

10. Let AB and CD be two parallel chords of a circle with centre O.
    Let POQ be a diameter such that $\angle PEB = 90°$.
    Then, $\angle PFD = \angle PEB = 90°$ [corres. $\angle$s].
    Thus, $PF \perp CD$, and so $OF \perp CD \Rightarrow CF = FD$.



11. If possible, let two different circles intersect at three distinct points A, B, C. Then, these points are clearly noncollinear. So, a unique circle can be drawn to pass through these points. This is a contradiction.

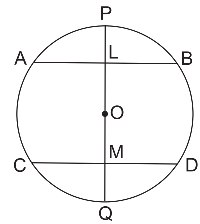
12. $OA = 10$ cm, $AD = \frac{1}{2}AB = 6$ cm.
    $OD^2 = OA^2 - AD^2 \Rightarrow OD = 8$ cm.
    $O'A = 8$ cm, $O'D^2 = O'A^2 - AD^2 \Rightarrow O'D = \sqrt{28}$ cm $= 2\sqrt{7}$ cm.
    $\therefore \quad OO' = (OD + O'D)$.

13. Join PQ. Now, PQ is a common chord of congruent circles. So, arc PCQ = arc PDQ.
    $\therefore \quad \angle QAP = \angle QBP \Rightarrow QA = QB$.

14. Let AB and CD be two chords of a circle with centre O. Suppose a diameter POQ bisects them at L and M. Then,

$OL \perp AB$ and $OM \perp CD$.

Thus, $\angle ALM = \angle LMD \Rightarrow AB \parallel CD$.



15. Join AP. Let PQ intersect AB at L.

Then, $AB = (5-3)$ cm $= 2$ cm.

$PQ \perp AB$ and $AL = \frac{1}{2}AB = 1$ cm.

$\therefore \quad PL = \sqrt{AP^2 - AL^2}$ cm $= \sqrt{(25-1)} = 2\sqrt{6}$ cm $\Rightarrow PQ = (2 \times PL) = 2 \times 2\sqrt{6} = 4\sqrt{6}$ cm.

16. $OB = BC \Rightarrow \angle BOC = \angle BCO = y°$.

Ext. $\angle OBA = \angle BOC + \angle BCO = (2y)°$.

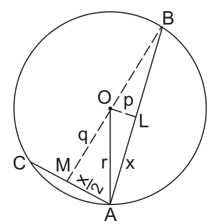
$OA = OB \Rightarrow \angle OAB = \angle OBA = (2y)°$.

Ext. $\angle AOD = \angle OAC + \angle ACO = \angle OAB + \angle BCO = (3y)° \Rightarrow x = 3y$.

17. Let $AC = x$. Then, $AB = 2x$.

Let $OL \perp AB$ and $OM \perp AC$.

Then, $OL = p$, $OM = q$ and $OA = r$.



Also, $AM = \frac{1}{2}AC = \frac{x}{2}$ and $AL = \frac{1}{2}AB = x$.

Clearly, $r^2 = p^2 + x^2$ ... (i)

and $r^2 = q^2 + \left(\frac{x}{2}\right)^2$. ... (ii)

$\therefore \quad q^2 = r^2 - \frac{x^2}{4} \Rightarrow 4q^2 = 4r^2 - x^2$

$\Rightarrow 4q^2 = 4r^2 - (r^2 - p^2)$ [using (i)]

$\Rightarrow 4q^2 = 3r^2 + p^2$.

Hence, $4q^2 = 3r^2 + p^2$.

18. $AB = AC \Rightarrow \frac{1}{2}AB = \frac{1}{2}AC \Rightarrow MB = NC$.

Also, $\angle PMB = \angle QNC = 90°$.

Equal chords are equidistant from the centre $\Rightarrow OM = ON$.

Also, $OP = OQ =$ radius $\Rightarrow OP - OM = OQ - ON \Rightarrow PM = QN$.

$\therefore \quad \triangle MPB \cong \triangle NQC \Rightarrow PB = QC$.

19. Draw $OL \perp AB$ and $OM \perp CD$.

$\triangle OLB \cong \triangle OMC$ [$\because \angle OBL = \angle OCL$, $OB = OC = r$ and $\angle OLB = \angle OMC = 90°$].

$\therefore \quad OL = OM$.

But, the chords equidistant from the centre are equal.

Hence, $AB = CD$.

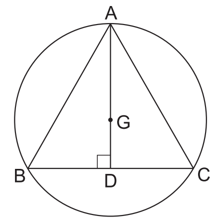
20. Let $\triangle ABC$ be an equilateral triangle of side 9 cm. Let AD be one of its medians.

Then, $AD \perp BC$ and $BD = 4.5$ cm.

$\therefore \quad AD = \sqrt{AB^2 - BD^2} = \sqrt{(9)^2 - \left(\frac{9}{2}\right)^2} \text{ cm} = \frac{9\sqrt{3}}{2} \text{ cm.}$

In an equilateral triangle, the centroid and circumcentre coincide and $AG : GD = 2 : 1$.

$\therefore \quad \text{radius} = AG = \frac{2}{3}AD = \left(\frac{2}{3} \times \frac{9\sqrt{3}}{2}\right) \text{cm} = 3\sqrt{3} \text{ cm.}$



21. Join OA, OB and OC.

$\triangle OAB \cong \triangle OAC \quad [\because OA = OA, OB = OC = r, AB = AC]$

$\Rightarrow \quad \angle OAB = \angle OAC.$

22. $\triangle OXP \cong \triangle OYR. \quad [\because OP = OR, OX = OY = r \text{ and } \angle OPX = \angle ORY = 90°]$

$\Rightarrow \quad PX = RY \Rightarrow PQ - PX = QR - RY \qquad [\because PQ = QR]$

$\Rightarrow \quad QX = QY.$

23. Draw $OM \perp PQ$ and $O'N \perp PQ$. Then,

$OM \perp \text{chord } PA \Rightarrow PM = MA$

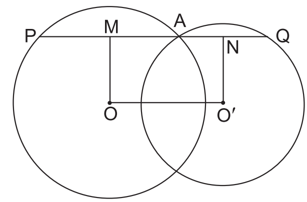
$\Rightarrow PA = 2PM = 2MA.$

Similarly, $AQ = 2AN$.

$\therefore \quad PQ = PA + AQ = 2MA + 2AN$

$= 2(MA + AN) = 2MN$

$= 2OO'.$



RESULTS ON ANGLES SUBTENDED BY ARCS

THEOREM 1 *The angle subtended by an arc of a circle at the centre is double the angle subtended by it at any point on the remaining part of the circle.*

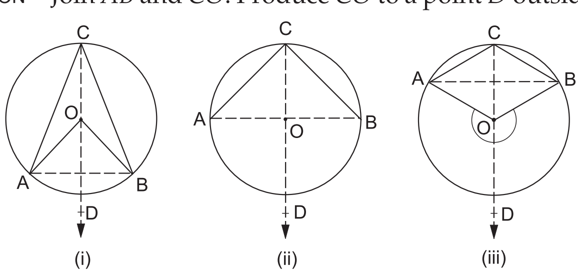
[CBSE 2002]

GIVEN A circle $C(O, r)$ in which arc AB subtends $\angle AOB$ at the centre and $\angle ACB$ at any point C on the remaining part of the circle.

TO PROVE $\angle AOB = 2\angle ACB$, when $\widehat{AB}$ is a minor arc or a semicircle.

Reflex $\angle AOB = 2\angle ACB$, when $\widehat{AB}$ is a major arc.

CONSTRUCTION Join AB and CO. Produce CO to a point D outside the circle.



PROOF Clearly, there are three cases.

Case I $\overset{\frown}{AB}$ is a minor arc [Fig. (i)].

Case II $\overset{\frown}{AB}$ is a semicircle [Fig. (ii)].

Case III $\overset{\frown}{AB}$ is a major arc [Fig. (iii)].

We know that when one side of a triangle is produced then the exterior angle so formed is equal to the sum of the interior opposite angles.

$\therefore \quad \angle AOD = \angle OAC + \angle OCA$

and $\quad \angle BOD = \angle OBC + \angle OCB.$

But, $\quad \angle OAC = \angle OCA \quad [\because \; OC = OA = r]$

and $\quad \angle OBC = \angle OCB \quad [\because \; OC = OB = r].$

$\therefore \quad \angle AOD = 2\angle OCA$ and $\angle BOD = 2\angle OCB.$

In Fig. (i) and Fig. (ii),

$$\angle AOD + \angle BOD = 2\angle OCA + 2\angle OCB$$

$\Rightarrow \quad \angle AOB = 2(\angle OCA + \angle OCB)$

$\Rightarrow \quad \angle AOB = 2\angle ACB.$

In Fig. (iii),

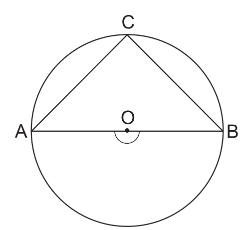
$$\angle AOD + \angle BOD = 2\angle OCA + 2\angle OCB$$

$\Rightarrow \quad$ reflex $\angle AOB = 2(\angle OCA + \angle OCB)$

$\Rightarrow \quad$ reflex $\angle AOB = 2\angle ACB.$

THEOREM 2 *The angle in a semicircle is a right angle.*

GIVEN AB is a diameter of a circle $C(O, r)$ and $\angle ACB$ is an angle in a semicircle.



TO PROVE $\angle ACB = 90°.$

PROOF We know that the angle subtended by an arc at the centre of a circle is twice the angle formed by it at any point on the remaining part of the circle.

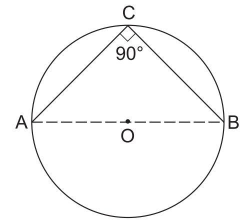
$\therefore \quad \angle AOB = 2\angle ACB$

[angle subtended by $\overset{\frown}{AB}$ at $O = 2 \times$ angle formed by it at C]

$\Rightarrow \quad 2\angle ACB = \angle AOB = 180° \qquad [\because \; \angle AOB$ is a straight angle]

$\Rightarrow \quad ACB = 90°.$

THEOREM 3 (Converse of Theorem 2) *The arc of a circle subtending a right angle at any point of the circle in its alternate segment is a semicircle.*



GIVEN An arc AB of a circle with centre O, which subtends $\angle ACB$ at a point C of $\overparen{BA}$, other than A and B such that $\angle ACB = 90°$.

TO PROVE $\overparen{AB}$ is a semicircle.

CONSTRUCTION Join OA and OB.

PROOF Since $\overparen{AB}$ subtends $\angle AOB$ at the centre and $\angle ACB$ at the remaining part of the circle, we have

$$\angle AOB = 2\angle ACB = (2 \times 90°) = 180° \quad [\because \ \angle ACB = 90° \text{ (given)}].$$

This shows that AO and OB are in the same straight line.

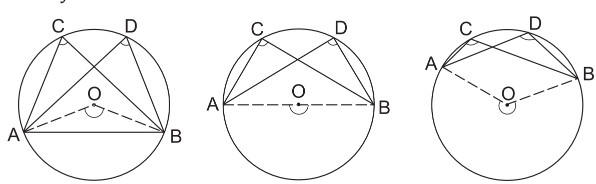
Hence, $\overparen{AB}$ is a semicircle.

THEOREM 4 *Angles in the same segment of a circle are equal.*

GIVEN A circle $C(O, r)$ in which $\angle ACB$ and $\angle ADB$ are two angles made by $\overparen{AB}$ in the same segment $ACDB$ of the circle.

TO PROVE $\angle ACB = \angle ADB$.

CONSTRUCTION Join OA and OB.



(i) (ii) (iii)

PROOF We know that

angle made by an arc at the centre of a circle

= 2 × (angle made by the arc at any point on its remaining part). ... (i)

Case I *When $\overparen{AB}$ is a minor arc [Fig. (i)]*

Using (i), we have

$$\angle AOB = 2\angle ACB \quad \text{and} \quad \angle AOB = 2\angle ADB$$

$\Rightarrow$ $2\angle ACB = 2\angle ADB$ [each equal to $\angle AOB$]

$\Rightarrow$ $\angle ACB = \angle ADB$.

Case II *When $\overparen{AB}$ is a semicircle [Fig. (ii)]*

We know that the angle in a semicircle is 1 right angle.

$\therefore \quad \angle ACB = 90^\circ$ and $\angle ADB = 90^\circ$.

$\therefore \quad \angle ACB = \angle ADB$.

Case III *When $\widehat{AB}$ is a major arc [Fig. (iii)]*

Using (i), we have

reflex $\angle AOB = 2\angle ACB$ and reflex $\angle AOB = 2\angle ADB$

$\Rightarrow \quad 2\angle ACB = 2\angle ADB$ [each equal to reflex $\angle AOB$]

$\Rightarrow \quad \angle ACB = \angle ADB$.

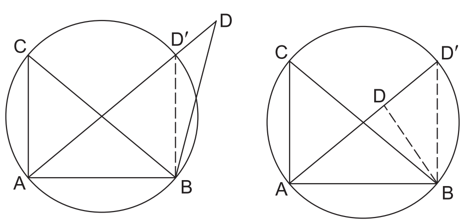
Hence, in all the cases we have $\angle ACB = \angle ADB$.

THEOREM 5 *If a line segment joining two points subtends equal angles at two other points lying on the same side of the line segment then the four points are concyclic, i.e., lie on the same circle.*

GIVEN AB is a line segment and C, D are two points lying on the same side of AB such that $\angle ACB = \angle ADB$.

TO PROVE A, B, C, D lie on the same circle.

CONSTRUCTION Draw the circle through three noncollinear points A, B and C.



PROOF If D lies on the circle passing through A, B and C then clearly the result follows.

If possible, suppose D does not lie on this circle.

Then, this circle will intersect AD or AD produced in D'.

Join $D'B$.

Now, $\angle ACB = \angle ADB$ [given]

and $\angle ACB = \angle AD'B$ [angles in the same segment].

$\therefore \quad \angle ADB = \angle AD'B$.

But, an exterior angle of a triangle can never be equal to its interior opposite angle.

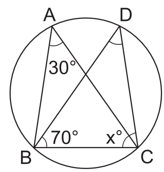
So, $\angle ADB = \angle AD'B$ is true only when D' coincides with D.

Thus, D lies on the circle passing through A, B and C.

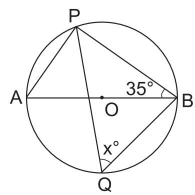
Hence, the points A, B, C, D are concyclic.

SOLVED EXAMPLES

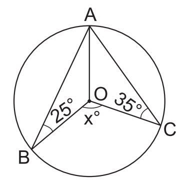
EXAMPLE 1 *If O is the centre of a circle, find the value of x in each of the following figures:*



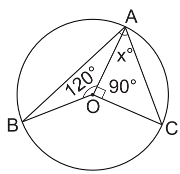
(i)



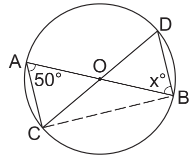
(ii)



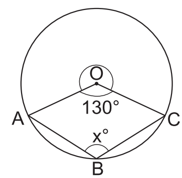
(iii)



(iv)



(v)



(vi)

SOLUTION (i) $\angle BDC = \angle BAC = 30°$ [$\angle$s in the same segment].

In $\triangle BCD$, the sum of the angles is $180°$.

$\therefore \quad \angle CBD + \angle BCD + \angle BDC = 180°$

$\Rightarrow \quad 70° + x° + 30° = 180°$

$\Rightarrow \quad x = 80.$

(ii) $\angle APB = 90°$ [angle in a semicircle].

$\angle BAP + \angle APB + \angle ABP = 180°$

$\Rightarrow \quad \angle BAP + 90° + 35° = 180°$

$\Rightarrow \quad \angle BAP = (180° - 125°) = 55°.$

Now, $x° = \angle BQP = \angle BAP = 55°$ [$\angle$s in the same segment].

Hence, $x = 55$.

(iii) $OA = OB \Rightarrow \angle OAB = \angle OBA = 25°.$

$OC = OA \Rightarrow \angle OAC = \angle OCA = 35°.$

$\therefore \quad \angle BAC = \angle OAB + \angle OAC = (25° + 35°) = 60°.$

Now, $x° = \angle BOC = 2\angle BAC = (2 \times 60°) = 120°.$

$\therefore \quad x = 120.$

(iv) We know that the sum of all angles at a point is $360°$.

$\therefore \quad 90° + 120° + \angle BOC = 360° \Rightarrow \angle BOC = 150°.$

$\angle BOC = 2\angle BAC \Rightarrow 2x = 150 \Rightarrow x = 75.$

(v) $\angle ODB = \angle OAC = 50°$ [angles in the same segment]

$OB = OD \Rightarrow \angle OBD = \angle ODB = 50° = x°$

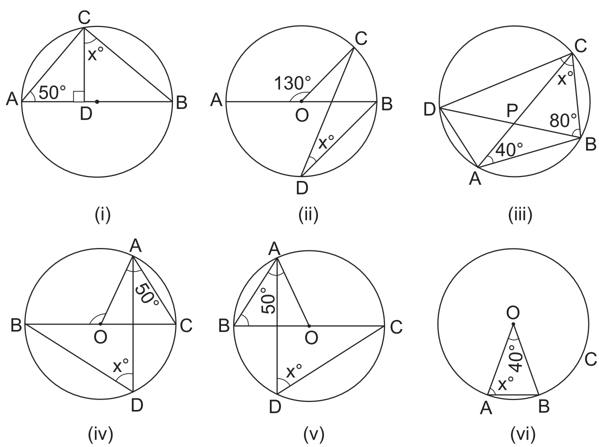
$\Rightarrow x = 50.$

(vi) Reflex $\angle AOC = (360° - 130°) = 230°.$

$\therefore \quad \angle ABC = \frac{1}{2} \times \text{reflex } \angle AOC = \left(\frac{1}{2} \times 230°\right) = 115° = x°.$

$\therefore \quad x = 115.$

EXAMPLE 2 *If O is the centre of the circle, find the value of x in each of the following figures:*



SOLUTION (i) In $\triangle ADC$, we have

$\angle CAD + \angle ADC + \angle ACD = 180°$

$\Rightarrow \quad 50° + 90° + \angle ACD = 180°$

$\Rightarrow \quad \angle ACD = 40°.$

But, $\angle ACB = 90°$ [angle in a semicircle].

$\therefore \quad \angle ACD + x° = 90°$

$\Rightarrow \quad 40° + x° = 90°$

$\Rightarrow \quad x = 50.$

(ii) Since AOB is a straight line, we have

$\angle AOC + \angle BOC = 180°$

$\Rightarrow \quad 130° + \angle BOC = 180°$

$\Rightarrow \quad \angle BOC = 180° - 130° = 50°.$

Now, $\angle BDC = \frac{1}{2}\angle BOC = 25° \Rightarrow x = 25.$

(iii) $\angle BDC = \angle BAC = 40^\circ$ [$\angle$s in the same segment].

$\angle BDC + \angle DBC + \angle BCD = 180^\circ$

$\therefore \quad 40^\circ + 80^\circ + x^\circ = 180^\circ \Rightarrow x = 60.$

(iv) $OA = OC \Rightarrow \angle OCA = \angle OAC = 50^\circ.$

Side CO of $\triangle OCA$ is produced to B.

$\therefore \quad$ ext. $\angle AOB = \angle OAC + \angle OCA = (50^\circ + 50^\circ) = 100^\circ.$

Now, $\angle BDA = \frac{1}{2}\angle AOB = \left(\frac{1}{2} \times 100^\circ\right) = 50^\circ.$

Hence, $x = 50.$

(v) In $\triangle OBA$, we have $OA = OB$.

$\therefore \quad \angle OBA = \angle OAB = 50^\circ \Rightarrow \angle CBA = 50^\circ.$

Now, $\angle CDA = x^\circ = \angle CBA = 50^\circ$ [$\angle$s in the same segment].

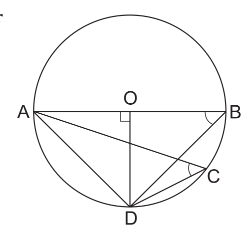
$\therefore \quad x = 50.$

(vi) $OA = OB$ [each equal to the radius of the circle]

$\Rightarrow \quad \angle OBA = \angle OAB = x^\circ$

$\therefore \quad x^\circ + x^\circ + 40^\circ = 180^\circ \Rightarrow 2x^\circ = 140^\circ \Rightarrow x = 70.$

EXAMPLE 3 *In the given figure, AB is a diameter of the circle C(O, r) and the radius OD is perpendicular to AB. If C is any point on* $\overset{\frown}{DB}$, *find* $\angle BAD$ *and* $\angle ACD$.



SOLUTION $OA = OD = r \Rightarrow \angle OAD = \angle ODA. \quad \ldots$(i)

In $\triangle OAD$, we have

$\angle OAD + \angle ODA + \angle AOD = 180^\circ$

[sum of the $\angle$s of a triangle]

$\Rightarrow \quad 2\angle OAD + 90^\circ = 180^\circ$ [$\angle AOD = 90^\circ$ and using (i)]

$\Rightarrow \quad \angle OAD = 45^\circ$

$\Rightarrow \quad \angle BAD = 45^\circ$ [$\because \angle BAD = \angle OAD$].

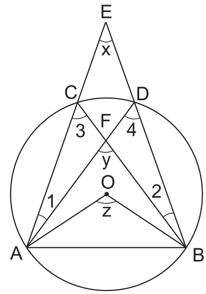
Similarly, by taking $\triangle OBD$, we can prove that $\angle ABD = 45^\circ$.

$\therefore \quad \angle ACD = \angle ABD = 45^\circ$ [angles in the same segment].

Hence, $\angle BAD = 45^\circ$ and $\angle ACD = 45^\circ$.

EXAMPLE 4 *In the given figure, O is the centre of a circle. Prove that*

$$\angle x + \angle y = \angle z.$$



SOLUTION In $\triangle ACF$, side CF is produced to B.

$\therefore \quad \angle y = \angle 1 + \angle 3$ [ext. $\angle$ = sum of int. opp. $\angle$s]. ... (i)

In $\triangle AED$, side ED is produced to B.

$\therefore \quad \angle 1 + \angle x = \angle 4.$ ... (ii)

From (i) and (ii), we have

$$\angle 1 + \angle x + \angle y = \angle 1 + \angle 3 + \angle 4$$

$$\Rightarrow \quad \angle x + \angle y = \angle 3 + \angle 4$$

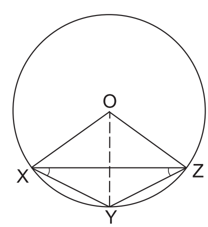
$$= 2\angle 3 \quad [\because \angle 4 = \angle 3, \text{ angles in the same segment}]$$

$$= \angle z \quad [\because \angle AOB = 2\angle ACB].$$

Hence, $\angle x + \angle y = \angle z$.

EXAMPLE 5 *In the given figure, O is the centre of the circle. Prove that*

$$\angle XOZ = 2(\angle XZY + \angle YXZ).$$



SOLUTION Join OY.

Now, arc XY subtends $\angle XOY$ at the centre and $\angle XZY$ at a point Z on the remaining part of the circle.

$\therefore \quad \angle XOY = 2\angle XZY.$ ... (i)

Similarly, arc YZ subtends $\angle YOZ$ at the centre and $\angle YXZ$ at a point X on the remaining part of the circle.

$\therefore \quad \angle YOZ = 2\angle YXZ.$ ... (ii)

Adding (i) and (ii), we get

$$\angle XOY + \angle YOZ = 2(\angle XZY + \angle YXZ)$$

$$\Rightarrow \quad \angle XOZ = 2(\angle XZY + \angle YXZ).$$

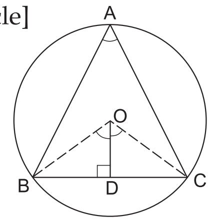
EXAMPLE 6 *If O is the circumcentre of a $\triangle ABC$ and $OD \perp BC$, prove that $\angle BOD = \angle A$.*

SOLUTION GIVEN A $\triangle ABC$ whose circumcentre is O and $OD \perp BC$.

TO PROVE $\angle BOD = \angle A$.

CONSTRUCTION Join OB and OC.

PROOF In the right △s OBD and OCD, we have



$OB = OC$ [radii of the same circle]

$OD = OD$ [common]

$\therefore$ $\triangle OBD \cong \triangle OCD$ [by RHS-congruence]

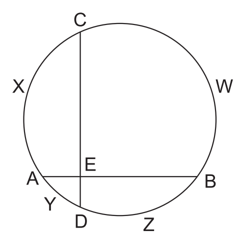
$\Rightarrow$ $\angle BOD = \angle COD$ [c.p.c.t.]

$\Rightarrow$ $\angle BOD = \frac{1}{2}\angle BOC$

$\Rightarrow$ $\angle BOD = \angle A$ $\left[\because \angle A = \frac{1}{2}\angle BOC\right]$.

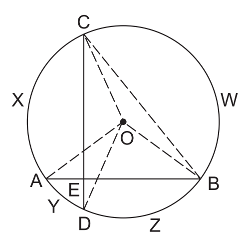
EXAMPLE 7 *In the given figure, two chords AB and CD of a circle $AYDZBWCX$ intersect at right angles. Prove that*

arc CXA + arc DZB = arc AYD + arc BWC
= semicircle.



SOLUTION GIVEN Two chords AB and CD of a circle $AYDZBWCX$, intersecting at a point such that $AB \perp CD$.

CONSTRUCTION Let O be the centre of the given circle. Join OA, OB, OC, OD and BC.



PROOF In $\triangle EBC$, we have

$\angle EBC + \angle ECB + \angle CEB = 180°$

$\Rightarrow$ $\angle EBC + \angle ECB = 180° - \angle CEB$
$= (180° - 90°) = 90°$

$\Rightarrow$ $\angle ABC + \angle DCB = 90°$. … (i)

Since the angle subtended by an arc of a circle at its centre is double the angle subtended by it at any point of the remaining part of the circle, we have

$\angle AOC = 2\angle ABC$ and $\angle DOB = 2\angle DCB$

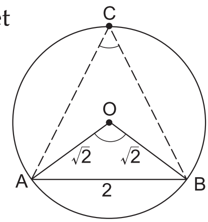
$\Rightarrow$ $\angle AOC + \angle DOB = 2(\angle ABC + \angle DCB) = (2 \times 90°) = 180°$
[using (i)]

$\Rightarrow$ arc CXA + arc DZB = semicircle.

Semilarly, arc AYD + arc BWC = semicircle.

EXAMPLE 8 *A circle has radius* $\sqrt{2}$ cm. *It is divided into two segments by a chord of length* 2 cm. *Prove that the angle subtended by the chord at a point in the major segment is* 45°.

SOLUTION Let O be the centre of the given circle and let AB be a chord of length 2 cm.



$\therefore \quad OA = OB = \sqrt{2}$ cm.

Clearly, $OA^2 + OB^2 = \{(\sqrt{2})^2 + (\sqrt{2})^2\}\ \text{cm}^2$

$= 4\ \text{cm}^2 = (2\ \text{cm})^2 = AB^2$.

$\therefore \quad OA^2 + OB^2 = AB^2$.

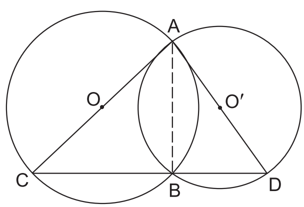
Consequently, $\angle AOB = 90°$.

Let C be a point in the major segment of the circle. Join CA and CB.

Since the angle subtended by an arc of a circle at its centre is twice the angle subtended by it at any point on the remaining part of the circle, we have

$$\angle ACB = \frac{1}{2}\angle AOB = \left(\frac{1}{2} \times 90°\right) = 45°.$$

EXAMPLE 9 *In the figure given below, two circles intersect at A and B, and AC, AD are respectively the diameters of the circles. Prove that the points C, B, D are collinear.*



SOLUTION GIVEN Two circles $C(O, r)$ and $C(O', r')$ with AC and AD as diameters respectively, intersecting at A and B.

TO PROVE Points C, B, D are collinear.

CONSTRUCTION Join CB, BD and AB.

PROOF We know that an angle in a semicircle measures 90°.

$\therefore \quad \angle ABC = 90° \quad \text{and} \quad \angle ABD = 90°$

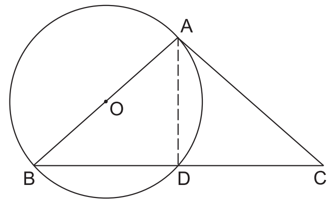
$\Rightarrow \quad \angle ABC + \angle ABD = 90° + 90° = 180°$

$\Rightarrow \quad CBD$ is a straight line

$\Rightarrow \quad$ points C, B, D are collinear.

EXAMPLE 10 *Prove that the circle drawn on any one of the equal sides of an isosceles triangle as diameter, bisects the third side of the triangle.*

SOLUTION GIVEN A $\triangle ABC$ in which $AB = AC$ and a circle is drawn with AB as diameter, intersecting BC at D.



TO PROVE $BD = CD$.

CONSTRUCTION Join AD.

PROOF We know that an angle in a semicircle is a right angle.

$\therefore \quad \angle ADB = 90°.$ … (i)

Also, BDC being a straight line, we have

$\angle ADB + \angle ADC = 180°$

$\Rightarrow \quad 90° + \angle ADC = 180°$ [using (i)]

$\Rightarrow \quad \angle ADC = 90°.$ … (ii)

Now, in $\triangle ADB$ and $\triangle ADC$, we have

$AB = AC$ [given]

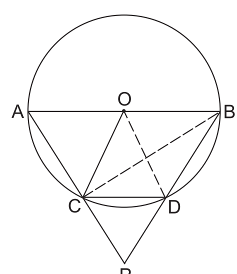
$\angle ADB = \angle ADC$ [each equal to 90°]

$AD = AD$ [common]

$\therefore \quad \triangle ADB \cong \triangle ADC$ [by RHS-congruence].

Hence, $BD = CD$.

EXAMPLE 11 *In the given figure, AB is a diameter of a circle C(O, r) and chord CD = radius OC. If AC and BD when produced meet at P, prove that $\angle APB$ is constant.*



SOLUTION Join OD and BC.

Now, $CD = OC = OD$ [each equal to r]

$\Rightarrow \quad \triangle OCD$ is equilateral

$\Rightarrow \quad \angle COD = 60°.$ … (i)

Also, $\angle ACB = 90°$ [angle in a semicircle] … (ii)

And, $\angle CBD = \frac{1}{2} \times \angle COD = \left(\frac{1}{2} \times 60°\right) = 30°$ … (iii)

[$\because$ angle made by $\overset{\frown}{CD}$ at the centre

= 2 × angle at any point on its remaining part].

Now, side PC of $\triangle BPC$ is produced to A.

$\therefore$ exterior angle = sum of interior opposite angles

$\Rightarrow$ $\angle ACB = \angle APB + \angle CBD$

$\Rightarrow$ $90^\circ = \angle APB + 30^\circ$ [using (ii) and (iii)]

$\Rightarrow$ $\angle APB = 60^\circ$ = constant.

Hence, $\angle APB$ is constant.

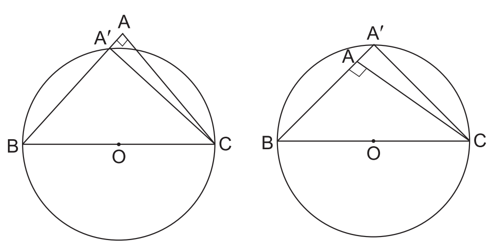
EXAMPLE 12 *Prove that the midpoint of the hypotenuse of a right triangle is equidistant from its vertices.*

SOLUTION GIVEN A $\triangle ABC$ in which $\angle BAC = 90^\circ$ and O is the midpoint of BC.

TO PROVE $OA = OB = OC$.

CONSTRUCTION Draw a circle with O as centre and OB as radius.

PROOF Clearly, this circle will pass through B as well as C. If possible, suppose this circle does not pass through A. Let it meet BA or BA produced at A'.



Then, $\angle BA'C = 90^\circ$ [angle in a semicircle].

But, $\angle BAC = 90^\circ$ [given].

$\therefore$ $\angle BA'C = \angle BAC$. ... (i)

But, this is wrong since an exterior angle of a triangle cannot be equal to its interior opposite angle.

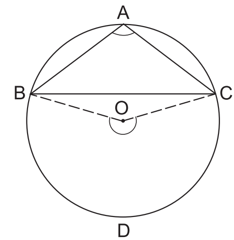
This can be true only when A' coincides with A.

Thus, the above circle which passes through B and C must also pass through A.

$\therefore$ $OA = OB = OC$.

EXAMPLE 13 *Prove that the angle in a segment smaller than a semicircle is greater than a right angle.*

SOLUTION GIVEN $\angle BAC$ in a segment smaller than a semicircle in circle $C(O, r)$.



TO PROVE $\angle BAC > 90°$.

CONSTRUCTION Join BO and CO.

PROOF We know that the angle subtended by an arc of a circle at the centre is double the angle subtended by it at any point on the remaining part of the circle.

Here, arc BDC subtends reflex $\angle BOC$ at the centre and $\angle BAC$ at a point A on the remaining part of the circle.

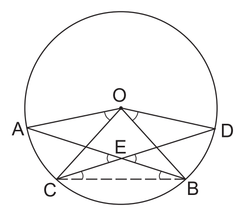
$\therefore$ reflex $\angle BOC = 2\angle BAC$.

But, arc BDC being a major arc, we have reflex $\angle BOC > 180°$.

$\therefore$ $2\angle BAC > 180° \Rightarrow \angle BAC > 90°$.

EXAMPLE 14 *In the given figure, O is the centre of the given circle, and chords AB and CD intersect at a point E inside the circle. Prove that*

$$\angle AOC + \angle BOD = 2\angle AEC.$$



SOLUTION Join CB.

We know that the angle made by $\overset{\frown}{AC}$ at the centre O is double the angle made by it at a point B on the remaining part of the circle.

$\therefore$ $\angle AOC = 2\angle ABC$. ... (i)

Similarly, the angle made by $\overset{\frown}{BD}$ at the centre O is double the angle made by it at a point C on the remaining part of the circle.

$\therefore$ $\angle BOD = 2\angle BCD$. ... (ii)

$\therefore$ $\angle AOC + \angle BOD = 2(\angle ABC + \angle BCD)$ [adding (i) and (ii)]

$= 2(\angle EBC + \angle BCE)$

$= 2(180° - \angle CEB)$

[$\because$ sum of $\angle$s of $\triangle ECB$ is 180°]

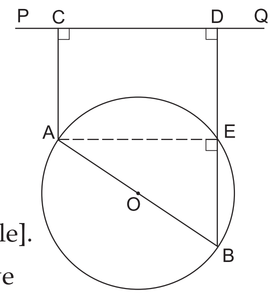
$= 2[180° - (180° - \angle AEC)]$

[$\because$ $\angle CEB = 180° - \angle AEC$]

$= 2\angle AEC$.

Hence, $\angle AOC + \angle BOD = 2\angle AEC$.

EXAMPLE 15 *In the adjoining figure, AB is a diameter of the circle with centre O. If AC and BD are perpendiculars on a line PQ, and BD meets the circle at E, prove that AC = ED.*



SOLUTION Join AE.

Now, $\angle AEB = 90°$ [angle in a semicircle].

But, BED being a straight line, we have

$$\angle AEB + \angle AED = 180°$$

$\Rightarrow \quad 90° + \angle AED = 180°$

$\Rightarrow \quad \angle AED = 90°.$

Now, $\angle EAC + \angle ACD + \angle CDE + \angle AED = 360°$

[sum of the $\angle$s of quad. $EACD$ is 360°]

$\Rightarrow \quad \angle EAC + 90° + 90° + 90° = 360°$

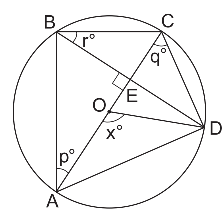
$\Rightarrow \quad \angle EAC = 90°.$

Thus, each angle of quad. $EACD$ is 90°.

So, $EACD$ is a rectangle.

$\therefore \quad AC = ED$ [opposite sides of a rectangle].

EXAMPLE 16 *In the adjoining figure, AC is the diameter of a circle with centre O and chord BD ⊥ AC, intersecting each other at E. Find out the values of p, q, r in terms of x.*



SOLUTION (i) Arc AD subtends $\angle AOD$ at the centre and $\angle ABD$ at a point B on the remaining part of the circle.

$$\therefore \quad \angle ABD = \frac{1}{2}\angle AOD = \frac{x°}{2}\cdot$$

Now, $\angle AEB = 90°$ [$\because$ $BD \perp AC$].

From $\triangle AEB$, we get

$$p° = 180° - (\angle ABE + \angle AEB)$$

$$\Rightarrow \quad p° = 180° - (\angle ABD + \angle AEB) = 180° - \left(\frac{x}{2} + 90°\right)$$

$$= \left(90° - \frac{x°}{2}\right)\cdot$$

$$\therefore \quad p = 90 - \frac{x}{2}\cdot$$

(ii) Arc AD subtends $\angle AOD$ at the centre and $\angle ACD$ at a point C of the circle.

$$\therefore \quad \angle ACD = \frac{1}{2}\angle AOD = \frac{x^\circ}{2} \Rightarrow q = \frac{x}{2}.$$

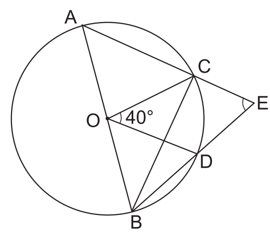
(iii) $\angle ABC = 90^\circ$ [angle in a semicircle]

$$\Rightarrow \quad \angle ABE + \angle CBE = 90^\circ \Rightarrow \angle ABD + \angle CBE = 90^\circ$$

$$\Rightarrow \quad \frac{x^\circ}{2} + \angle CBE = 90^\circ \Rightarrow \angle CBE = \left(90^\circ - \frac{x^\circ}{2}\right)$$

$$\Rightarrow \quad r = \left[90 - \frac{x}{2}\right].$$

EXAMPLE 17 *In the given figure, AB is a diameter of the circle with centre O, AC and BD produced meet at E and* $\angle COD = 40^\circ$. *Calculate* $\angle CED$.



SOLUTION Since ACE is a straight line, we have

$$\angle ACB + \angle BCE = 180^\circ$$

$$\Rightarrow \quad 90^\circ + \angle BCE = 180^\circ$$

[$\because$ $\angle ACB$ is in a semicircle]

$$\Rightarrow \quad \angle BCE = 90^\circ.$$

Also, $\angle DBC = \frac{1}{2}\angle COD = \left(\frac{1}{2} \times 40^\circ\right) = 20^\circ$

[angle at centre = 2 × angle at a point on a circle]

$$\Rightarrow \quad \angle EBC = \angle DBC = 20^\circ.$$

Now, in $\triangle EBC$, we have

$$\angle EBC + \angle BCE + \angle CEB = 180^\circ$$

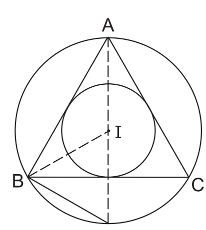
$$\Rightarrow \quad 20^\circ + 90^\circ + \angle CED = 180^\circ \qquad [\because \angle CEB = \angle CED]$$

$$\Rightarrow \quad \angle CED = 180^\circ - 110^\circ = 70^\circ.$$

Hence, $\angle CED = 70^\circ$.

EXAMPLE 18 *In the given figure, I is the incentre of* $\triangle ABC$ *and AI when produced meets the circumcircle of* $\triangle ABC$ *in D. If* $\angle BAC = 56^\circ$ *and* $\angle ACB = 64^\circ$, *calculate*

(i) $\angle DBC$, *(ii)* $\angle IBC$, *(iii)* $\angle ADB$ *and (iv)* $\angle BID$



SOLUTION Join BI.

In $\triangle BAC$, we have

$\angle BAC + \angle ACB + \angle ABC = 180^\circ$ [sum of the $\angle$s of a $\triangle$]

$\Rightarrow$ $56^\circ + 64^\circ + \angle ABC = 180^\circ$

$\Rightarrow$ $\angle ABC = 180^\circ - 120^\circ = 60^\circ$.

Now, $\angle DAC = \frac{1}{2}\angle BAC = \left(\frac{1}{2} \times 56^\circ\right) = 28^\circ$

[$\because$ AD is the bisector of $\angle A$].

(i) $\angle DBC = \angle DAC = 28^\circ$ [angles in the same segment].

(ii) $\angle IBC = \frac{1}{2}\angle ABC = \left(\frac{1}{2} \times 60^\circ\right) = 30^\circ$

[$\because$ BI is the bisector of $\angle ABC$].

(iii) $\angle ADB = \angle ACB = 64^\circ$ [angles in the same segment].

(iv) In $\triangle BDI$, we have

$\angle DBI + \angle BDI + \angle BID = 180^\circ$ [sum of the $\angle$s of a triangle]

$\Rightarrow$ $\angle DBC + \angle IBC + \angle ADB + \angle BID = 180^\circ$

[$\because$ $\angle DBI = \angle DBC + \angle IBC$ and $\angle BDI = \angle ADB$]

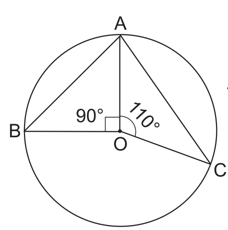
$\Rightarrow$ $28^\circ + 30^\circ + 64^\circ + \angle BID = 180^\circ$

$\Rightarrow$ $\angle BID = 180^\circ - 122^\circ = 58^\circ$.

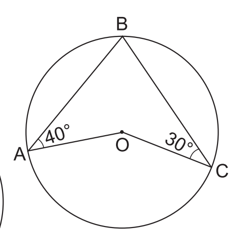
EXERCISE 12B

1. (i) In Figure (1), O is the centre of the circle. If $\angle OAB = 40^\circ$ and $\angle OCB = 30^\circ$, find $\angle AOC$.

(ii) In Figure (2), A, B and C are three points on the circle with centre O such that $\angle AOB = 90^\circ$ and $\angle AOC = 110^\circ$. Find $\angle BAC$.



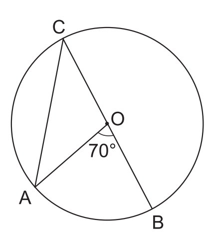
(2)



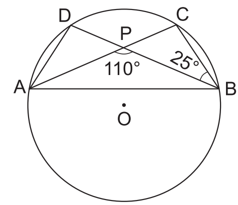
(1)

2. In the given figure, O is the centre of the circle and $\angle AOB = 70^\circ$.

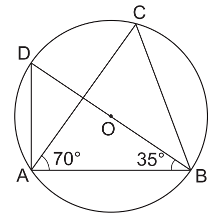
Calculate the values of (i) $\angle OCA$, (ii) $\angle OAC$.



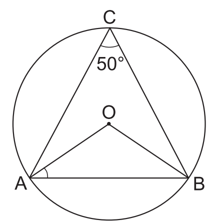
3. In the given figure, O is the centre of the circle. If $\angle PBC = 25°$ and $\angle APB = 110°$, find the value of $\angle ADB$.



4. In the given figure, O is the centre of the circle. If $\angle ABD = 35°$ and $\angle BAC = 70°$, find $\angle ACB$.

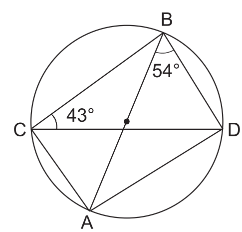


5. In the given figure, O is the centre of the circle. If $\angle ACB = 50°$, find $\angle OAB$.

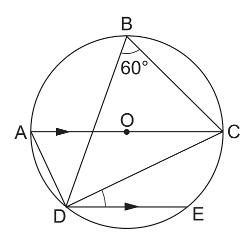


6. In the given figure, $\angle ABD = 54°$ and $\angle BCD = 43°$, calculate

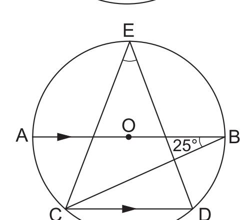
(i) $\angle ACD$, (ii) $\angle BAD$, (iii) $\angle BDA$.



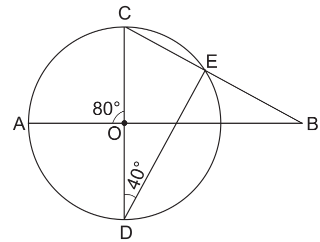
7. In the adjoining figure, DE is a chord parallel to diameter AC of the circle with centre O. If $\angle CBD = 60°$, calculate $\angle CDE$.



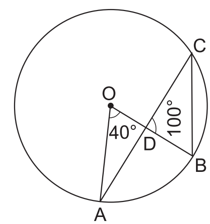
8. In the adjoining figure, O is the centre of a circle. Chord CD is parallel to diameter AB. If $\angle ABC = 25°$, calculate $\angle CED$.



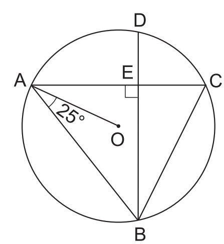
9. In the given figure, AB and CD are straight lines through the centre O of a circle. If $\angle AOC = 80°$ and $\angle CDE = 40°$, find (i) $\angle DCE$, (ii) $\angle ABC$.



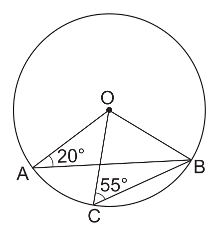
10. In the given figure, O is the centre of a circle, $\angle AOB = 40°$ and $\angle BDC = 100°$, find $\angle OBC$.



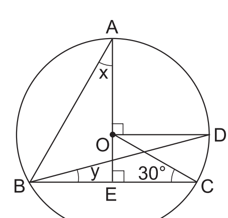
11. In the adjoining figure, chords AC and BD of a circle with centre O, intersect at right angles at E. If $\angle OAB = 25°$, calculate $\angle EBC$.



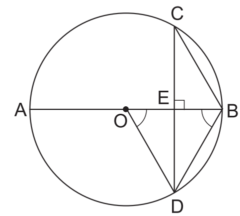
12. In the given figure, O is the centre of a circle in which $\angle OAB = 20°$ and $\angle OCB = 55°$. Find (i) $\angle BOC$, (ii) $\angle AOC$.



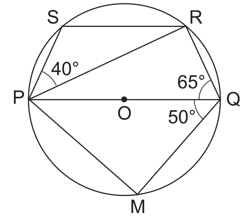
13. In the given figure, O is the centre of the circle and $\angle BCO = 30°$. Find x and y.



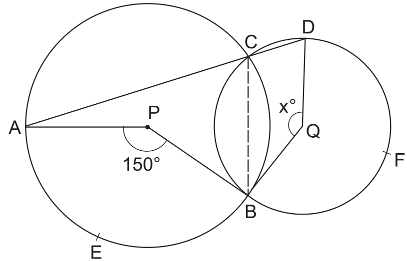
14. In the given figure, O is the centre of the circle, $BD = OD$ and $CD \perp AB$. Find $\angle CAB$.



15. In the given figure, PQ is a diameter of a circle with centre O. If $\angle PQR = 65°$, $\angle SPR = 40°$ and $\angle PQM = 50°$, find $\angle QPR$, $\angle QPM$ and $\angle PRS$.

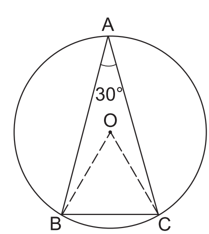


16. In the figure given below, P and Q are centres of two circles, intersecting at B and C, and ACD is a straight line.



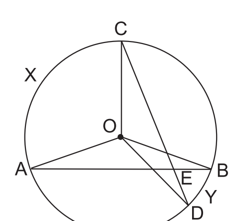
If $\angle APB = 150°$ and $\angle BQD = x°$, find the value of x.

17. In the given figure, $\angle BAC = 30°$. Show that BC is equal to the radius of the circumcircle of $\triangle ABC$ whose centre is O.



18. In the given figure, AB and CD are two chords of a circle, intersecting each other at a point E. Prove that

$$\angle AEC = \frac{1}{2}(\text{angle subtended by arc } CXA \text{ at the centre} + \text{angle subtended by arc } DYB \text{ at the centre}).$$



ANSWERS (EXERCISE 12B)

1. (i) $\angle AOC = 140°$ (ii) $\angle BAC = 80°$ **2.** (i) $\angle OCA = 35°$ (ii) $\angle OAC = 35°$

3. $\angle ADB = 85°$ **4.** $\angle ACB = 55°$ **5.** $\angle OAB = 40°$

6. (i) $\angle ACD = 54°$ (ii) $\angle BAD = 43°$ (iii) $\angle BDA = 83°$ **7.** $\angle CDE = 30°$

8. $\angle CED = 40°$ **9.** (i) $\angle DCE = 50°$ (ii) $\angle ABC = 30°$ **10.** $\angle OBC = 60°$

11. $\angle EBC = 25°$ **12.** (i) $\angle BOC = 70°$ (ii) $\angle AOC = 70°$

13. $x = 30°, y = 15°$ **14.** $\angle CAB = 30°$

15. $\angle QPR = 25°, \angle QPM = 40°, \angle PRS = 25°$ **16.** $x = 150$

HINTS TO SOME SELECTED QUESTIONS

3. $\angle BPC = (180° - 110°) = 70°$.

$\angle ACB = \angle PCB = 180° - (70° + 25°) = 85°$.

$\angle ADB = \angle ACB = 85°$ [$\measuredangle$s in the same segment].

4. $\angle BAD = 90° \Rightarrow \angle ADB = 180° - (90° + 35°) = 55°$

$\Rightarrow \angle ACB = \angle ADB = 55°$.

7. $\angle CAD = \angle CBD = 60°$ [$\measuredangle$s in the same segment].

$\angle ADC = 90°$ [angle in a semicircle].

$\therefore \quad \angle ACD = 180° - (60° + 90°) = 30°$

$\Rightarrow \quad \angle CDE = \angle ACD = 30°$ [alt. int. $\measuredangle$s].

8. Join CO and DO.

$\angle BCD = \angle ABC = 25°$ [alt. int. $\measuredangle$s].

$\angle BOD = 2\angle BCD = 50°$ [$\because$ arc BD makes $\angle BOD$ at the centre and $\angle BCD$ at a point on the circle].

Similarly, $\angle AOC = 2\angle ABC = 50°$.

$\angle AOC + \angle COD + \angle BOD = 180° \Rightarrow \angle COD = 80°$.

$\angle CED = \frac{1}{2}\angle COD = 40°$.

9. $\angle CED = 90°$.

$\angle DCE = 180° - (40° + 90°) = 50°$.

$\angle BOC = 180° - 80° = 100°$.

$\therefore \quad \angle OBC = 180° - (100° + 50°) = 30°$.

10. $\angle DCB = \frac{1}{2}\angle AOB = \left(\frac{1}{2} \times 40°\right) = 20°$.

In $\triangle DBC$, we have

$\angle BDC + \angle DCB + \angle DBC = 180°$

$\Rightarrow \quad 100° + 20° + \angle DBC = 180°$

$\Rightarrow \quad \angle DBC = 60° \Rightarrow \angle OBC = 60°$.

11. $OA = OB \Rightarrow \angle OBA = \angle OAB = 25°$.

$\therefore \quad \angle OAB + \angle OBA + \angle AOB = 180° \Rightarrow \angle AOB = 130°$.

$\therefore \quad \angle ADB = \frac{1}{2}\angle AOB = 65° \Rightarrow \angle ECB = 65°$ [$\angle$s in the same segment].

Now, $\angle EBC + \angle BEC + \angle ECB = 180°$

$\Rightarrow \quad \angle EBC + 90° + 65° = 180° \Rightarrow \angle EBC = 25°$.

12. $OB = OC \Rightarrow \angle OBC = \angle OCB = 55°$.

$\therefore \quad \angle BOC = 180° - (\angle OCB + \angle OBC)$
$= [180° - (55° + 55°)] = 70°$.

$OA = OB \Rightarrow \angle OBA = \angle OAB = 20°$

$\Rightarrow \angle AOB = 180° - (\angle OAB + \angle OBA)$

$\Rightarrow \angle AOB = 180° - (20° + 20°) = 140°$.

$\therefore \quad \angle AOC = \angle AOB - \angle BOC = 140° - 70° = 70°$.

13. $\angle AOD = \angle OEC = 90°$ (given).

But, these are corresponding $\angle$s.

$\therefore \quad OD \parallel BC$.

Now, $OD \parallel BC$ and OC is the transversal.

$\therefore \quad \angle DOC = \angle BCO = 30°$.

$\therefore \quad \angle DBC = \frac{1}{2}\angle DOC = 15° \Rightarrow y = 15°$.

$\angle AOD = 90°$ (given).

Now, $\angle ABD = \frac{1}{2}\angle AOD = \left(\frac{1}{2} \times 90°\right) = 45°$.

In $\triangle ABE$, we have

$\angle ABE = (\angle ABD + \angle DBC) = (45° + 15°) = 60°, \ \angle AEB = 90°$.

$\therefore \quad \angle BAE = 180° - (90° + 60°) = 30° \Rightarrow x = 30°$.

14. $BD = OD$ and $OD = OB$ (radii of a circle)

$\Rightarrow \quad BD = OD = OB$

$\Rightarrow \quad \triangle BDO$ is equilateral $\Rightarrow \angle BDO = 60°$.

Since the altitude of an equilateral triangle bisects the vertical angle, so $\angle BDE = \angle ODE = 30°$.

$\therefore \quad \angle CAB = \angle BDE = 30°$ (angles in the same segment).

16. Join BC.

We know that the angle made by an arc at the centre is twice the angle made by this arc at a point on the remaining part of the circle.

$\therefore \quad \angle ACB = \frac{1}{2}\angle APB = \left(\frac{1}{2} \times 150°\right) = 75°$.

Now, ACD being a straight line, we have

$\angle ACB + \angle BCD = 180°$

$\Rightarrow \quad 75° + \angle BCD = 180° \Rightarrow \angle BCD = 180° - 75° = 105°$.

Similarly, for the second circle, we have

$\angle BCD = \frac{1}{2}$ reflex $\angle BQD$.

[angle made by the major arc BFD at the centre $= 2\angle BCD$]

$\Rightarrow \quad 105^\circ = \frac{1}{2}(360^\circ - x^\circ) \Rightarrow x = 150.$

Hence, $x = 150$.

17. Join OB and OC.

$\angle BOC = 2\angle BAC = 2 \times 30^\circ = 60^\circ$.

$OB = OC \Rightarrow \angle OBC = \angle OCB = x^\circ$ (say).

$\therefore$ in $\triangle OBC$, we have

$\angle BOC + \angle OBC + \angle OCB = 180^\circ \Rightarrow 60^\circ + 2x^\circ = 180^\circ \Rightarrow x = 60.$

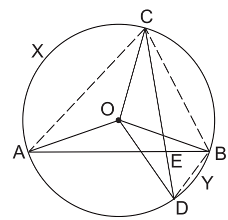
Thus, each angle of $\triangle BOC$ is 60°.

$\therefore$ $\triangle BOC$ is an equilateral triangle.

$\therefore$ $OB = OC = BC$.

Hence, BC is the radius of the circumcircle.

18. Join AC, BC and BD.



Since AC is a chord, we have

$\angle AOC = 2\angle ABC.$ ... (i)

Similarly, $\angle DOB = 2\angle DCB.$ ... (ii)

$\therefore$ $\angle AOC + \angle DOB = 2(\angle ABC + \angle DCB) = 2\angle AEC.$

Hence, $\angle AEC = \frac{1}{2}(\angle AOC + \angle DOB)$.

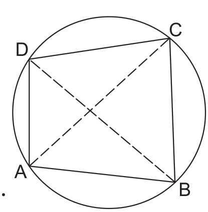
RESULTS ON CYCLIC QUADRILATERALS

THEOREM 1 *The sum of either pair of the opposite angles of a cyclic quadrilateral is* 180°.

Or *The opposite angles of a cyclic quadrilateral are supplementary.*

GIVEN A cyclic quadrilateral $ABCD$.

TO PROVE $\angle A + \angle C = 180^\circ$ and $\angle B + \angle D = 180^\circ$.



CONSTRUCTION Join AC and BD.

PROOF We have

$\angle ACB = \angle ADB$ [$\angle$s in the same segment]

and $\angle BAC = \angle BDC$ [$\angle$s in the same segment].

$\therefore$ $\angle ACB + \angle BAC = \angle ADB + \angle BDC$

$\Rightarrow$ $\angle ACB + \angle BAC = \angle ADC$

$\Rightarrow$ $\angle ACB + \angle BAC + \angle ABC = \angle ADC + \angle ABC$

[adding $\angle ABC$ on both sides]

$\Rightarrow$ $\angle ADC + \angle ABC = 180^\circ$ [the sum of the $\angle$s of $\triangle ABC$ is 180°]

$\Rightarrow$ $\angle B + \angle D = 180^\circ$. ... (i)

Now, $\angle A + \angle B + \angle C + \angle D = 360^\circ$

[sum of the $\angle$s of a quadrilateral is 360°]

$\Rightarrow \quad \angle A + \angle C = 360^\circ - (\angle B + \angle D) = 360^\circ - 180^\circ = 180^\circ$ [using (i)].

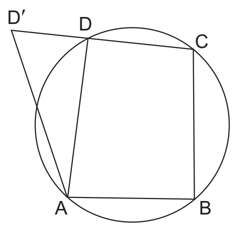
Hence, $\angle A + \angle C = 180^\circ$ and $\angle B + \angle D = 180^\circ$.

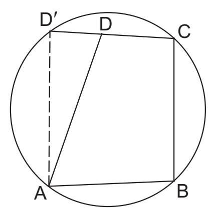
THEOREM 2 (Converse of Theorem 1) *If a pair of opposite angles of a quadrilateral is supplementary then the quadrilateral is cyclic.*

GIVEN A quad. $ABCD$ in which $\angle B + \angle D = 180^\circ$.

TO PROVE $ABCD$ is a cyclic quadrilateral.

CONSTRUCTION If possible, let $ABCD$ be not a cyclic quadrilateral. Draw a circle, passing through three noncollinear points A, B, C. Let this circle meet CD or CD produced in D'. Join $D'A$.





PROOF $\angle ABC + \angle ADC = 180^\circ$ [given]

$\angle ABC + \angle AD'C = 180^\circ$ [opposite ∠s of a cyclic quad.]

$\therefore \quad \angle ABC + \angle ADC = \angle ABC + \angle AD'C$

$\Rightarrow \quad \angle ADC = \angle AD'C$.

This is not possible, since an exterior angle of a triangle can never be equal to its interior opposite angle.

So, $\angle ADC = \angle AD'C$ is possible only when D' coincides with D.

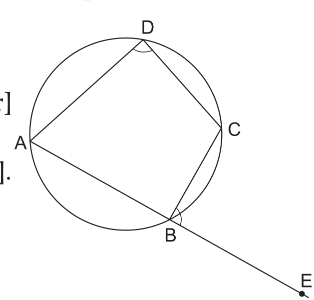
Hence, the circle passing through A, B, C must pass through D also.

Hence, $ABCD$ is a cyclic quadrilateral.

THEOREM 3 (Exterior-angle property of a cyclic quadrilateral) *If one side of a cyclic quadrilateral is produced then the exterior angle so formed is equal to the interior opposite angle.*

GIVEN A cyclic quadrilateral $ABCD$ whose side AB is produced to E.

TO PROVE $\angle CBE = \angle ADC$.



PROOF We have

$\angle ABC + \angle CBE = 180^\circ$ [linear pair]

and $\angle ABC + \angle ADC = 180^\circ$

[opposite ∠s of a cyclic quad.].

$\therefore \quad \angle ABC + \angle CBE = \angle ABC + \angle ADC$

$\Rightarrow \quad \angle CBE = \angle ADC$.

Hence, $\angle CBE = \angle ADC$.

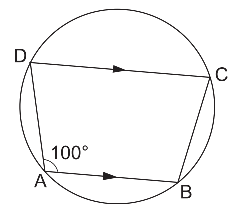
SUMMARY OF THE RESULTS

1. The opposite angles of a cyclic quadrilateral are supplementary.
2. If a pair of opposite angles of a quadrilateral is supplementary then the quadrilateral is cyclic.
3. If one side of a cyclic quadrilateral is produced, then the exterior angle so formed is equal to the interior opposite angle.

SOLVED EXAMPLES

EXAMPLE 1 *In the given figure, ABCD is a cyclic quadrilateral in which $AB \parallel DC$. If $\angle BAD = 100°$, find*

(i) $\angle BCD$, (ii) $\angle ADC$, (iii) $\angle ABC$.



SOLUTION (i) We know that the opposite angles of a cyclic quadrilateral are supplementary.

$\therefore \quad \angle BAD + \angle BCD = 180°$

$\Rightarrow \quad 100° + \angle BCD = 180°$

$\Rightarrow \quad \angle BCD = 180° - 100° = 80°.$

(ii) $AB \parallel DC$ and DA is the transversal.

$\therefore \quad \angle ADC + \angle BAD = 180° \quad$ [sum of int. opp. $\angle$s]

$\Rightarrow \quad \angle ADC + 100° = 180°$

$\Rightarrow \quad \angle ADC = 180° - 100° = 80°.$

(iii) Using the fact that the opposite angles of a cyclic quadrilateral are supplementary, we have

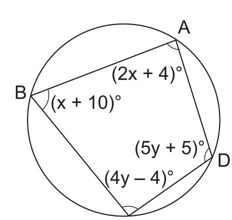
$\angle ABC + \angle ADC = 180°$

$\Rightarrow \quad \angle ABC + 80° = 180° \quad [\because \angle ADC = 80°]$

$\Rightarrow \quad \angle ABC = 180° - 80° = 100°.$

$\therefore \quad \angle BCD = 80°, \angle ADC = 80°$ and $\angle ABC = 100°$.

EXAMPLE 2 *In the given figure, ABCD is a cyclic quadrilateral in which $\angle A = (2x + 4)°$, $\angle B = (x + 10)°$, $\angle C = (4y - 4)°$ and $\angle D = (5y + 5)°$. Find the values of x and y. Hence, find the measure of each angle.*



SOLUTION We know that the opposite angles of a cyclic quadrilateral are supplementary.

$\therefore \quad \angle A + \angle C = 180°$ and $\angle B + \angle D = 180°$

$\Rightarrow \quad (2x + 4) + (4y - 4) = 180$ and $(x + 10) + (5y + 5) = 180$

$\Rightarrow \quad 2x + 4y = 180$ and $x + 5y = 165$

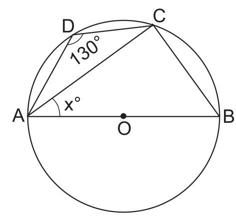
$\Rightarrow \quad x + 2y = 90 \quad \ldots$ (i) and $x + 5y = 165 \quad \ldots$ (ii).

On solving (i) and (ii), we get $x = 40$ and $y = 25$.

$\therefore \quad \angle A = (2 \times 40 + 4)° = 84°, \angle B = (40° + 10°) = 50°$

$\angle C = (4 \times 25 - 4)° = 96°$ and $\angle D = (5 \times 25 + 5)° = 130°$.

EXAMPLE 3 *In the given figure, O is the centre of a circle and $\angle ADC = 130°$. If $\angle BAC = x°$, then find the value of x.*



SOLUTION Since $ABCD$ is a cyclic quadrilateral, we have

$\angle ABC + \angle ADC = 180°$

$\Rightarrow \quad \angle ABC + 130° = 180°$

$\Rightarrow \quad \angle ABC = 180° - 130° = 50°$.

Also, $\angle ACB = 90°$ [angle in a semicircle].

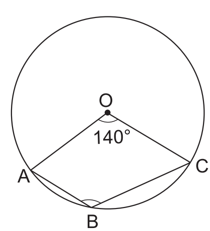
We know that the sum of the angles of a triangle is 180°.

$\therefore$ in $\triangle ABC$, we have $\angle BAC + \angle ACB + \angle ABC = 180°$

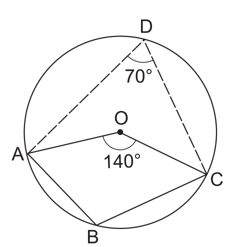
$\Rightarrow \quad x + 90 + 50 = 180 \Rightarrow x = 180 - 140 = 40$.

Hence, $x = 40$.

EXAMPLE 4 *In the given figure, O is the centre of a circle and $\angle AOC = 140°$. Find $\angle ABC$.*



SOLUTION Take a point D on the remaining part of the circumference. Join DA and DC, as shown in the figure.



We know that the angle at the centre of a circle is twice the angle at any point on the remaining part of the circumference.

$\therefore \quad \angle ADC = \frac{1}{2}\angle AOC = \left(\frac{1}{2} \times 140°\right) = 70°$.

We know that the opposite angles of a cyclic qudrilateral are supplementary.

$\therefore$ from cyclic quadrilateral $ABCD$, we have

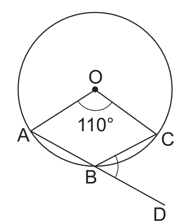
$\angle ABC + \angle ADC = 180°$

$\Rightarrow \quad \angle ABC + 70° = 180°$

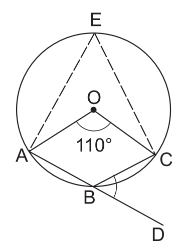
$\Rightarrow \quad \angle ABC = 180° - 70° = 110°.$

Hence, $\angle ABC = 110°$.

EXAMPLE 5 *In the given figure, O is the centre of a circle, $\angle AOC = 110°$ and side AB has been produced to a point D. Find $\angle CBD$.*



SOLUTION Take a point E on the remaining part of the circumference. Join EA and EC.



We know that the angle subtended by an arc at the centre of a circle is twice the angle subtended at a point on the remaining part of the circumference.

$$\therefore \quad \angle AEC = \frac{1}{2}\angle AOC = \left(\frac{1}{2} \times 110°\right) = 55°.$$

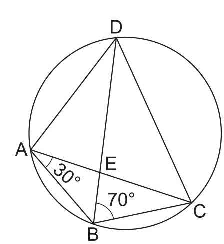
Now, $ABCE$ is a cyclic quadrilateral whose side AB has been produced to a point D.

But, the exterior angle of a cyclic quadrilateral is equal to its interior opposite angle.

$\therefore \quad \angle CBD = \angle AEC = 55°.$

Hence, $\angle CBD = 55°$.

EXAMPLE 6 *In the given figure, ABCD is a cyclic quadrilateral whose diagonals intersect at a point E. If $\angle DBC = 70°$ and $\angle BAC = 30°$, find $\angle BCD$. Further, if $AB = BC$, find $\angle ECD$.*



SOLUTION We have

$\angle BDC = \angle BAC = 30°$ [angles in the same segment]

Now, in $\triangle BCD$, we have

$$\angle DBC + \angle BCD + \angle BDC = 180^\circ$$

[$\because$ sum of the angles of a $\triangle$ is 180°]

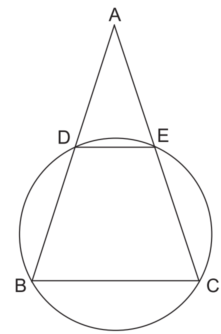
$\Rightarrow \quad 70^\circ + \angle BCD + 30^\circ = 180^\circ$

$\Rightarrow \quad \angle BCD = 180^\circ - 100^\circ = 80^\circ$.

Now, $AB = BC \Rightarrow \angle BCA = \angle BAC = 30^\circ$.

$\therefore \quad \angle ECD = \angle BCD - \angle BCA = 80^\circ - 30^\circ = 50^\circ$.

EXAMPLE 7 *In an isosceles $\triangle ABC$ with $AB = AC$, a circle passing through B and C intersects the sides AB and AC at D and E respectively. Prove that $DE \parallel BC$.*



SOLUTION $AB = AC$

$\Rightarrow \quad \angle ABC = \angle ACB$. ...(i)

Side BD of the cyclic quadrilateral $BCED$ is produced to A.

$\therefore \quad \angle ADE = \angle ACB$ ... (ii)

[ext. $\angle ADE$ = int. opp. $\angle C$].

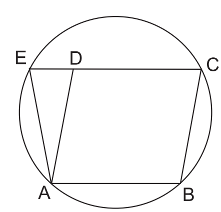
From (i) and (ii), we get

$$\angle ABC = \angle ADE.$$

But, these are corresponding angles.

Hence, $DE \parallel BC$.

EXAMPLE 8 *In the given figure, ABCD is a parallelogram. The circle through A, B, C intersects CD produced at E. Prove that $AD = AE$.*



SOLUTION It is given that $ABCE$ is a cyclic quadrilateral.

$\therefore \quad \angle ABC + \angle AED = 180^\circ$. ... (i)

Also, EDC is a straight line.

$\therefore \quad \angle ADE + \angle ADC = 180^\circ$.

But, $\angle ADC = \angle ABC$ [opposite $\angle$s of a ‖gm].

$\therefore \quad \angle ADE + \angle ABC = 180^\circ$. ... (ii)

Thus, $\angle ABC + \angle AED = \angle ADE + \angle ABC$ [from (i) and (ii)]

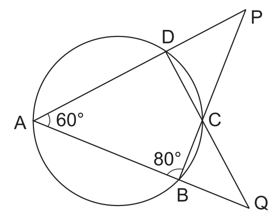
$\Rightarrow \quad \angle AED = \angle ADE$

$\Rightarrow \quad AD = AE \quad$ [opposite sides of equal $\angle$s of $\triangle ADE$]

Hence, $AD = AE$.

EXAMPLE 9 *In the given figure, $\angle A = 60^\circ$ and $\angle ABC = 80^\circ$.*

Find (i) $\angle DPC$ and (ii) $\angle BQC$.



SOLUTION (i) Side AD of cyclic quad. $ABCD$ has been produced to P.

$\therefore \quad \angle PDC = \angle ABC = 80^\circ$

[$\because$ ext. $\angle PDC$ = int. opp $\angle ABC$].

Again, side BC of quad. $ABCD$ has been produced to P.

$\therefore \quad \angle PCD = \angle BAD = 60^\circ$ [$\because$ ext. $\angle PCD$ = int. opp $\angle BAD$].

In $\triangle PCD$, we have

$\angle PDC + \angle PCD + \angle DPC = 180^\circ$

[$\because$ sum of the angles of a $\triangle$ is 180°].

$\Rightarrow \quad 80^\circ + 60^\circ + \angle DPC = 180^\circ$

$\Rightarrow \quad \angle DPC = 180^\circ - 140^\circ = 40^\circ$.

Hence, $\angle DPC = 40^\circ$.

(ii) Since $ABCD$ is a cyclic quadrilateral, we have

$\angle ABC + \angle ADC = 180^\circ$

$\Rightarrow \quad 80^\circ + \angle ADC = 180^\circ$

$\Rightarrow \quad \angle ADC = 180^\circ - 80^\circ = 100^\circ$.

Again, side DC of cyclic quad. $ABCD$ is produced to Q.

$\therefore \quad \angle QCB = \angle BAD = 60^\circ$

[$\because$ ext. $\angle QCB$ = int. opp. $\angle BAD$].

Also, side AB of cyclic quad. $ABCD$ has been produced to Q.

$\therefore \quad \angle QBC = \angle ADC = 100^\circ$

[$\because$ ext. $\angle QBC$ = int. opp. $\angle ADC$].

We know that the sum of the angles of a triangle is 180°.

In $\triangle QCB$, we have

$\angle QCB + \angle QBC + \angle BQC = 180^\circ$

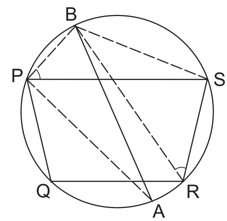
$\Rightarrow \quad 60^\circ + 100^\circ + \angle BQC = 180^\circ$

$\Rightarrow \quad \angle BQC = 180^\circ - 160^\circ = 20^\circ$.

Hence, $\angle BQC = 20^\circ$.

<u>EXAMPLE 10</u> *The bisectors of opposite angles $\angle P$ and $\angle R$ of a cyclic quadrilateral PQRS intersect the corresponding circle at A and B respectively.*

Prove that AB is a diameter of the circle.



SOLUTION GIVEN A cyclic quad. $PQRS$ in which PA and RB are the bisectors of $\angle P$ and $\angle R$ respectively.

TO PROVE AB is a diameter of the circle which passes through the points P, Q, R and S.

CONSTRUCTION Join PB and BS.

PROOF Since $PQRS$ is a cyclic quadrilateral, we have

$$\angle QPS + \angle QRS = 180^\circ$$

$$\Rightarrow \quad \frac{1}{2}\angle QPS + \frac{1}{2}\angle QRS = 90^\circ$$

$$\Rightarrow \quad \angle APS + \angle BRS = 90^\circ$$

$$\Rightarrow \quad \angle APS + \angle BPS = 90^\circ$$

$$[\because \angle BRS = \angle BPS, \text{ ∠s in the same segment}]$$

$$\Rightarrow \quad \angle APB = 90^\circ \quad [\because \angle APS + \angle BPS = \angle APB]$$

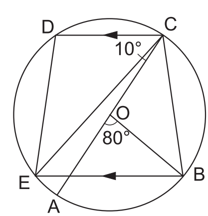
$\Rightarrow$ $\angle APB$ is in a semicircle

$\Rightarrow$ AB is a diameter of the circle which passes through the points P, Q, R and S.

Hence, AB is a diameter of the circle.

<u>EXAMPLE 11</u> *In the given figure, AC is a diameter of a circle with centre O. If $CD \parallel BE$, $\angle AOB = 80^\circ$ and $\angle ACE = 10^\circ$, find*

(i) $\angle BEC$, (ii) $\angle BCD$ and (iii) $\angle CED$.



SOLUTION (i) Since AOC is a straight line, we have

$$\angle AOB + \angle BOC = 180^\circ$$

$$\Rightarrow \quad 80^\circ + \angle BOC = 180^\circ$$

$$\Rightarrow \quad \angle BOC = 180^\circ - 80^\circ = 100^\circ.$$

Now, arc BC makes $\angle BOC = 100^\circ$ at the centre and $\angle BEC$ at a point E on the remaining part of the circle.

$$\therefore \quad \angle BEC = \frac{1}{2}\angle BOC = \left(\frac{1}{2} \times 100^\circ\right) = 50^\circ.$$

(ii) Arc AB makes $\angle AOB$ at the centre and $\angle ACB$ at a point C on the remaining part of the circle.

$$\therefore \quad \angle ACB = \frac{1}{2}\angle AOB = \left(\frac{1}{2} \times 80^\circ\right) = 40^\circ.$$

Now, $CD \parallel BE$ and CE is the transversal.

$\therefore \quad \angle ECD = \angle BEC = 50^\circ$ [alt. int. $\angle$s]

$\therefore \quad \angle BCD = \angle ACB + \angle ACE + \angle ECD$

$= 40^\circ + 10^\circ + 50^\circ = 100^\circ.$

(iii) $DEBC$ is a cyclic quadrilateral.

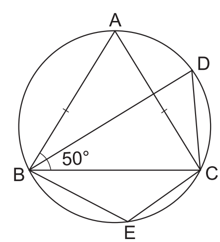
$\therefore \quad \angle BED + \angle BCD = 180^\circ$

$\Rightarrow \quad \angle BEC + \angle CED + \angle BCD = 180^\circ$

$\Rightarrow \quad 50^\circ + \angle CED + 100^\circ = 180^\circ$

$\Rightarrow \quad \angle CED = 180^\circ - 150^\circ = 30^\circ.$

EXAMPLE 12 *In the given figure, $\triangle ABC$ is isosceles with $AB = AC$ and $\angle ABC = 50^\circ$. Find $\angle BDC$ and $\angle BEC$.*



SOLUTION $AB = AC \Rightarrow \angle ACB = \angle ABC = 50^\circ.$

But, $\angle ABC + \angle ACB + \angle BAC = 180^\circ$

$\Rightarrow \quad 50^\circ + 50^\circ + \angle BAC = 180^\circ$

$\Rightarrow \quad \angle BAC = 180^\circ - 100^\circ = 80^\circ.$

$\therefore \quad \angle BDC = \angle BAC = 80^\circ$ [$\angle$s in the same segment].

Now, $BECD$ being a cyclic quadrilateral, we have

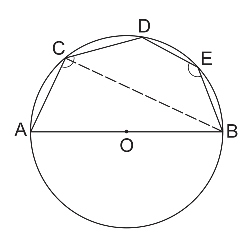
$\angle BEC + \angle BDC = 180^\circ$

$\Rightarrow \quad \angle BEC + 80^\circ = 180^\circ$

$\Rightarrow \quad \angle BEC = 180^\circ - 80^\circ = 100^\circ.$

Hence, $\angle BDC = 80^\circ$ and $\angle BEC = 100^\circ$.

EXAMPLE 13 *In the given figure, AOB is a diameter of a circle with centre O. Write down the numerical value of $\angle ACD + \angle DEB$.*



SOLUTION Join BC.

Then, $\angle ACB = 90^\circ$

[angle in a semicircle].

Now, $DCBE$ is a cyclic quadrilateral.

$\therefore \quad \angle BCD + \angle DEB = 180^\circ$

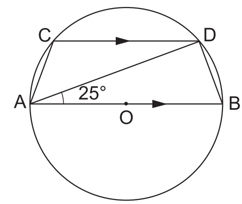
$\Rightarrow \quad \angle ACB + \angle BCD + \angle DEB = 90° + 180° \qquad [\because \angle ACB = 90°]$

$\Rightarrow \quad \angle ACD + \angle DEB = 270° \qquad [\because \angle ACB + \angle BCD = \angle ACD]$.

$\therefore$ the numerical value of $(\angle ACD + \angle DEB)$ is 270°.

EXAMPLE 14 *In the given figure, AB is a diameter of the circle and $CD \parallel AB$. If $\angle DAB = 25°$, calculate*

(i) $\angle ACD$ and (ii) $\angle CAD$.



SOLUTION Join AC and BD.

We know that an angle in a semicircle is 90°.

$\therefore \quad \angle ADB = 90°$.

Also, the sum of the angles of a triangle is 180°.

$\therefore$ in $\triangle ABD$, we have

$\angle BAD + \angle ADB + \angle ABD = 180°$

$\Rightarrow \quad 25° + 90° + \angle ABD = 180°$

$\Rightarrow \quad \angle ABD = 180° - 115° = 65°$.

Since $ABDC$ is a cyclic quadrilateral, we have

$\angle ABD + \angle ACD = 180° \Rightarrow \angle ACD = 180° - \angle ABD$

$\Rightarrow \angle ACD = 180° - 65° = 115°$.

Also, $AB \parallel CD$ and AD is a transversal.

$\therefore \quad \angle ADC = \angle BAD = 25° \qquad$ [alt. int. $\angle$s].

Now, in $\triangle ACD$, we have

$\angle ACD + \angle CAD + \angle ADC = 180°$

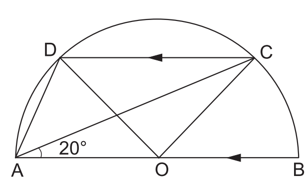
$\Rightarrow \quad 115° + \angle CAD + 25° = 180°$

$\Rightarrow \quad \angle CAD = 180° - 140° = 40°$.

Hence, (i) $\angle ACD = 115°$ (ii) $\angle CAD = 40°$.

EXAMPLE 15 *In the given figure, AB is a diameter of a circle with centre O and $CD \parallel BA$. If $\angle BAC = 20°$, find*

(i) $\angle BOC$, (ii) $\angle COD$, (iii) $\angle CAD$ and (iv) $\angle ADC$.



SOLUTION (i) Arc BC subtends $\angle BOC$ at the centre and $\angle BAC$ at a point on the remaining part of the circumference.

$\therefore \quad \angle BOC = 2 \times \angle ABC = 2 \times 20° = 40°$.

(ii) $CD \parallel BA$ and OC cuts them.

$\therefore \quad \angle OCD = \angle BOC = 40^\circ$ [alt. int. ∠s].

$\therefore \quad \angle ODC = \angle OCD = 40^\circ$ [$\because$ $OC = OD$ = radius].

Now, in $\triangle OCD$, we have

$$\angle COD + \angle OCD + \angle ODC = 180^\circ$$

$\Rightarrow \quad \angle COD + 40^\circ + 40^\circ = 180^\circ$ [$\because$ $\angle OCD = \angle ODC = 40^\circ$]

$\Rightarrow \quad \angle COD = 180^\circ - 80^\circ = 100^\circ$.

(iii) Since the angle at the centre is double the angle at a point on the remaining part of the circumference, we have

$$\angle CAD = \frac{1}{2}\angle COD = \left(\frac{1}{2} \times 100^\circ\right) = 50^\circ.$$

(iv) Since $BA \parallel CD$ and AC cuts them, we have

$\angle ACD = \angle CAB = 20^\circ$ [alt. int ∠s].

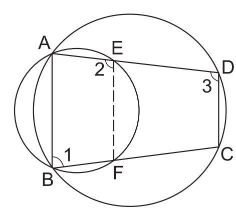
In $\triangle ACD$, we have

$$\angle CAD + \angle ACD + \angle ADC = 180^\circ$$

$\Rightarrow \quad 50^\circ + 20^\circ + \angle ADC = 180^\circ$

$\Rightarrow \quad \angle ADC = 180^\circ - 70^\circ = 110^\circ$.

EXAMPLE 16 *In the given figure, ABCD is a cyclic quadrilateral. A circle passing through A and B, meets AD and BC in E and F respectively. Prove that EF ∥ DC.*



SOLUTION Join EF.

Now, $ABFE$ being a cyclic quadrilateral, we have

$\angle 1 + \angle 2 = 180^\circ$. … (i)

Also, $ABCD$ being a cyclic quadrilateral, we have

$\angle 1 + \angle 3 = 180^\circ$. … (ii)

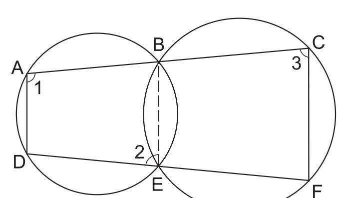
From (i) and (ii), we get

$$\angle 1 + \angle 2 = \angle 1 + \angle 3 \Rightarrow \angle 2 = \angle 3.$$

But, these are corresponding angles.

$\therefore \quad EF \parallel DC$.

EXAMPLE 17 *In the given figure A, B, C and D, E, F are two sets of collinear points. Prove that AD ∥ CF.*



SOLUTION Join BE.

Now, $ABED$ is a cyclic quadrilateral.

$\therefore \quad \angle 1 + \angle 2 = 180°.$ … (i)

Again, $BCFE$ is a cyclic quadrilateral.

$\therefore$ exterior angle = interior opposite angle.

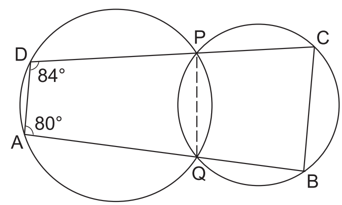
$\therefore \quad \angle 2 = \angle 3.$ … (ii)

From (i) and (ii), we get $\angle 1 + \angle 3 = 180°$.

But these are interior angles on the same side of the transversal.

Hence, $AD \parallel CF$.

EXAMPLE 18 *In the given figure, two circles intersect at P and Q. If $\angle A = 80°$ and $\angle D = 84°$, calculate (i) $\angle QBC$ and (ii) $\angle BCP$.*



SOLUTION Join PQ.

Now, quad. $AQPD$ is cyclic

$\Rightarrow \quad \angle QAD + \angle QPD = 180°$

$\Rightarrow \quad 80° + \angle QPD = 180°$

$\Rightarrow \quad \angle QPD = 180° - 80° = 100°.$

(i) The side CP of quad. $QBCP$ is produced to D.

$\therefore \quad \angle QBC = \angle QPD = 100°$ [ext. $\angle$ = int. opp. $\angle$].

(ii) Side AQ of cyclic quad. $PDAQ$ is produced to B.

$\therefore \quad \angle PQB = \angle ADP = 84°$ [ext. $\angle$ = int. opp. $\angle$].

In the cyclic quad. $QBCP$, we have

$\angle PQB + \angle BCP = 180°$

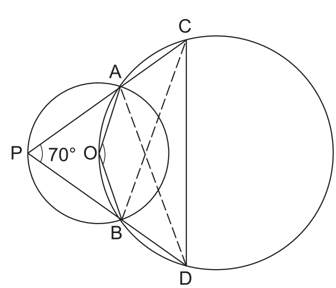
$\Rightarrow \quad 84° + \angle BCP = 180°$

$\Rightarrow \quad \angle BCP = 180° - 84° = 96°.$

EXAMPLE 19 *In the given figure, two circles intersect at A and B. The centre of the smaller circle is O and lies on the circumference of the larger circle.*

If PAC and PBD are straight lines and $\angle APB = 70°$, find

(i) $\angle AOB$, (ii) $\angle ACB$ and (iii) $\angle ADB$.



SOLUTION (i) Minor arc AB subtends $\angle AOB$ at the centre and $\angle APB$ at a point on the remaining part of the circle.

$\therefore \quad \angle AOB = 2\angle APB = 2 \times 70° = 140°.$

(ii) $AOBC$ is a cyclic quadrilateral

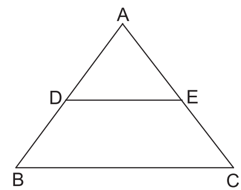
$\therefore \quad \angle AOB + \angle ACB = 180°$

$\Rightarrow \quad 140° + \angle ACB = 180°$

$\Rightarrow \quad \angle ACB = 180° - 140° = 40°.$

(iii) $\angle ADB = \angle ACB = 40°$ [$\angle$s in the same segment].

EXAMPLE 20 *ΔABC is an isosceles triangle in which $AB = AC$. If D and E are midpoints of AB and AC respectively, prove that the points B, C, E and D are concyclic.*



SOLUTION In order to show that the points B, C, E and D are concyclic, we must prove that $\angle ECB + \angle EDB = 180°$.

Since D and E are the midpoints of AB and AC respectively, we have $DE \parallel BC$.

Now, $DE \parallel BC$

$\Rightarrow \quad \angle ABC = \angle ADE$

$\Rightarrow \quad \angle ACB = \angle ADE \quad \dots \text{(i)} \quad [\because \ AB = AC \Rightarrow \angle ABC = \angle ACB].$

$\therefore \quad \angle ECB + \angle EDB = \angle ACB + \angle EDB$

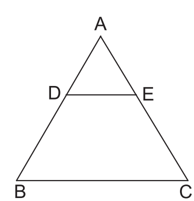
$= \angle ADE + \angle EDB$ [using (i)]

$= 180°$

[$\because$ $\angle ADE$ and $\angle EDB$ form a linear pair].

Hence, the points B, C, E and D are concyclic.

EXAMPLE 21 *D and E are points on equal sides AB and AC of an isosceles ΔABC such that $AD = AE$. Prove that the points B, C, E and D are concyclic.*



SOLUTION In order to prove that the points B, C, E and D are concyclic, we must show that $\angle ECB + \angle EDB = 180°$.

Now, $AB = AC \Rightarrow \angle ABC = \angle ACB. \quad \dots \text{(i)}$

And, $AD = AE \Rightarrow \angle AED = \angle ADE. \quad \dots \text{(ii)}$

We know that the sum of the angles of a triangle is 180°.

$\therefore$ in $\triangle ABC$ and $\triangle ADE$, we have

$\angle A + \angle ABC + \angle ACB = 180°$ and $\angle A + \angle ADE + \angle AED = 180°$

$\Rightarrow$ $\angle A + 2\angle ACB = 180°$ and $\angle A + 2\angle ADE = 180°$

[using (i) and (ii)]

$\Rightarrow$ $\angle A + 2\angle ACB = \angle A + 2\angle ADE$

$\Rightarrow$ $2\angle ACB = 2\angle ADE \Rightarrow \angle ACB = \angle ADE$

$\Rightarrow$ $\angle ECB = \angle ADE$ ... (iii) [$\because \angle ACB = \angle ECB$].

$\therefore$ $\angle ECB + \angle EDB = \angle ADE + \angle EDB$ [using (iii)]

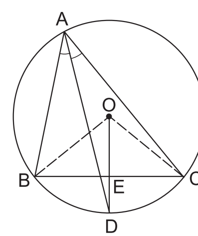
$= 180°$ [$\because \angle ADE + \angle EDB =$ a straight $\angle$].

Thus, $\angle ECB + \angle EDB = 180°$.

This shows that the points B, C, E and D are concyclic.

EXAMPLE 22 *Prove that if the bisector of any angle of a triangle and the perpendicular bisector of its opposite side intersect then they will intersect on the circumcircle of the triangle.*

SOLUTION Let O be the circumcentre of the given $\triangle ABC$. Then, the perpendicular bisector of BC passes through the point O. Let it cut the circumcircle of $\triangle ABC$ at D and BC at E. Join OB and OC.



In order to show that the perpendicular bisector of BC and the bisector of $\angle A$ of $\triangle ABC$ intersect at D, it is sufficient to show that AD is the bisector of $\angle A$.

Clearly, arc BC makes $\angle BOC$ at the centre and $\angle A$ at a point A on the remaining part of the circle.

$\therefore$ $\angle BOC = 2\angle A$. ... (i)

In $\triangle OEB$ and $\triangle OEC$, we have $\angle OEB = \angle OEC = 90°$.

Hyp. OB = hyp. OC = radius of the circle.

$OE = OE$ (common)

$\therefore$ $\triangle OEB \cong \triangle OEC$

$\therefore$ $\angle BOE = \angle COE = \angle A$ [$\because \angle BOC = 2\angle A$].

Now, arc DC makes $\angle A$ at the centre and therefore it will make $\frac{1}{2}\angle A$ at the point A on remaining part of the circle.

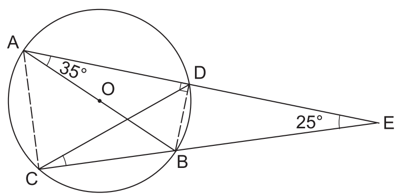
$\therefore \quad \angle CAD = \frac{1}{2}\angle A.$

This shows that AD is the bisector of $\angle A$.

Thus, the bisector of $\angle A$ and the perpendicular bisector of side BC, intersect at a point D lying on the circumcircle of $\triangle ABC$.

Hence, the result follows.

EXAMPLE 23 *In the given figure, AB is a diameter of a circle with centre O. If ADE and CBE are straight lines, meeting at E such that* $\angle BAD = 35°$ *and* $\angle BED = 25°$, *find (i)* $\angle DBC$, *(ii)* $\angle BCD$ *and (iii)* $\angle CDB$.



SOLUTION Join AC and BD.

Now, $\angle ADB = 90°$ (angle in a semicircle).

Since ADE is a straight line, we have

$$\angle ADB + \angle EDB = 180° \Rightarrow 90° + \angle EDB = 180°$$
$$\Rightarrow \angle EDB = 180° - 90° = 90°.$$

In $\triangle DBE$, we have

$$\angle DBE + \angle BED + \angle EDB = 180°$$
$$\Rightarrow \angle DBE + 25° + 90° = 180°$$
$$\Rightarrow \angle DBE = 180° - 115° = 65°.$$

(i) Since CBE is a straight line, we have

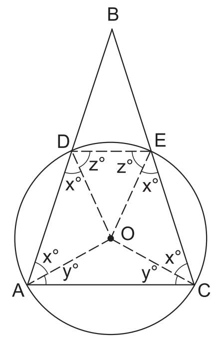
$$\angle DBC + \angle DBE = 180° \Rightarrow \angle DBC + 65° = 180°$$
$$\Rightarrow \angle DBC = 180° - 65° = 115°.$$

(ii) $\angle BCD = \angle BAD = 35°$ [angles in the same segment].

(iii) In $\triangle DBC$, we have

$$\angle DBC + \angle BCD + \angle CDB = 180°$$
$$\Rightarrow 115° + 35° + \angle CDB = 180°$$
$$\Rightarrow \angle CDB = (180° - 150°) = 30°.$$

EXAMPLE 24 *Let the vertex of an $\angle ABC$ be located outside a circle and let the sides of the angle intersect equal chords AD and CE with the circle. Prove that $\angle ABC$ is equal to half of the difference of the angles subtended by the chords AC and DE at the centre.*



SOLUTION Let O be the centre of the given circle. Let ADB and CEB be the two chords of the circle such that $AD = CE$.

Join OA, OC, OD, OE and DE.

Then, we have to prove that $\angle ABC = \frac{1}{2}(\angle AOC - \angle DOE)$.

In $\triangle OAD$ and $\triangle OCE$, we have

$OA = OC$ [radii of the same circle]

$OD = OE$ [radii of the same circle]

$AD = CE$ [given]

$\therefore$ $\triangle OAD \cong \triangle OCE$ [by SSS-congruence].

So, $\angle AOD = \angle COE$, $OA = OC$ and $OD = OE$.

$\therefore$ $\angle OAD = \angle ODA = \angle OCE = \angle OEC = x°$ (say).

In $\triangle OAC$, we have

$OA = OC \Rightarrow \angle OAC = \angle OCA = y°$ (say).

$\therefore$ $\angle AOC = (180° - 2y°)$. ... (i)

In $\triangle ODE$, we have

$OD = OE \Rightarrow \angle ODE = \angle OED = z°$ (say).

$\therefore$ $\angle DOE = (180° - 2z°)$. ... (ii)

Clearly, $ACED$ is a cyclic quadrilateral.

$\therefore$ $\angle C + \angle D = 180° \Rightarrow (x° + y°) + (x° + z°) = 180°$

$\Rightarrow 2x° + y° + z° = 180°$. ... (iii)

Now, in $\triangle BDE$, we have

$\angle BDE + \angle BED + \angle DBE = 180°$

$\Rightarrow$ $\{180° - (x° + z°)\} + \{180° - (x° + z°)\} + \angle DBE = 180°$

$\Rightarrow$ $\angle DBE = 2x° + 2z° - 180°$

$= 180° - (y° + z°) + 2z° - 180°$ [using (iii)]

$\Rightarrow$ $\angle DBE = z° - y°$

$\Rightarrow \quad \angle ABC = z^\circ - y^\circ$ ... (iv) [$\because \angle DBE = \angle ABC$].

From (i) and (ii), we get

$$(\angle AOC - \angle DOE) = (180^\circ - 2y^\circ) - (180^\circ - 2z^\circ) = 2(z^\circ - y^\circ)$$

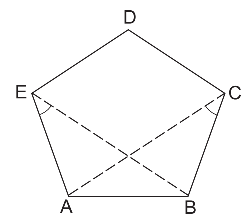
$$\Rightarrow \quad \frac{1}{2}(\angle AOC - \angle DOE) = z^\circ - y^\circ. \quad \text{... (v)}$$

From (iv) and (v), we have

$$\frac{1}{2}(\angle AOC - \angle DOE) = \angle ABC.$$

EXAMPLE 25 *Prove that any four vertices of a regular pentagon are concyclic.*

SOLUTION GIVEN A regular pentagon $ABCDE$.



TO PROVE Every set of four vertices of $ABCDE$ is a set of points lying on a circle.

PROOF First we show that the points A, B, C, E lie on a circle.

Join AC and BE.

In $\triangle ABC$ and $\triangle BAE$, we have

	$AB = BA$	[common]
	$BC = AE$	[sides of a regular pentagon]
	$\angle ABC = \angle BAE$	[each equal to 108°]
$\therefore$	$\triangle ABC \cong \triangle BAE$	[by SAS-congruence]
$\Rightarrow$	$\angle BCA = \angle AEB$	[c.p.c.t.].

Thus, AB subtends equal angles at two points C and E on the same side of AB.

$\therefore$ the points A, B, C, E are concyclic [by Theorem 5 at Page 444].

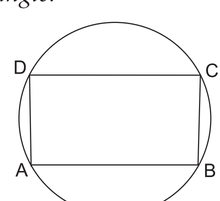
Similarly, every set of four vertices of pentagon $ABCDE$ is a set of concyclic points.

SOME RESULTS ON CYCLIC QUADRILATERALS

THEOREM 1 *Prove that every cyclic parallelogram is a rectangle.*

GIVEN A $\parallel$gm $ABCD$ inscribed in a circle.

TO PROVE $ABCD$ is a rectangle.



PROOF We have

$\angle A + \angle C = 180^\circ$ ... (i)

[opp. $\angle$s of a cyclic quad.].

But, $\angle A = \angle C$ ... (ii) [opposite $\angle$s of a $\|$gm].

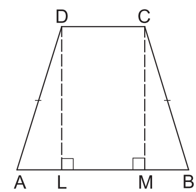
$\therefore \quad \angle A = \angle C = 90°$ [from (i) and (ii)].

But, a $\|$gm one of whose angles is 90° is a rectangle.

Hence, $ABCD$ is a rectangle.

THEOREM 2 *Prove that an isosceles trapezium is always cyclic.*

Or *If two nonparallel sides of a trapezium are equal, prove that it is cyclic.*



GIVEN A trapezium $ABCD$ in which $AB \parallel DC$ and $AD = BC$.

TO PROVE $\angle A + \angle C = 180°$, and $\angle B + \angle D = 180°$.

CONSTRUCTION Draw $DL \perp AB$ and $CM \perp AB$.

PROOF From the right $\triangle$s ALD and BMC, we have

$AD = BC$ [given]

$DL = CM$ [distance between two parallels]

$\therefore \quad \triangle ALD \cong \triangle BMC$. [RHS-congruence]

$\Rightarrow \quad \angle A = \angle B$ ... (i) and $\angle ADL = \angle BCM$ ... (ii)

$\Rightarrow \quad \angle C = \angle BCD = \angle BCM + 90°$

$= \angle ADL + 90° = \angle ADC = \angle D$ [using (ii)]

$\Rightarrow \quad \angle C = \angle D$. ... (iii)

Now, $\angle A + \angle B + \angle C + \angle D = 360°$ [sum of the $\angle$s of a quad. is 360°]

$\Rightarrow \quad 2(\angle A + \angle C) = 360°$ and $2(\angle B + \angle D) = 360°$

[using (i) and (iii)]

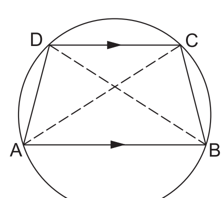
$\Rightarrow \quad \angle A + \angle C = 180°$ and $\angle B + \angle D = 180°$.

Hence, quad. $ABCD$ is cyclic.

THEOREM 3 *Prove that a cyclic trapezium is always isosceles and its diagonals are equal.*

Or *If two sides of a cyclic quadrilateral are parallel, prove that its remaining two sides are equal and the diagonals are equal.*

GIVEN A cyclic trapezium in which $AB \parallel DC$.



TO PROVE $AD = BC$ and $AC = BD$.

PROOF $AB \parallel DC$ and BC is a transversal.

$\therefore \quad \angle ABC + \angle BCD = 180°$ [sum of int. $\angle$s].

But, $\angle ABC + \angle ADC = 180°$

[opposite $\angle$s of a cyclic quad.].

$\therefore \quad \angle ABC + \angle ADC = \angle ABC + \angle BCD$

$\Rightarrow \quad \angle ADC = \angle BCD.$ … (i)

Now, in $\triangle ADC$ and $\triangle BCD$, we have

$\angle ADC = \angle BCD$ [proved in (i)]

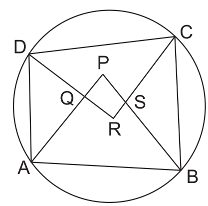
$\angle DAC = \angle CBD$ [$\measuredangle$s in the same segment]

$DC = CD$ [common]

$\therefore \quad \triangle ADC \cong \triangle BCD.$

Hence, $AD = BC$ and $AC = BD$.

THEOREM 4 *Prove that the quadrilateral formed by angle bisectors of a cyclic quadrilateral is also cyclic.*



GIVEN A cyclic quad. $ABCD$ in which AP, BP, CR and DR are the bisectors of $\angle A, \angle B, \angle C$ and $\angle D$ respectively, forming quad. $PQRS$.

TO PROVE $PQRS$ is a cyclic quadrilateral.

PROOF Since the sum of the angles of a triangle is 180°, from $\triangle ABP$ and $\triangle CDR$, we have

$$\left.\begin{aligned}\angle APB + \angle PAB + \angle PBA = 180^\circ \\ \angle CRD + \angle RCD + \angle RDC = 180^\circ\end{aligned}\right\}$$

$$\Rightarrow \begin{cases}\angle APB + \frac{1}{2}\angle A + \frac{1}{2}\angle B = 180^\circ & \text{… (i)} \\ \angle CRD + \frac{1}{2}\angle C + \frac{1}{2}\angle D = 180^\circ. & \text{… (ii)}\end{cases}$$

Adding (i) and (ii), we get

$$\angle APB + \angle CRD + \frac{1}{2}(\angle A + \angle B + \angle C + \angle D) = 360^\circ$$

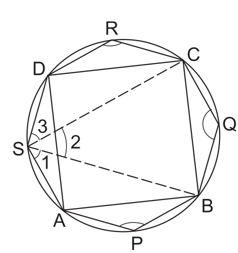
$\Rightarrow \quad \angle APB + \angle CRD = 180^\circ$ $[\because \angle A + \angle B + \angle C + \angle D = 360^\circ]$

$\Rightarrow \quad \angle QPS + \angle QRS = 180^\circ.$

Thus, two opposite angles of quad. $PQRS$ are supplementary.

Hence, $PQRS$ is cyclic.

THEOREM 5 *Prove that the sum of the angles in the four segments exterior to a cyclic quadrilateral is equal to six right angles.*



GIVEN A cyclic quadrilateral $ABCD$, and $\angle P, \angle Q$, $\angle R$ and $\angle S$ are in the four segments exterior to it.

TO PROVE $\angle P + \angle Q + \angle R + \angle S = 6$ rt. $\measuredangle$s.

CONSTRUCTION Join BS and CS.

PROOF $APBS$ is a cyclic quadrilateral.

$\therefore \quad \angle 1 + \angle P = 180^\circ.$ … (i)

Again, $SBQC$ is a cyclic quadrilateral.

$\therefore \quad \angle 2 + \angle Q = 180°$. … (ii)

Also, $SCRD$ is a cyclic quadrilateral.

$\therefore \quad \angle 3 + \angle R = 180°$. … (iii)

From (i), (ii) and (iii), we get

$$(\angle 1 + \angle 2 + \angle 3) + (\angle P + \angle Q + \angle R) = 540°$$

$\Rightarrow \quad \angle P + \angle Q + \angle R + \angle S = 540° = 6$ rt. $\angle$s $\quad [\because \angle 1 + \angle 2 + \angle 3 = \angle S]$.

Hence, $\angle P + \angle Q + \angle R + \angle S = 6$ rt. $\angle$s.

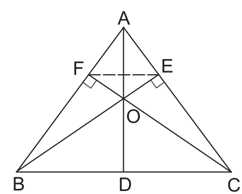
THEOREM 6 *Prove that the altitudes of a triangle are concurrent.*

Or *Prove that the perpendiculars from the vertices of a triangle on the opposite sides are concurrent.*

GIVEN A $\triangle ABC$ in which $BE \perp AC$ and $CF \perp AB$ such that BE and CF intersect at a point O. AO is joined and produced to meet BC at D.

TO PROVE $AD \perp BC$.

CONSTRUCTION Join FE.



PROOF $BE \perp AC$ and $CF \perp AB$

$\Rightarrow \quad \angle BEC = \angle BFC = 90°$

$\Rightarrow \quad BC$ makes equal angles at E and F

$\Rightarrow \quad BCEF$ is a cyclic quadrilateral

$\Rightarrow \quad \angle ECB + \angle BFE = 180°$

$\Rightarrow \quad \angle ACD + \angle BFE = 180° \quad [\because \angle ECB = \angle ACD]$

$\Rightarrow \quad \angle ACD + (\angle BFO + \angle OFE) = 180°$

$\Rightarrow \quad \angle ACD + 90° + \angle OFE = 180° \quad [\because \angle BFO = 90°]$

$\Rightarrow \quad \angle ACD + \angle OFE = 90°$. … (i)

Again, $\angle OFA + \angle OEA = 90° + 90° = 180°$

$\Rightarrow \quad EOFA$ is a cyclic quadrilateral

$\Rightarrow \quad \angle OFE = \angle OAE \quad$ [$\angle$s in the same segment]. … (ii)

From (i) and (ii), we have

$$\angle ACD + \angle OAE = 90°$$

$\Rightarrow \quad \angle ACD + \angle DAC = 90°$ … (iii) $\quad [\because \angle OAE = \angle DAC]$.

We know that the sum of the angles of a triangle is 180°.

$\therefore \quad$ in $\triangle ADC$, we have

$$\angle ACD + \angle DAC + \angle ADC = 180°$$

$\Rightarrow \quad 90° + \angle ADC = 180° \quad$ [using (iii)]

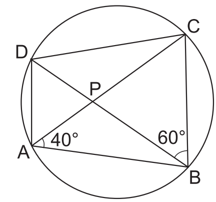
$\Rightarrow \quad \angle ADC = 90°$

$\Rightarrow \quad AD \perp BC$.

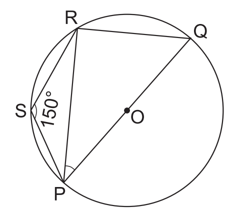
Hence, the altitudes AD, BE and CF are concurrent.

EXERCISE 12C

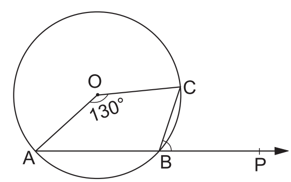
1. In the given figure, *ABCD* is a cyclic quadrilateral whose diagonals intersect at *P* such that $\angle DBC = 60°$ and $\angle BAC = 40°$.
Find (i) $\angle BCD$, (ii) $\angle CAD$.



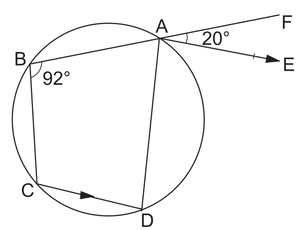
2. In the given figure, *POQ* is a diameter and *PQRS* is a cyclic quadrilateral. If $\angle PSR = 150°$, find $\angle RPQ$.



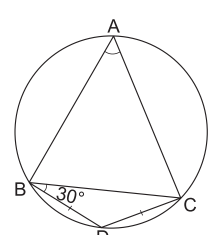
3. In the given figure, *O* is the centre of the circle and arc *ABC* subtends an angle of 130° at the centre. If *AB* is extended to *P*, find $\angle PBC$.



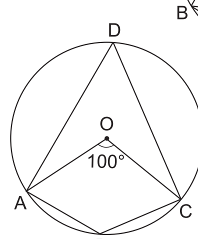
4. In the given figure, *ABCD* is a cyclic quadrilateral in which *AE* is drawn parallel to *CD*, and *BA* is produced to *F*. If $\angle ABC = 92°$ and $\angle FAE = 20°$, find $\angle BCD$.



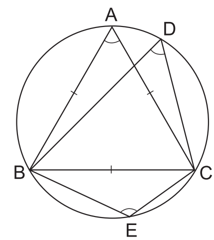
5. In the given figure, $BD = DC$ and $\angle CBD = 30°$, find $\angle BAC$.



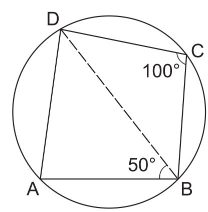
6. In the given figure, *O* is the centre of the given circle and measure of arc *ABC* is 100°. Determine $\angle ADC$ and $\angle ABC$.



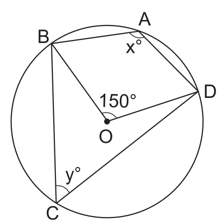
7. In the given figure, $\triangle ABC$ is equilateral. Find (i) $\angle BDC$, (ii) $\angle BEC$.



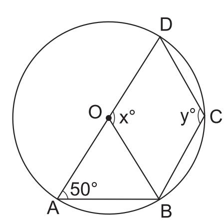
8. In the adjoining figure, $ABCD$ is a cyclic quadrilateral in which $\angle BCD = 100°$ and $\angle ABD = 50°$. Find $\angle ADB$.



9. In the given figure, O is the centre of a circle and $\angle BOD = 150°$. Find the values of x and y.

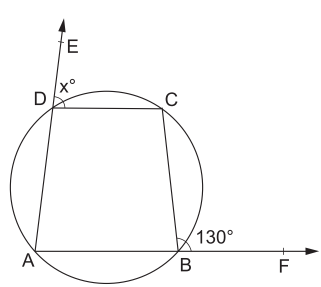


10. In the given figure, O is the centre of the circle and $\angle DAB = 50°$. Calculate the values of x and y.



11. In the given figure, sides AD and AB of cyclic quadrilateral $ABCD$ are produced to E and F respectively.

If $\angle CBF = 130°$ and $\angle CDE = x°$, find the value of x.



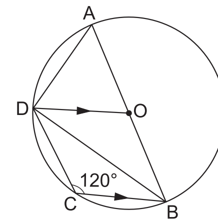
12. In the given figure, AB is a diameter of a circle with centre O and $DO \parallel CB$.

If $\angle BCD = 120°$, calculate

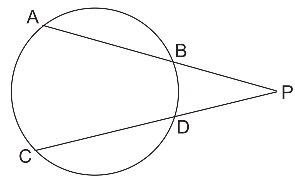
(i) $\angle BAD$, (ii) $\angle ABD$,

(iii) $\angle CBD$, (iv) $\angle ADC$.

Also, show that $\triangle AOD$ is an equilateral triangle.

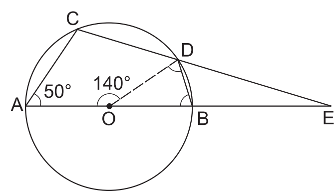


13. Two chords AB and CD of a circle intersect each other at P outside the circle. If $AB = 6$ cm, $BP = 2$ cm and $PD = 2.5$ cm, find CD.



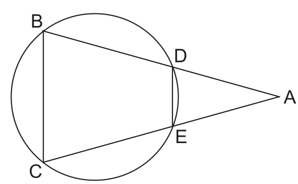
14. In the given figure, O is the centre of a circle. If $\angle AOD = 140°$ and $\angle CAB = 50°$, calculate

(i) $\angle EDB$, (ii) $\angle EBD$.

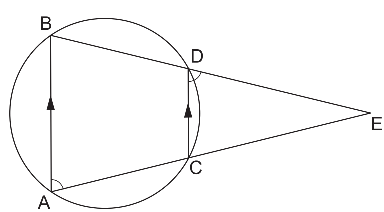


15. In the given figure, $\triangle ABC$ is an isosceles triangle in which $AB = AC$ and a circle passing through B and C intersects AB and AC at D and E respectively.

Prove that $DE \parallel BC$.

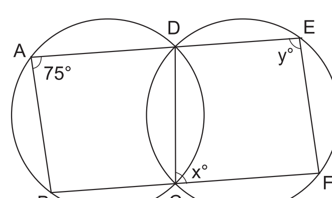


16. In the given figure, AB and CD are two parallel chords of a circle. If BDE and ACE are straight lines, intersecting at E, prove that $\triangle AEB$ is isosceles.

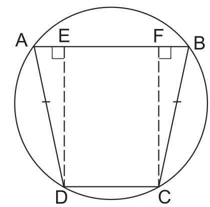


17. In the given figure, $\angle BAD = 75°$, $\angle DCF = x°$ and $\angle DEF = y°$.

Find the values of x and y.



18. In the given figure, $ABCD$ is a quadrilateral in which $AD = BC$ and $\angle ADC = \angle BCD$. Show that the points A, B, C, D lie on a circle.



19. Prove that the perpendicular bisectors of the sides of a cyclic quadrilateral are concurrent.

20. Prove that the circles described with the four sides of a rhombus as diameters pass through the point of intersection of its diagonals.

21. $ABCD$ is a rectangle. Prove that the centre of the circle through A, B, C, D is the point of intersection of its diagonals.

22. Give a geometrical construction for finding the fourth point lying on a circle passing through three given points, without finding the centre of the circle. Justify the construction.

23. In a cyclic quadrilateral $ABCD$, if $(\angle B - \angle D) = 60°$, show that the smaller of the two is 60°.

24. The diagonals of a cyclic quadrilateral are at right angles. Prove that the perpendicular from the point of their intersection on any side when produced backwards, bisects the opposite side.

25. On a common hypotenuse AB, two right triangles ACB and ADB are situated on opposite sides. Prove that $\angle BAC = \angle BDC$.

26. $ABCD$ is a quadrilateral such that A is the centre of the circle passing through B, C and D. Prove that $\angle CBD + \angle CDB = \frac{1}{2}\angle BAD$.

ANSWERS (EXERCISE 12C)

1. (i) $\angle BCD = 80°$ (ii) $\angle CAD = 60°$ **2.** $\angle RPQ = 60°$ **3.** $\angle PBC = 65°$

4. $\angle BCD = 108°$ **5.** $\angle BAC = 60°$ **6.** $\angle ADC = 50°, \angle ABC = 130°$

7. (i) $\angle BDC = 60°$ (ii) $\angle BEC = 120°$ **8.** $\angle ADB = 50°$

9. $x = 105, y = 75$ **10.** $x = 100, y = 130$ **11.** $x = 50$

12. (i) $\angle BAD = 60°$ (ii) $\angle ABD = 30°$ (iii) $\angle CBD = 30°$ (iv) $\angle ADC = 120°$

13. $CD = 3.9$ cm **14.** (i) $\angle EDB = 50°$ (ii) $\angle EBD = 110°$

17. $x = 75, y = 105$

HINTS TO SOME SELECTED QUESTIONS

1. $\angle BDC = \angle BAC = 40°$ [angles in the same segment].
   $\angle BCD = 180° - (40° + 60°) = 80°$.
   $\angle CAD = \angle CBD = 60°$ [angles in the same segment].
2. $\angle PSR + \angle PQR = 180° \Rightarrow \angle PQR = 30°$.
   $\angle PRQ = 90°$ [angle in a semicircle]
   $\therefore \quad 30° + 90° + \angle RPQ = 180° \Rightarrow \angle RPQ = 60°$.
4. $\angle ABC + \angle ADC = 180° \Rightarrow \angle ADC = 88°$.
   $AE \parallel CD \Rightarrow \angle EAD = \angle ADC = 88°$
   $\Rightarrow \angle DAF = \angle EAF + \angle EAD = 20° + 88° = 108°$.
   $\angle BCD = \angle DAF = 108°$ [$\because$ ext. $\angle$ of a cyclic quad. = int. opp. $\angle$].
5. $BD = DC \Rightarrow \angle BCD = \angle CBD = 30°$.
   $\angle BDC = 180° - (30° + 30°) = 120°$.
   $\angle BAC + \angle BDC = 180° \Rightarrow \angle BAC = 60°$.
9. Reflex $\angle BOD = (360° - 150°) = 210°$.
   $x° = \frac{1}{2}$ reflex $\angle BOD = 105°$.
   Now, $x° + y° = 180° \Rightarrow y° = 75$.
10. $ABCD$ is a cyclic quadrilateral.
    $\therefore \quad 50° + y° = 180° \Rightarrow y = 180 - 50 = 130$.
    $OA = OB \Rightarrow \angle OBA = \angle OAB = 50°$
    $\therefore \quad \angle AOB = 180° - (50° + 50°) = 80°$.
    $\therefore \quad x = 180 - 80 = 100$.
11. In a cyclic quadrilateral, we have, exterior angle = interior opposite angle.
    $\therefore \quad \angle CBF = \angle CDA = (180° - x°)$
    $\Rightarrow \quad 130 = 180 - x \Rightarrow x = 50$.
12. $\angle BCD + \angle BAD = 180° \Rightarrow \angle BAD = 60°$.
    $\angle BAD = 90°$ [angle in a semicircle]
    $\angle BDA + \angle BAD + \angle ABD = 180° \Rightarrow \angle ABD = 30°$.
    $OD = OA \Rightarrow \angle ODA = \angle OAD = \angle BAD = 60°$.
    $\therefore \quad \angle ODB = 90° - \angle ODA = 30°$.
    $\therefore \quad \angle CBD = \angle ODB = 30° \Rightarrow \angle CDB = 180° - (120° + 30°) = 30°$
    and $\angle ADC = \angle ADB + \angle CDB = (90° + 30°) = 120°$.
13. $AP \times BP = CP \times DP$
    $\Rightarrow 8 \times 2 = (CD + 2.5) \times 2.5$
    $\Rightarrow CD = 3.9$ cm.
14. $\angle BOD = 180° - \angle AOD = 180° - 140° = 40°$.
    $OB = OD \Rightarrow \angle OBD = \angle ODB = 70°$.
    $\angle CAB + \angle BDC = 180°$ [$\because$ $ABDC$ is cyclic]
    $\Rightarrow \quad \angle CAB + \angle ODB + \angle ODC = 180°$
    $\Rightarrow \quad 50° + 70° + \angle ODC = 180°$

$\Rightarrow \quad \angle ODC = 60°$.

$\therefore \quad \angle EDB = 180° - (60° + 70°) = 50°$

and $\angle EBD = 180° - \angle OBD = 180° - 70° = 110°$.

15. $AB = AC \Rightarrow \angle ACB = \angle ABC$.

$\therefore \quad$ ext. $\angle ADE = \angle ACB = \angle ABC$.

Thus, $\angle ADE = \angle ABC \Rightarrow DE \parallel BC$.

16. Ext. $\angle EDC = \angle A$, ext. $\angle DCE = \angle B$.

$AB \parallel CD \Rightarrow \angle EDC = \angle B$ and $\angle DCE = \angle A$.

$\therefore \quad \angle A = \angle B$.

17. $\angle BAD + \angle BCD = 180° \Rightarrow \angle BCD = 105°$.

$\therefore \quad x = 180 - 105 = 75$.

$x + y = 180 \Rightarrow y = 180 - 75 = 105$.

18. Draw $DE \perp AB$ and $CF \perp AB$.

In $\triangle ADE$ and $\triangle BCF$, we have

$\angle ADE = \angle ADC - 90° \Rightarrow \angle BCD - 90° = \angle BCF \quad [\because \angle ADC = \angle BCD$ (given)],

$AD = BC \quad$ [given]

and $\angle AED = \angle BFC = 90°$.

$\therefore \quad \triangle ADE \cong \triangle BCF \quad$ [by RHS-congruence]

$\Rightarrow \quad \angle A = \angle B$.

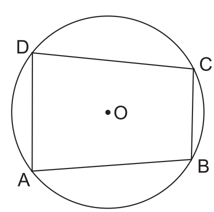
$\therefore \quad \angle A + \angle B + \angle C + \angle D = 360°$

$\Rightarrow \quad 2\angle B + 2\angle D = 360° \quad [\because \angle C = \angle D$ (given)]

$\Rightarrow \quad \angle B + \angle D = 180°$ and therefore, $\angle A + \angle C = 180°$.

Hence, the points A, B, C, D are concyclic.

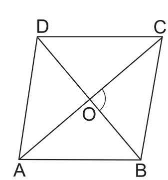
19. Let $ABCD$ be a cyclic quadrilateral and let O be the centre of the circle, passing through A, B, C, D. Then, each of AB, BC, CD and DA being a chord of the circle, its right bisector must pass through O. Hence, the right bisectors of AB, BC, CD and DA pass through O.



20. Let the diagonals AC and BD of a rhombus $ABCD$ intersect at O. But, the diagonals of a rhombus bisect each other at right angles. So, $\angle BOC = 90°$.

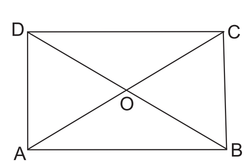
$\therefore \quad \angle BOC$ lies in a semicircle.

Thus, the circle drawn with BC as diameter will pass through O. Similarly, all the circles described with AB, AD and CD as diameters, will pass through O.

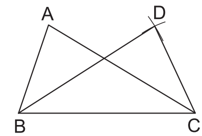


21. Let O be the point of intersection of the diagonals AC and BD of rect. $ABCD$. Since the diagonals of a rectangle are equal and bisect each other, we have, $OA = OB = OC = OD$.

Hence, O is the centre of the circle through A, B, C, D.



22. Let A, B, C be the given points. With B as centre and radius equal to AC draw an arc. With C as centre and AB as radius draw another arc, intersecting the previous arc at D. Then, D is the desired point.



PROOF Join BD and CD.

$\triangle ABC \cong \triangle DCB$ [$\because$ $AB = DC$, $AC = DB$ and $BC = CB$]

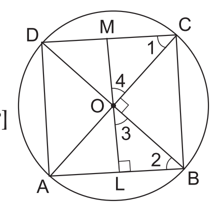
$\Rightarrow$ $\angle BAC = \angle CDB$.

Thus, BC subtends equal angles $\angle BAC$ and $\angle CDB$ on the same side of it.

$\therefore$ points A, B, C, D are cyclic.

23. $\angle B - \angle D = 60$ and $\angle B + \angle D = 180° \Rightarrow \angle B = 120°$ and $\angle D = 60°$.

24. Let $ABCD$ be a cyclic quadrilateral whose diagonals AC and BD intersect at O at right angles. Let $OL \perp AB$ such that LO produced meets CD at M.



Then, we have to prove that $CM = MD$.

Clearly, $\angle 1 = \angle 2$ [$\angle$s in the same segment]

$\angle 2 + \angle 3 = 90°$ [$\because$ $\angle OLB = 90°$]

$\angle 3 + \angle 4 = 90°$ [$\because$ LOM is a straight line and $\angle BOC = 90°$]

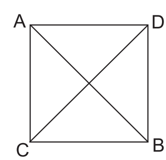
$\therefore$ $\angle 2 + \angle 3 = \angle 3 + \angle 4 \Rightarrow \angle 2 = \angle 4$.

Thus, $\angle 1 = \angle 2$ and $\angle 2 = \angle 4 \Rightarrow \angle 1 = \angle 4$.

$\therefore$ $OM = CM$ and similarly, $OM = MD$.

Hence, $CM = MD$.

25. Clearly, $\angle ACB = 90°$ and $\angle ADB = 90°$.



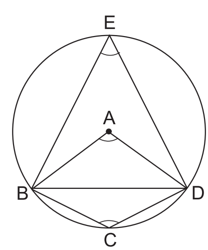
Thus, the opposite angles of quad. $ACBD$ are supplementary.

$\therefore$ $ACBD$ is a cyclic quadrilateral.

This means that a circle passes through the points A, C, B and D.

$\therefore$ $\angle BAC = \angle BDC$ [$\angle$s in the same segment].

26. With A as centre and AB as radius, draw a circle passing through the points B, C and D.



Take a point E on the circle outside arc BCD. Join BE, DE and BD.

Clearly, $\angle BAD = 2\angle BED$. ... (i)

Now, $EBCD$ is a cyclic quadrilateral.

$\therefore$ $\angle BED + \angle BCD = 180°$

$\Rightarrow$ $\angle BCD = 180° - \angle BED$

$\Rightarrow$ $\angle BCD = 180° - \frac{1}{2}\angle BAD$ ... (ii) [using (i)].

In $\triangle BCD$, we have

$\angle CBD + \angle BCD + \angle CDB = 180°$

$\Rightarrow$ $\angle CBD + \angle CDB = 180° - \angle BCD$

$= 180° - \left(180° - \frac{1}{2}\angle BAD\right) = \frac{1}{2}\angle BAD$ [using (ii)].

MULTIPLE-CHOICE QUESTIONS (MCQ)

Choose the correct answer in each of the following:

1. The radius of a circle is 13 cm and the length of one of its chords is 10 cm. The distance of the chord from the centre is

(a) 11.5 cm (b) 12 cm

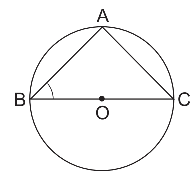
(c) $\sqrt{69}$ cm (d) 23 cm

2. A chord is at a distance of 8 cm from the centre of a circle of radius 17 cm. The length of the chord is

(a) 25 cm (b) 12.5 cm

(c) 30 cm (d) 9 cm

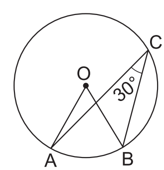
3. In the given figure, BOC is a diameter of a circle and $AB = AC$. Then, $\angle ABC = ?$



(a) 30° (b) 45°

(c) 60° (d) 90°

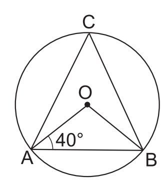
4. In the given figure, O is the centre of a circle and $\angle ACB = 30°$. Then, $\angle AOB = ?$



(a) 30° (b) 15°

(c) 60° (d) 90°

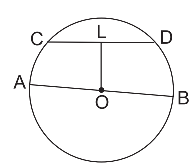
5. In the given figure, O is the centre of a circle. If $\angle OAB = 40°$ and C is a point on the circle then $\angle ACB = ?$



(a) 40° (b) 50°

(c) 80° (d) 100°

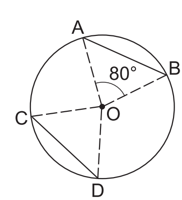
6. In the given figure, AOB is a diameter of a circle with centre O such that $AB = 34$ cm and CD is a chord of length 30 cm. Then, the distance of CD from AB is



(a) 8 cm (b) 15 cm

(c) 18 cm (d) 6 cm

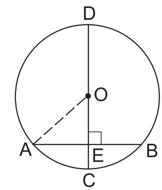
7. AB and CD are two equal chords of a circle with centre O such that $\angle AOB = 80°$. Then, $\angle COD = ?$



(a) 100° (b) 80°

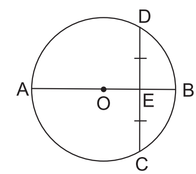
(c) 120° (d) 40°

8. In the given figure, CD is the diameter of a circle with centre O and CD is perpendicular to chord AB. If $AB = 12$ cm and $CE = 3$ cm then radius of the circle is



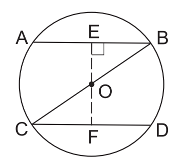
(a) 6 cm (b) 9 cm
(c) 7.5 cm (d) 8 cm

9. In the given figure, O is the centre of a circle and diameter AB bisects the chord CD at a point E such that $CE = ED = 8$ cm and $EB = 4$ cm. The radius of the circle is



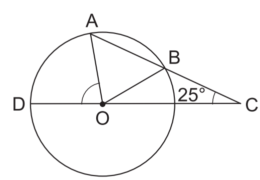
(a) 10 cm (b) 12 cm
(c) 6 cm (d) 8 cm

10. In the given figure, BOC is a diameter of a circle with centre O. If AB and CD are two chords such that $AB \parallel CD$ and $AB = 10$ cm then $CD = ?$



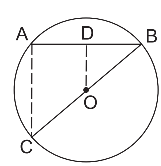
(a) 5 cm (b) 12.5 cm
(c) 15 cm (d) 10 cm

11. In the given figure, AB is a chord of a circle with centre O and AB is produced to C such that $BC = OB$. Also, CO is joined and produced to meet the circle in D. If $\angle ACD = 25°$ then $\angle AOD = ?$



(a) 50° (b) 75°
(c) 90° (d) 100°

12. In the given figure, AB is a chord of a circle with centre O and BOC is a diameter. If $OD \perp AB$ such that $OD = 6$ cm then $AC = ?$



(a) 9 cm (b) 12 cm
(c) 15 cm (d) 7.5 cm

13. An equilateral triangle of side 9 cm is inscribed in a circle. The radius of the circle is

(a) 3 cm (b) $3\sqrt{2}$ cm (c) $3\sqrt{3}$ cm (d) 6 cm

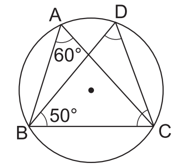
14. The angle in a semicircle measures

(a) 45° (b) 60° (c) 90° (d) 36°

15. Angles in the same segment of a circle are

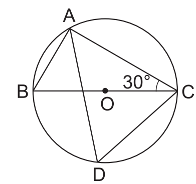
(a) equal (b) complementary
(c) supplementary (d) none of these

16. In the given figure, $\triangle ABC$ and $\triangle DBC$ are inscribed in a circle such that $\angle BAC = 60°$ and $\angle DBC = 50°$. Then, $\angle BCD = ?$



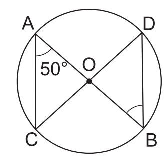
(a) 50° (b) 60°
(c) 70° (d) 80°

17. In the given figure, BOC is a diameter of a circle with centre O. If $\angle BCA = 30°$ then $\angle CDA = ?$



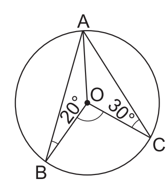
(a) 30° (b) 45°
(c) 60° (d) 50°

18. In the given figure, O is the centre of a circle. If $\angle OAC = 50°$ then $\angle ODB = ?$



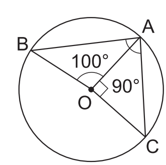
(a) 40° (b) 50°
(c) 60° (d) 75°

19. In the given figure, O is the centre of a circle in which $\angle OBA = 20°$ and $\angle OCA = 30°$. Then, $\angle BOC = ?$



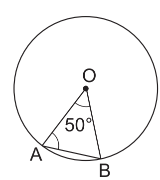
(a) 50° (b) 90°
(c) 100° (d) 130°

20. In the given figure, O is the centre of a circle. If $\angle AOB = 100°$ and $\angle AOC = 90°$ then $\angle BAC = ?$



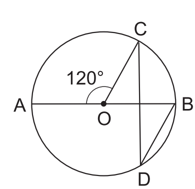
(a) 85° (b) 80°
(c) 95° (d) 75°

21. In the given figure, O is the centre of a circle. Then, $\angle OAB = ?$



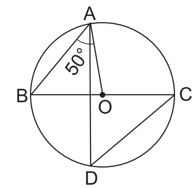
(a) 50° (b) 60°
(c) 55° (d) 65°

22. In the given figure, O is the centre of a circle and $\angle AOC = 120°$. Then, $\angle BDC = ?$



(a) 60° (b) 45°
(c) 30° (d) 15°

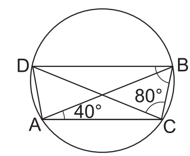
23. In the given figure, O is the centre of a circle and $\angle OAB = 50°$. Then, $\angle CDA = ?$



(a) 40° (b) 50°

(c) 75° (d) 25°

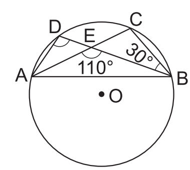
24. In the given figure, AB and CD are two intersecting chords of a circle. If $\angle CAB = 40°$ and $\angle BCD = 80°$ then $\angle CBD = ?$



(a) 80° (b) 60°

(c) 50° (d) 70°

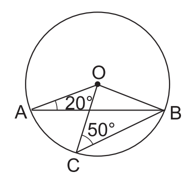
25. In the given figure, O is the centre of a circle and chords AC and BD intersect at E. If $\angle AEB = 110°$ and $\angle CBE = 30°$ then $\angle ADB = ?$



(a) 70° (b) 60°

(c) 80° (d) 90°

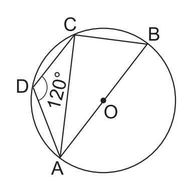
26. In the given figure, O is the centre of a circle in which $\angle OAB = 20°$ and $\angle OCB = 50°$. Then, $\angle AOC = ?$



(a) 50° (b) 70°

(c) 20° (d) 60°

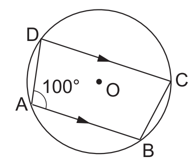
27. In the given figure, AOB is a diameter and $ABCD$ is a cyclic quadrilateral. If $\angle ADC = 120°$ then $\angle BAC = ?$



(a) 60° (b) 30°

(c) 20° (d) 45°

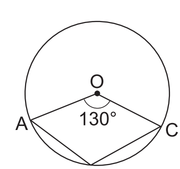
28. In the given figure, $ABCD$ is a cyclic quadrilateral in which $AB \parallel DC$ and $\angle BAD = 100°$. Then, $\angle ABC = ?$



(a) 80° (b) 100°

(c) 50° (d) 40°

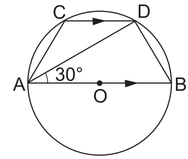
29. In the given figure, O is the centre of a circle and $\angle AOC = 130°$. Then, $\angle ABC = ?$



(a) 50° (b) 65°

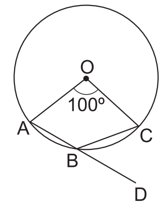
(c) 115° (d) 130°

30. In the given figure, AOB is a diameter of a circle and $CD \parallel AB$. If $\angle BAD = 30°$ then $\angle CAD = ?$



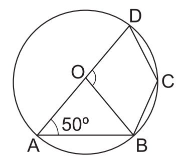
(a) 30° (b) 60°
(c) 45° (d) 50°

31. In the given figure, O is the centre of a circle in which $\angle AOC = 100°$. Side AB of quad. $OABC$ has been produced to D. Then, $\angle CBD = ?$



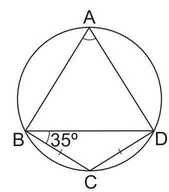
(a) 50° (b) 40°
(c) 25° (d) 80°

32. In the given figure, O is the centre of a circle and $\angle OAB = 50°$. Then, $\angle BOD = ?$



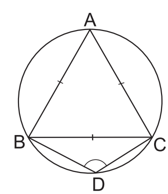
(a) 130° (b) 50°
(c) 100° (d) 80°

33. In the given figure, $ABCD$ is a cyclic quadrilateral in which $BC = CD$ and $\angle CBD = 35°$. Then, $\angle BAD = ?$



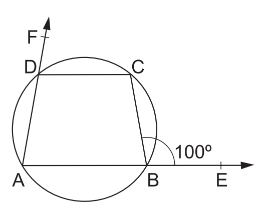
(a) 65° (b) 70°
(c) 110° (d) 90°

34. In the given figure, equilateral $\triangle ABC$ is inscribed in a circle and $ABDC$ is a quadrilateral, as shown. Then, $\angle BDC = ?$



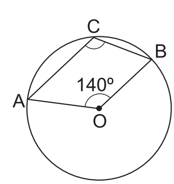
(a) 90° (b) 60°
(c) 120° (d) 150°

35. In the given figure, sides AB and AD of quad. $ABCD$ are produced to E and F respectively. If $\angle CBE = 100°$ then $\angle CDF = ?$



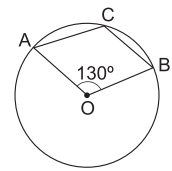
(a) 100° (b) 80°
(c) 130° (d) 90°

36. In the given figure, O is the centre of a circle and $\angle AOB = 140°$. Then, $\angle ACB = ?$



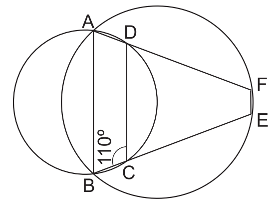
(a) 70° (b) 80°
(c) 110° (d) 40°

37. In the given figure, O is the centre of a circle and $\angle AOB = 130°$. Then, $\angle ACB = ?$



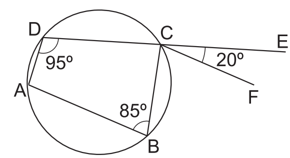
(a) 50° (b) 65°
(c) 115° (d) 155°

38. In the given figure, $ABCD$ and $ABEF$ are two cyclic quadrilaterals. If $\angle BCD = 110°$ then $\angle BEF = ?$



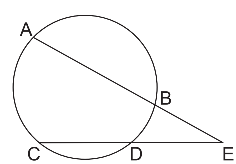
(a) 55° (b) 70°
(c) 90° (d) 110°

39. In the given figure, $ABCD$ is a cyclic quadrilateral in which DC is produced to E and CF is drawn parallel to AB such that $\angle ADC = 95°$ and $\angle ECF = 20°$. Then, $\angle BAD = ?$



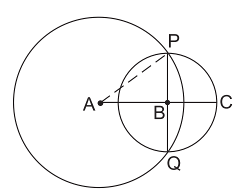
(a) 95° (b) 85°
(c) 105° (d) 75°

40. Two chords AB and CD of a circle intersect each other at a point E outside the circle. If $AB = 11$ cm, $BE = 3$ cm and $DE = 3.5$ cm, then $CD = ?$



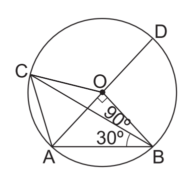
(a) 10.5 cm (b) 9.5 cm
(c) 8.5 cm (d) 7.5 cm

41. In the given figure, A and B are the centres of two circles having radii 5 cm and 3 cm respectively and intersecting at points P and Q respectively. If $AB = 4$ cm then the length of common chord PQ is



(a) 3 cm (b) 6 cm
(c) 7.5 cm (d) 9 cm

42. In the given figure, $\angle AOB = 90°$ and $\angle ABC = 30°$. Then, $\angle CAO = ?$



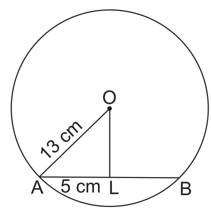
(a) 30° (b) 45°
(c) 60° (d) 90°

ANSWERS (MCQ)

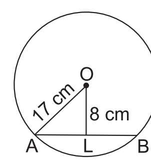
1. (b)	**2.** (c)	**3.** (b)	**4.** (c)	**5.** (b)	**6.** (a)	**7.** (b)	**8.** (c)
9. (a)	**10.** (d)	**11.** (b)	**12.** (b)	**13.** (c)	**14.** (c)	**15.** (a)	**16.** (c)
17. (c)	**18.** (b)	**19.** (c)	**20.** (a)	**21.** (d)	**22.** (c)	**23.** (b)	**24.** (b)
25. (c)	**26.** (d)	**27.** (b)	**28.** (b)	**29.** (c)	**30.** (a)	**31.** (a)	**32.** (c)
33. (b)	**34.** (c)	**35.** (b)	**36.** (c)	**37.** (c)	**38.** (d)	**39.** (c)	**40.** (c)
41. (b)	**42.** (c)						

HINTS TO SOME SELECTED QUESTIONS

1. Let O be the centre of a circle with radius $OA = 13$ cm.
Let AB be the chord of length 10 cm. Draw $OL \perp AB$.
Then, L is the midpoint of AB. Thus, $AL = 5$ cm.
$$OL^2 = OA^2 - AL^2 = (13)^2 - (5)^2 = 169 - 25 = 144$$
$\Rightarrow \quad OL = \sqrt{144} = 12$ cm.
Distance of the chord from the centre = 12 cm.



2. Here $OA = 17$ cm and $OL = 8$ cm.
$\therefore \quad AL^2 = OA^2 - OL^2 = (17)^2 - (8)^2 = 289 - 64 = 225.$
$\Rightarrow \quad AL = \sqrt{225} = 15$
$\Rightarrow \quad AB = (2 \times AL) = (2 \times 15)$ cm $= 30$ cm.

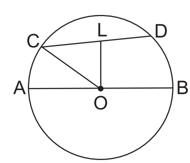


3. Since an angle in a semicircle is a right angle, $\angle BAC = 90°$.
$\therefore \quad \angle ABC + \angle ACB = 90°.$
Now, $AB = AC \Rightarrow \angle ABC = \angle ACB = 45°.$

4. Clearly, $\angle AOB = (2 \times \angle AOC) = (2 \times 30°) = 60°.$

5. $OA = OB \Rightarrow \angle OBA = \angle OAB = 40°.$
$\therefore \quad \angle AOB = 180° - (40° + 40°) = 100°.$
$\therefore \quad \angle ACB = \frac{1}{2}\angle AOB = \frac{1}{2} \times 100° = 50°.$

6. Join OC. Then, OC = radius = 17 cm.
$CL = \frac{1}{2}CD = \left(\frac{1}{2} \times 30\right)$ cm $= 15$ cm.
$OL^2 = OC^2 - CL^2 = (17)^2 - (15)^2 = (17 + 15)(17 - 15) = (32 \times 2) = 64$
$\Rightarrow \quad OL = \sqrt{64} = 8$ cm.
$\therefore \quad$ distance of CD from $AB = 8$ cm.



7. Since equal chords of a circle subtend equal angles at the centre, so $\angle COD = \angle AOB = 80°.$

8. Let $OA = OC = r$ cm. Then, $OE = (r - 3)$ cm and $AE = \frac{1}{2}AB = 6$ cm.
Now, $OA^2 = OE^2 + AE^2 \Rightarrow r^2 = (r - 3)^2 + 6^2$
$\Rightarrow \quad 6r = 45 \Rightarrow r = \frac{45}{6} = 7.5$ cm.

9. Let the radius of the circle be r cm.

   Then, $OD = OB = r$ cm, $OE = (r-4)$ cm, $ED = 8$ cm.

   Now, $OD^2 = OE^2 + ED^2 \Rightarrow r^2 = (r-4)^2 + 8^2$

   $\therefore \quad 8r = 80 \Rightarrow r = 10$ cm.

10. Draw $OE \perp AB$ and $OF \perp CD$.

    Then, $\triangle OEB \cong \triangle OFC$

    [$\because$ $OB = OC = r$, $\angle BOE = \angle COF$(vert. opp. $\angle$s) and $\angle OEB = \angle OFC = 90°$].

    $\therefore \quad OE = OF$.

    But, the chords equidistant from the centre are equal.

    $\therefore \quad CD = AB = 10$ cm.

11. $OB = BC \Rightarrow \angle BOC = \angle BCO = 25°$

    Ext. $\angle OBA = \angle BOC + \angle BCO = 25° + 25° = 50°$.

    $OA = OB \Rightarrow \angle OAB = \angle OBA = 50°$.

    In $\triangle AOC$, side CO has been produced to D.

    $\therefore \quad$ ext. $\angle AOD = \angle OAC + \angle ACO = \angle OAB + \angle BCO = 50° + 25° = 75°$.

12. $OD \perp AB \Rightarrow D$ is the midpoint of AB.

    Also, O is the midpoint of BC.

    Now, in $\triangle BAC$, D is the midpoint of AB and O is the midpoint of BC.

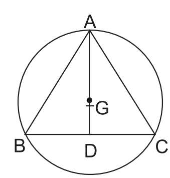
    $\therefore \quad OD = \frac{1}{2}AC \Rightarrow AC = 2 \times OD = (2 \times 6)\text{ cm} = 12$ cm.

13. Let $\triangle ABC$ be an equilateral triangle of side 9 cm.

    Let AD be one of its medians.

    Then, $AD \perp BC$ and $BD = 4.5$ cm.

    $\therefore \quad AD = \sqrt{AB^2 - BD^2} = \sqrt{(9)^2 - \left(\frac{9}{2}\right)^2} = \sqrt{\frac{243}{4}} = \frac{9\sqrt{3}}{2}$ cm.

    Let G be the centroid of $\triangle ABC$.

    Then, $AG : GD = 2 : 1$.

    $\therefore \quad$ radius $= AG = \frac{2}{3}AD = \left(\frac{2}{3} \times \frac{9\sqrt{3}}{2}\right)$ cm $= 3\sqrt{3}$ cm.

14. The angle in a semicircle measures 90°.

15. The angles in the same segment of a circle are equal.

16. $\angle BDC = \angle BAC = 60°$ ($\angle$s in the same segment of a circle).

    In $\triangle BDC$, $\angle DBC + \angle BDC + \angle BCD = 180°$.

    $\therefore \quad 50° + 60° + \angle BCD = 180° \Rightarrow \angle BCD = 70°$.

17. $\angle BAC = 90°$ (angle in a semicircle).

    In $\triangle ABC$, $\angle BAC + \angle ABC + \angle BCA = 180°$.

    $\therefore \quad 90° + \angle ABC + 30° = 180° \Rightarrow \angle ABC = 60°$.

    $\therefore \quad \angle CDA = \angle ABC = 60°$ (angles in the same segment of a circle).

18. $\angle ODB = \angle OAC = 50°$ ($\angle$s in the same segment).

    Now, $OB = OD$ (radii of the same circle)

    $\Rightarrow \quad \angle OBD = \angle ODB = 50°$.

19. $\left.\begin{array}{l}\text{In } \triangle OAB, OA = OB \Rightarrow \angle OAB = \angle OBA = 20^\circ \\ \text{In } \triangle OAC, OA = OC \Rightarrow \angle OAC = \angle OCA = 30^\circ\end{array}\right\} \Rightarrow \angle BAC = 20^\circ + 30^\circ = 50^\circ.$

$\therefore \quad \angle BOC = 2 \times \angle BAC = 2 \times 50^\circ = 100^\circ.$

20. $\angle BOC + \angle BOA + \angle AOC = 360^\circ \Rightarrow \angle BOC = 360^\circ - 190^\circ = 170^\circ$

$\therefore \quad \angle BAC = \frac{1}{2} \times 170^\circ = 85^\circ.$

21. $OA = OB$ (radii of the same circle) $\Rightarrow \angle OAB = \angle OBA = x^\circ.$

Then, $x + x + 50 = 180 \Rightarrow 2x = 130 \Rightarrow x = 65.$

22. $\angle COB = 180^\circ - \angle AOC = 180^\circ - 120^\circ = 60^\circ.$

Now, arc BC subtends $\angle BOC$ at the centre and $\angle BDC$ at a point D of the remaining part of the circle.

$\therefore \quad \angle COB = 2\angle BDC \Rightarrow \angle BDC = \frac{1}{2}\angle COB = \frac{1}{2} \times 60^\circ = 30^\circ.$

23. $OA = OB \Rightarrow \angle OBA = \angle OAB = 50^\circ.$

$\therefore \quad \angle CDA = \angle OBA = 50^\circ$ ($\angle$s in the same segment).

24. $\angle CDB = \angle CAB = 40^\circ$ ($\angle$s in the same segment).

$\angle CDB + \angle BCD + \angle CBD = 180^\circ \Rightarrow 40^\circ + 80^\circ + \angle CBD = 180^\circ \Rightarrow \angle CBD = 60^\circ.$

25. $\angle AEB + \angle CEB = 180^\circ$

$\Rightarrow \quad 110^\circ + \angle CEB = 180^\circ \Rightarrow \angle CEB = 70^\circ.$

In $\triangle CEB$, $\angle CEB + \angle EBC + \angle ECB = 180^\circ$

$\Rightarrow \quad 70^\circ + 30^\circ + \angle ECB = 180^\circ \Rightarrow \angle ECB = 80^\circ.$

$\therefore \quad \angle ADB = \angle ACB = \angle ECB = 80^\circ$ ($\angle$s in the same segment).

26. $OA = OB \Rightarrow \angle OBA = \angle OAB = 20^\circ.$

In $\triangle OAB$, $\angle OAB + \angle OBA + \angle AOB = 180^\circ$

$\Rightarrow \quad 20^\circ + 20^\circ + \angle AOB = 180^\circ \Rightarrow \angle AOB = 140^\circ.$

$OB = OC \Rightarrow \angle OBC = \angle OCB = 50^\circ.$

In $\triangle OCB$, $\angle OCB + \angle OBC + \angle COB = 180^\circ$

$\Rightarrow \quad 50^\circ + 50^\circ + \angle COB = 180^\circ \Rightarrow \angle COB = 80^\circ.$

$\angle AOB = 40^\circ \Rightarrow \angle AOC + \angle COB = 140^\circ$

$\Rightarrow \angle AOC + 80^\circ = 140^\circ \Rightarrow \angle AOC = 60^\circ.$

27. $\angle ABC + \angle ADC = 180^\circ$ (opp. $\angle$s of cyclic quad.)

$\Rightarrow \quad \angle ABC + 120^\circ = 180^\circ \Rightarrow \angle ABC = 60^\circ.$

Also, $\angle ACB = 90^\circ$ (angle in a semicircle).

In $\triangle ABC$, $\angle BAC + \angle ACB + \angle ABC = 180^\circ$

$\Rightarrow \quad \angle BAC + 90^\circ + 60^\circ = 180^\circ \Rightarrow \angle BAC = 30^\circ.$

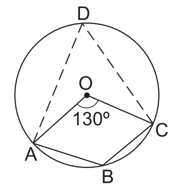
28. Since $ABCD$ is a cyclic quadrilateral, we have

$$\angle BAD + \angle BCD = 180^\circ \Rightarrow 100^\circ + \angle BCD = 180^\circ \Rightarrow \angle BCD = 80^\circ.$$

Now, $AB \parallel DC$ and CB is the transversal.

$\therefore \quad \angle ABC + \angle BCD = 180^\circ \Rightarrow \angle ABC + 80^\circ = 180^\circ \Rightarrow \angle ABC = 100^\circ.$

29. Take a point D on the remaining part of the circumference.
Join AD and CD. Then,



$$\angle ADC = \frac{1}{2}\angle AOC = \frac{1}{2} \times 130° = 65°.$$

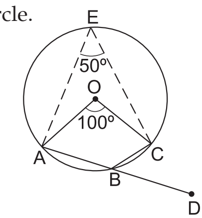
In cyclic quadrilateral $ABCD$, we have

$$\angle ABC + \angle ADC = 180° \Rightarrow \angle ABC + 65° = 180°$$
$$\Rightarrow \angle ABC = 180° - 65° = 115°.$$

30. $\angle ADC = \angle BAD = 30°$ [alt. int. $\angle$s].
$\angle ADB = 90°$ (angle in a semicircle).
$\therefore \quad \angle CDB = 30° + 90° = 120°.$
But, $ABCD$ being a cyclic quadrilateral, we have

$$\angle BAC + \angle CDB = 180° \Rightarrow \angle BAD + \angle CAD + \angle CDB = 180°$$
$\Rightarrow \quad 30° + \angle CAD + 120° = 180° \Rightarrow \angle CAD = 30°.$

31. Take a point E on the remaining part of circumference of the circle.
Join AE and CE.



$$\angle AEC = \frac{1}{2}\angle AOC = \frac{1}{2} \times 100° = 50°.$$

Now, side AB of cyclic quad. $ABCE$ has been produced to D.
So, ext. $\angle CBD = \angle AEC = 50°.$

32. $OA = OB \Rightarrow \angle OBA = \angle OAB = 50°.$
In $\triangle OAB$, $\angle OAB + \angle OBA + \angle AOB = 180° \Rightarrow 50° + 50° + \angle AOB = 180°.$
$\therefore \quad \angle AOB = 80°.$
$\therefore \quad \angle BOD = 180° - 80° = 100°.$

33. $BC = CD \Rightarrow \angle BDC = \angle CBD = 35°.$
In $\triangle BCD$, $\angle BCD + \angle BDC + \angle CBD = 180°$
$\Rightarrow \quad \angle BCD + 35° + 35° = 180° \Rightarrow \angle BCD = 110°.$
In cyclic quad. $ABCD$, $\angle BAD + \angle BCD = 180°.$
$\therefore \quad \angle BAD + 110° = 180° \Rightarrow \angle BAD = 70°.$

34. Since $\triangle ABC$ is an equilateral triangle, each of its angles is 60°.
$\therefore \quad \angle BAC = 60°.$
In cyclic quad. $ABDC$, we have
$\therefore \quad \angle BAC + \angle BDC = 180° \Rightarrow 60° + \angle BDC = 180° \Rightarrow \angle BDC = 120°.$

35. In cyclic quad. $ABCD$, int. opp. $\angle ADC$ = ext. $\angle CBE = 100°.$
$\therefore \quad \angle CDF = 180° - \angle ADC = 180° - 100° = 80°.$

36. $\angle BOC = 180° - 140° = 40°.$
$OB = OC \Rightarrow \angle OBC = \angle OCB = x°$ (say)
$\therefore \quad 40 + x + x = 180 \Rightarrow 2x = 140 \Rightarrow x = 70.$
$\therefore \quad \angle OBC = 70°.$
$\angle OBC + \angle ADC = 180° \Rightarrow \angle ADC = 180° - 70° = 110°.$

37. $\angle ABC = \frac{1}{2}\angle AOC = \frac{1}{2} \times 130° = 65°.$

$ABCD$ is a cyclic quadrilateral.

$\therefore \quad \angle ABC + \angle ADC = 180^\circ \Rightarrow 65^\circ + \angle ADC = 180^\circ$

$\Rightarrow \angle ADC = 180^\circ - 65^\circ = 115^\circ.$

38. Since $ABCD$ is a cyclic quadrilateral, we have

$\angle BAD + \angle BCD = 180^\circ \Rightarrow \angle BAD + 110^\circ = 180^\circ \Rightarrow \angle BAD = 70^\circ.$

Again, $ABEF$ is a cyclic quadrilateral.

$\therefore \quad \angle BAD + \angle BEF = 180^\circ \Rightarrow 70^\circ + \angle BEF = 180^\circ \Rightarrow \angle BEF = 110^\circ.$

39. $\angle ABC + \angle ADC = 180^\circ \Rightarrow \angle ABC + 95^\circ = 180^\circ \Rightarrow \angle ABC = 85^\circ.$

Now, $CF \parallel AB$ and CB is the transversal.

$\therefore \quad \angle BCF = \angle ABC = 85^\circ$ [alt. int. $\angle$s]

$\Rightarrow \quad \angle BCE = 85^\circ + 20^\circ = 105^\circ$

$\Rightarrow \quad \angle DCB = 180^\circ - 105^\circ = 75^\circ.$

Now, $\angle BAD + \angle BCD = 180^\circ \Rightarrow \angle BAD + 75^\circ = 180^\circ \Rightarrow \angle BAD = 105^\circ.$

40. Join AC. Then,

$AE : CE = DE : BE.$

$\therefore \quad AE \times BE = CE \times DE.$

Let $CD = x$ cm. Then,

$AE = (AB + BE) = (11 + 3)$ cm $= 14$ cm, $BE = 3$ cm, $CE = (x + 3.5)$ cm and $DE = 3.5$ cm.

$\therefore \quad 14 \times 3 = (x + 3.5) \times 3.5 \Rightarrow x + 3.5 = \dfrac{14 \times 3}{3.5} = 12$

$\Rightarrow x = (12 - 3.5) = 8.5$ cm.

$\therefore \quad CD = 8.5$ cm.

41. We know that the line joining their centres is the perpendicular bisector of the common chord. Join AP.

Then, $AP = 5$ cm, $BP = 3$ cm and $AB = 4$ cm.

And, $AP^2 = BP^2 + AB^2 \quad [\because 5^2 = 3^2 + 4^2]$

$\therefore \quad \triangle ABP$ is right-angled and $PQ = 2 \times BP = (2 \times 3)$ cm $= 6$ cm.

42. $\angle AOB = 2\angle ACB \Rightarrow \angle ACB = \frac{1}{2}\angle AOB = \left(\frac{1}{2} \times 90^\circ\right) = 45^\circ.$

$\angle COA = 2\angle CBA = (2 \times 30^\circ) = 60^\circ.$

$\therefore \quad \angle COD = 180^\circ - \angle COA = 180^\circ - 60^\circ = 120^\circ$

$\Rightarrow \quad \angle CAO = \frac{1}{2}\angle COD = \frac{1}{2} \times 120^\circ = 60^\circ.$

REVIEW OF FACTS AND FORMULAE

1. Two circles are congruent if and only if they have equal radii.
2. (i) Equal chords of a circle subtend equal angles at the centre.

   (ii) If two chords subtend equal angles at the centre of a circle then these chords are equal.

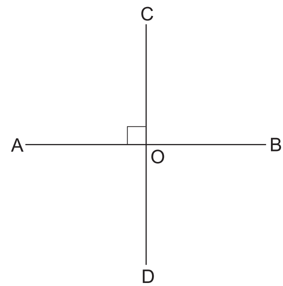
3. (i) Congruent arcs form equal chords.
   (ii) Equal chords cut congruent arcs.
4. (i) The perpendicular from the centre of a circle to a chord bisects the chord.
   (ii) The line drawn through the centre of a circle to bisect the chord, is perpendicular to the chord.
5. (i) Equal chords of a circle are equidistant from the centre.
   (ii) Two chords which are equidistant from the centre are always equal.
6. (i) Of any two chords of a circle, the one which is longer is nearer to the centre.
   (ii) Of any two chords of a circle, the one which is nearer to the centre is longer.
7. There is one and only one circle passing through three given non-collinear points.
8. (i) If two arcs of congruent circles are congruent then the corresponding chords are equal.
   (ii) If two chords of congruent circles are congruent then the corresponding arcs are congruent.
9. (i) Equal chords of congruent circles are equidistant from the centres.
   (ii) Chords of congruent circles which are equidistant from their centres are equal.
10. (i) Equal chords of congruent circles subtend equal angles at the centres.
    (ii) If the angles subtended at the centres by the two chords of congruent circles are equal then the chords are equal.
11. The angle subtended by an arc of a circle at the centre is double the angle subtended by it at any point on the remaining part of the circle.
12. (i) The angle in a semicircle is a right angle.
    (ii) The arc of a circle subtending a right angle at any point of the circle in its alternate segment is a semicircle.
13. Angles in the same segment of a circle are equal.
14. (i) The sum of either pair of the opposite angles of a cyclic quadrilateral is 180°.
    (ii) If a pair of opposite angles of a quadrilateral is supplementary then the quadrilateral is cyclic.
    (iii) Every cyclic parallelogram is a rectangle.

✿

13 Geometrical Constructions

PERPENDICULAR BISECTOR OF A LINE SEGMENT

Let AB be a given line segment and let O be the midpoint of AB. Then, a line segment CD passing through the point O and perpendicular to AB, is called the perpendicular bisector of AB.



Clearly, $OA = OB$ and $\angle AOC = \angle BOC = 90^\circ$.

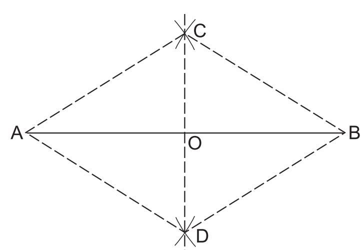
TO DRAW THE PERPENDICULAR BISECTOR OF A GIVEN LINE SEGMENT

EXAMPLE 1 *Draw a line segment* 6.4 cm *long and draw its perpendicular bisector. Measure the length of each part.*

STEPS OF CONSTRUCTION

(i) Draw a ling segment $AB = 6.4$ cm.

(ii) With A as centre and a radius equal to more than half of AB, draw two arcs, one above AB and the other below AB.



(iii) With B as centre and the same radius, draw two arcs, cutting the previously drawn arcs at points C and D respectively.

(iv) Join CD, intersecting AB at a point O.

Then, CD is the required perpendicular bisector of AB at the point O.

On measuring, we find that

$OA = 3.2$ cm and $OB = 3.2$ cm.

Also, $\angle AOC = \angle BOC = 90^\circ$.

Justification:

Join AC, AD, BC and BD.

In $\triangle CAD$ and $\triangle CBD$, we have

$AC = BC$ (arcs of equal radii)

$AD = BD$ (arcs of equal radii)

$CD = CD$ (common)

$\therefore$ $\triangle CAD \cong \triangle CBD$ (SSS-criteria)

$\therefore$ $\angle ACO = \angle BCO$ (c.p.c.t.).

Now, in $\triangle AOC$ and $\triangle BOC$, we have

$AC = BC$ (arcs of equal radii)

$\angle ACO = \angle BCO$ (proved above)

$CO = CO$ (common)

$\therefore$ $\triangle AOC \cong \triangle BOC$ (SAS-criteria).

Hence, $AO = BO$ and $\angle AOC = \angle BOC$.

But, $\angle AOC + \angle BOC = 180°$ (linear pair axiom)

$\therefore$ $\angle AOC = \angle BOC = 90°$.

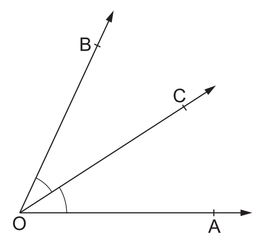
Hence, COD is the perpendicular bisector of $\angle AOB$.

BISECTOR OF A GIVEN ANGLE

Let $\angle AOB$ be a given angle. Then, a ray OC which divides $\angle AOB$ into two angles of equal measures, is called a bisector of $\angle AOB$.

In the given figure, $\angle AOC = \angle BOC$.

$\therefore$ OC is the bisector of $\angle AOB$.



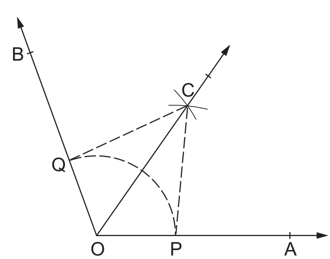
EXAMPLE 2 *Draw an angle of* $110°$ *with the help of a protractor and bisect it. Measure each of these angles. Justify your answer.*

STEPS OF CONSTRUCTION

(i) Draw $\angle AOB = 110°$ with the help of a protractor.

(ii) With O as centre and a convenient radius, draw an arc cutting OA at P and OB at Q.

(iii) With P as centre and a convenient radius, draw an arc.



(iv) With Q as centre and with the same radius, draw another arc, cutting the previous arc at a point C.

(v) Join OC and produce it beyond C.

Then, OC is the required bisector of $\angle AOB$.

Verification:

On measuring $\angle AOC$ and $\angle BOC$, you will find that

$\angle AOC = 55°$ and $\angle BOC = 55°$.

Thus, $\angle AOC = \angle BOC = 55°$ and therefore, OC is the bisector of $\angle AOB$.

Justification:

Join CP and CQ.

In $\triangle OPC$ and $\triangle OQC$, we have

$OP = OQ$ (radii of the same arc)

$PC = QC$ (arcs of equal radii)

$OC = OC$ (common)

$\therefore$ $\triangle OPC \cong \triangle OQC$.

Hence, $\angle POC = \angle QOC$ (c.p.c.t.).

Consequently, OC is the bisector of $\angle AOB$.

CONSTRUCTION OF ANGLES OF 60°, 30°, 15°, 120°, 90°, 45° USING RULER AND COMPASSES

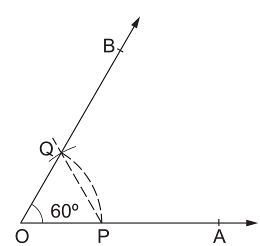
EXAMPLE 3 *Construct an angle of 60° using ruler and compasses only. Justify your construction.*

STEPS OF CONSTRUCTION

(i) Draw a ray OA.

(ii) With O as centre and a convenient radius, draw an arc, cutting OA at P.

(iii) With P as centre and the same radius as above, draw an arc, cutting the previous arc at Q.



(iv) Join OQ and produce it to a point B.

Then, $\angle AOB$ is the required angle of measure 60°.

Verification:

If you measure $\angle AOB$ by using a protractor, you will find that $\angle AOB = 60°$.

Justification:

Join PQ, as shown in the figure.

Then, $OQ = OP = PQ$ (by construction)

$\therefore \quad \triangle OPQ$ is an equilateral triangle.

Hence, $\angle POQ = 60^\circ$ and therefore, $\angle AOB = 60^\circ$.

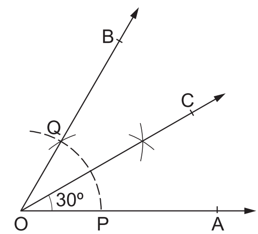
EXAMPLE 4 *Construct an angle of* 30° *using ruler and compasses only.*

STEPS OF CONSTRUCTION

(i) Draw an angle $\angle AOB = 60^\circ$, as indicated in Example 3.

(ii) Draw the bisector OC of $\angle AOB$, as indicated in Example 2.

Then, $\angle AOC = 30^\circ$ is the required angle.



EXAMPLE 5 *Construct an angle of* 15° *using ruler and compasses only.*

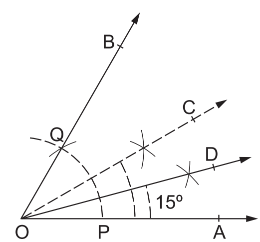
STEPS OF CONSTRUCTION

(i) Construct $\angle AOB = 60^\circ$, as indicated in Example 3.

(ii) Draw the bisector OC of $\angle AOB$, as indicated in Example 2.

(iii) Draw the bisector OD of $\angle AOC$, as indicated in Example 2.

Then, $\angle AOD = 15^\circ$ is the required angle.



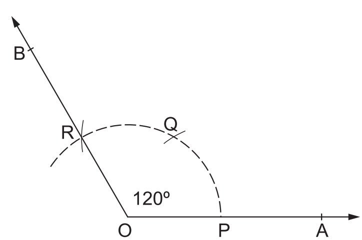
EXAMPLE 6 *Construct an angle of* 120°, *using ruler and compasses only.*

STEPS OF CONSTRUCTION

(i) Draw a ray OA.

(ii) With O as centre and a convenient radius, draw a long arc, cutting OA at P.

(iii) With P as centre and the same radius as above, draw an arc, cutting the previous arc at Q.



(iv) With Q as centre and the same radius, draw yet another arc, cutting the arc drawn in Step (ii) at R.

(v) Join OR and produce it to a point B.

Then, $\angle AOB$ is the required angle of measure 120°.

EXAMPLE 7 *Construct an angle of* 90°, *using ruler and compasses only.*

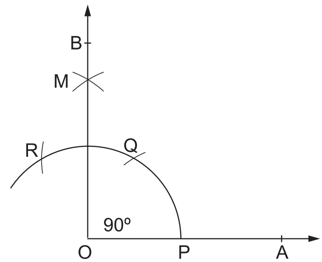
STEPS OF CONSTRUCTION

(i) Draw a ray OA.

(ii) With O as centre and a convenient radius, draw an arc cutting OA at P.

(iii) With P as centre and the same radius as above, draw an arc, cutting the above arc at Q.

(iv) With Q as centre and the same radius, draw another arc, cutting the arc drawn in Step (ii) at the point R.



(v) With Q and R as centres and the same radius, draw two arcs intersecting each other at M.

(vi) Join OM and produce it to a point B.

Then, $\angle AOB$ is the required angle of measure 90°.

EXAMPLE 8 *Construct an angle of* 45°, *using ruler and compasses only.*

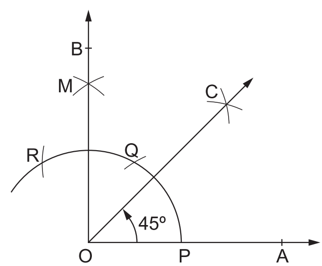
STEPS OF CONSTRUCTION

(i) Construct $\angle AOB = 90°$, as explained in Example 7.

(ii) Draw the bisector OC of $\angle AOB$.

Then, clearly, $\angle AOC = 45°$.

Hence, $\angle AOC$ is the required angle of measure 45°.



CONSTRUCTION OF TRIANGLE, RHOMBUS, ETC.

SOME FACTS ABOUT A TRIANGLE

(i) In any triangle, the angle opposite to the larger side is always greater than the angle opposite to the smaller side.

(ii) In any triangle, the side opposite to the larger angle is always greater than the side opposite to the smaller angle.

(iii) The sum of any two sides of a triangle is always greater than the third side.

(iv) The difference of any two sides of a triangle is always less than the third side.

(v) The sum of all the three angles of a triangle is always 180°.

(vi) In a right-angled triangle, the sum of the two acute angles is always 90°.

(vii) In a right-angled triangle, the hypotenuse is the largest side.

SOLVED EXAMPLES

EXAMPLE 1 *In each of the following cases, give reason to show that the construction of $\triangle ABC$ is not possible.*

(i) $AB = 6$ cm, $\angle A = 45°$ *and* $(BC + AC) = 5.8$ cm.

(ii) $AB = 5$ cm, $BC = 4$ cm *and* $AC = 9$ cm.

(iii) $AB = 5.4$ cm, $\angle B = 60°$ *and* $(BC - AC) = 6$ cm.

(iv) $BC = 4$ cm, $\angle A = 80°$, $\angle B = 50°$ *and* $\angle C = 60°$.

SOLUTION (i) Here $(BC + AC) = 5.8$ cm and $AB = 6$ cm.

$\therefore \quad (BC + AC) < AB$.

Thus, the sum of two sides is not greater than the third side.

Hence, the construction of $\triangle ABC$ is not possible.

(ii) Here, $(AB + BC) = (5 + 4)$ cm $= 9$ cm and $AC = 9$ cm.

$\therefore \quad (AB + BC) = AC$.

Thus, the sum of two sides is not greater than the third side.

Hence, the construction of $\triangle ABC$ is not possible.

(iii) Here, $(BC - AC) = 6$ cm and $AB = 5.4$ cm.

$\therefore \quad (BC - AC) > AB$.

Thus, the difference of two sides is not less than the third side.

Hence, the construction of $\triangle ABC$ is not possible.

(iv) Here, $\angle A + \angle B + \angle C = (80° + 50° + 60°) = 190°$.

But, we know that the sum of the angles of a triangle is always 180°.

Hence, the construction of $\triangle ABC$ is not possible in this case.

CONSTRUCTION OF TRIANGLES

EXAMPLE 2 *Construct a triangle whose sides are* 3.6 cm, 3 cm *and* 4.8 cm. *Bisect the smallest angle and measure each part.*

STEPS OF CONSTRUCTION

(i) Draw $AB = 3.6$ cm.

(ii) With A as centre and 3 cm as radius, draw an arc.

(iii) With B as centre and 4.8 cm as radius, draw another arc, cutting the previous arc at A.

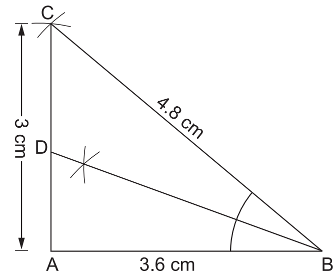
(iv) Join AC and BC.

Then, $\triangle ABC$ is the required triangle.

(v) Clearly, the angle opposite to the smallest side is smallest.

So, $\angle B$ is the smallest angle.

(vi) So, we draw BD, the bisector of $\angle B$.



On measuring, we find that $\angle ABD = \angle CBD = 20°$.

EXAMPLE 3 *Construct an equilateral triangle whose altitude is* 4 cm. *Give justification for your construction.*

STEPS OF CONSTRUCTION

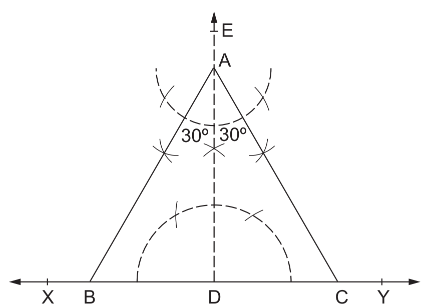
(i) Draw a line XY.

(ii) Mark any point D on XY.

(iii) From D, draw $DE \perp XY$.

(iv) From D, set off $DA = 4$ cm, cutting DE at A.

(v) Construct $\angle DAB = 30°$ and $\angle DAC = 30°$, meeting XY at B and C respectively.



Then, $\triangle ABC$ is the required equilateral triangle.

Verification:

On measuring, we find that

$$\angle A = \angle B = \angle C = 60°$$

and $AB = BC = CA = 4.5$ cm.

Justification:

In $\triangle DAB$, we have

$$\angle ABD + \angle BDA + \angle DAB = 180° \Rightarrow \angle ABD + 90° + 30° = 180°$$
$$\Rightarrow \angle ABD = 180° - 120° = 60°.$$

In $\triangle DAC$, we have

$$\angle ACD + \angle CDA + \angle DAC = 180° \Rightarrow \angle ACD + 90° + 30° = 180°$$
$$\Rightarrow \angle ACD = 180° - 120° = 60°.$$

In $\triangle ABC$, we have $\angle A = \angle B = \angle C = 60°$.

Hence, $\triangle ABC$ is an equilateral triangle.

TO CONSTRUCT A TRIANGLE WHOSE BASE, ONE BASE ANGLE AND SUM OF OTHER TWO SIDES ARE GIVEN

EXAMPLE 4 *Construct a* $\triangle ABC$ *in which* $BC = 6$ cm, $\angle B = 60°$ *and* $(AB + AC) = 9$ cm. *Measure AB and AC. Justify your answer.*

STEPS OF CONSTRUCTION

(i) Draw $BC = 6$ cm.

(ii) Construct $\angle CBX = 60°$.

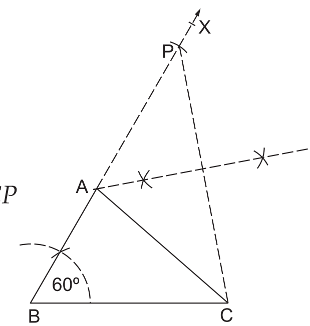
(iii) Along BX, set off $BP = 9$ cm.

(iv) Join CP.

(v) Draw the perpendicular bisector of CP to intersect BP at A.

(vi) Join AC.

Then, $\triangle ABC$ is the required triangle.



Verification:

On measuring, we find that

$$AB + AC = (4 + 5) \text{ cm} = 9 \text{ cm}.$$

Justification:

Clearly, A lies on the perpendicular bisector of CP.

$\therefore \quad AP = AC.$

Now, $BP = 9$ cm [see Step (iii)]

$\Rightarrow \quad AB + AP = 9$ cm

$\Rightarrow \quad AB + AC = 9$ cm $[\because AP = AC]$.

Hence, $\triangle ABC$ is the required triangle.

EXAMPLE 5 *Construct a right triangle whose base is* 4 cm *and the sum of its hypotenuse and the other side is* 8 cm.

STEPS OF CONSTRUCTION

(i) Draw a line segment $BC = 4$ cm.

(ii) Construct $\angle CBX = 90°$.

(iii) From B, set off $BD = 8$ cm.

(iv) Join CD.

(v) Draw the perpendicular bisector of CD, intersecting BD at A.

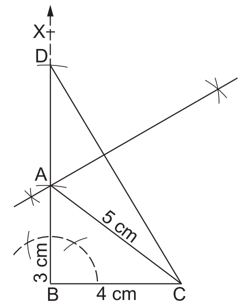
(vi) Join AC.

Then, $\triangle ABC$ is the required right triangle.

Verification:

On measuring, we find that

$AB + AC = (3 + 5)$ cm $= 8$ cm.



TO CONSTRUCT A TRIANGLE WHOSE BASE, ONE BASE ANGLE AND THE DIFFERENCE BETWEEN THE OTHER TWO SIDES IS GIVEN

EXAMPLE 6 *Construct a* $\triangle ABC$ *in which* $BC = 5.6$ cm, $\angle B = 30°$ *and* $(AB - AC) = 3$ cm. *Measure AB and AC. Justify your construction.*

STEPS OF CONSTRUCTION

(i) Draw $BC = 5.6$ cm.

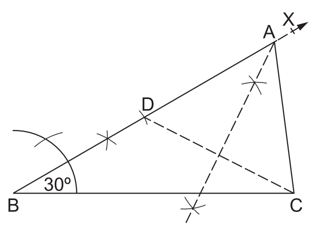
(ii) Construct $\angle CBX = 30°$.

(iii) Along BX, set off $BD = 3$ cm.

(iv) Join CD.

(v) Draw the right bisector of CD, meeting BD produced at A.

(vi) Join AC.



Then, $\triangle ABC$ is the required triangle.

Verification:

On measuring, we find that $AB = 6.1$ cm and $AC = 3.1$ cm.

$\therefore \quad (AB - AC) = (6.1 - 3.1)$ cm $= 3$ cm.

Justification:

Since A lies on the perpendicular bisector of CD, we have $AD = AC$.

Now, $BD = 3$ cm $\Rightarrow AB - AD = 3$ cm

$\Rightarrow AB - AC = 3$ cm $\quad [\because AD = AC]$.

Hence, $\triangle ABC$ is the required triangle.

EXAMPLE 7 *Construct a* $\triangle ABC$ *in which* $BC = 6$ cm, $\angle B = 60°$ *and* $(AC - AB) = 2$ cm. *Measure AC and AB. Justify your construction.*

STEPS OF CONSTRUCTION

(i) Draw $BC = 6$ cm.

(ii) Construct $\angle CBX = 60°$.

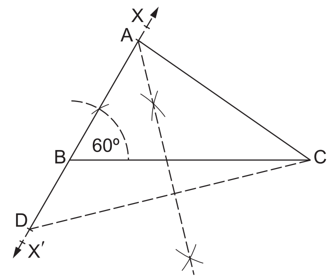
(iii) Produce XB downwards to a point X'.

(iv) Set off $BD = 2$ cm along BX'.

(v) Join CD.

(vi) Draw the perpendicular bisector of CD, meeting BX at A.

(vii) Join AC.



Then, $\triangle ABC$ is the required triangle.

Verification:

On measuring, we find that $AC = 5.4$ cm and $AB = 3.4$ cm.

$\therefore \quad (AC - AB) = (5.4 - 3.4)$ cm $= 2$ cm.

Justification:

Since A lies on the perpendicular bisector of CD, we have $AD = AC$.

Now, $BD = 2$ cm $\Rightarrow AD - AB = 2$ cm

$\Rightarrow AC - AB = 2$ cm $\quad [\because AD = AC]$.

Hence, $\triangle ABC$ is the required triangle.

TO CONSTRUCT A TRIANGLE WHOSE TWO SIDES AND A MEDIAN ARE GIVEN

EXAMPLE 8 *Construct a* $\triangle ABC$ *in which* $AB = 4.5$ cm, $BC = 6$ cm *and median* $AD = 4$ cm.

STEPS OF CONSTRUCTION

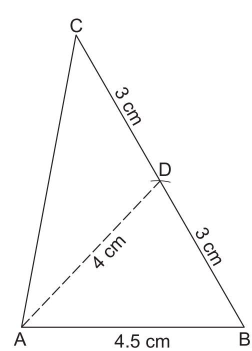
(i) Draw a line segment $AB = 4.5$ cm.

(ii) With A as centre and radius equal to 4 cm, draw an arc.

(iii) With B as centre and radius $= \frac{1}{2}BC = 3$ cm, draw another arc, cutting the previous arc at D.

(iv) Join BD and produce it to C such that $BC = 6$ cm.

(v) Join AC.



Then, $\triangle ABC$ is the required triangle.

EXAMPLE 9 *Construct a right-angled* $\triangle ABC$ *in which* $\angle A = 90°$, *hypotenuse* $BC = 6$ cm *and* $AB = 4$ cm. *Justify your construction.*

STEPS OF CONSTRUCTION

(i) Draw a line segment $BC = 6$ cm.

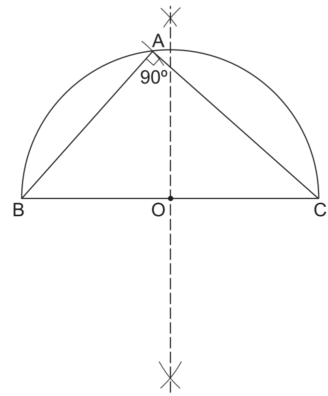
(ii) Draw the right bisector of BC, cutting BC at O.

(iii) With O as centre and radius OB, draw a semicircle on BC.

(iv) With B as centre and radius 4 cm, draw an arc, cutting the semicircle at A.

(v) Join AB and AC.

Then, $\triangle ABC$ is the required triangle, right angled at A.



Justification:

We know that an angle in a semicircle is a right angle and therefore, $\angle BAC = 90°$.

Also, $BA = 4$ cm and $BC = 6$ cm.

Hence, $\triangle ABC$ is the required triangle, right angled at A.

EXAMPLE 10 *Construct a* $\triangle ABC$ *whose perimeter is* 12 cm *and the base angles are* 60° *and* 70°. *Justify the construction.*

STEPS OF CONSTRUCTION

(i) Draw a line segment $PQ = 12$ cm.

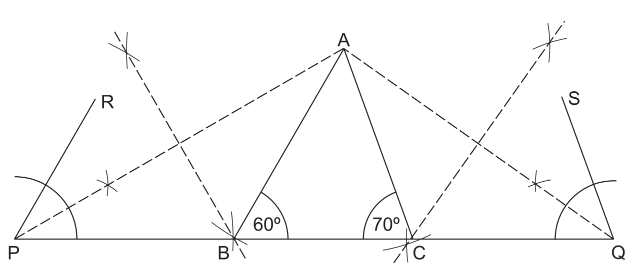
(ii) Make $\angle QPR = 60°$ and $\angle PQS = 70°$.

(iii) Draw the bisectors of $\angle QPR$ and $\angle PQS$ to meet at A.

(iv) Draw the perpendicular bisectors of PA and QA to meet PQ at B and C respectively.

(v) Join AB and AC.

Then, $\triangle ABC$ is the required triangle.



Justification:

Since B lies on the perpendicular bisector of AP, we have $BA = BP$.

Since C lies on the perpendicular bisector of AQ, we have $CA = CQ$.

Thus, $AB = PB$ and $AC = CQ$.

$\therefore \quad AB + BC + AC = PB + BC + CQ = PQ = 12$ cm.

Now, $BA = BP \Rightarrow \angle BPA = \angle BAP$

$\Rightarrow \angle ABC = \angle BPA + \angle BAP = 2\angle BPA$

$= 2 \times \frac{1}{2} \times 60^\circ = 60^\circ.$

And, $CA = CQ \Rightarrow \angle CQA = \angle CAQ$

$\Rightarrow \angle ACB = \angle CQA + \angle CAQ = 2\angle CQA$

$= 2 \times \frac{1}{2} \times 70^\circ = 70^\circ.$

Verification:

On measurement, we find that

$AB + BC + CA = (4.4 + 3.5 + 4.1)$ cm $= 12$ cm.

$\angle B = 60^\circ$ and $\angle C = 70^\circ$.

EXAMPLE 11 *Construct a* $\triangle ABC$ *whose perimeter is* 13.5 cm *and its sides are in the ratio* 2 : 3 : 4.

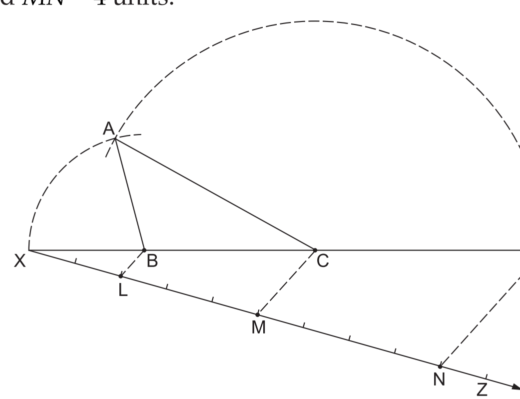
STEPS OF CONSTRUCTION

(i) Draw a line segment $XY = 13.5$ cm.

(ii) Draw a ray XZ, making an acute angle with XY and drawn in the downward direction.

(iii) From X, set off $(2 + 3 + 4) = 9$ equal distances along XZ.

(iv) Mark points L, M, N on XZ such that $XL = 2$ units, $LM = 3$ units and $MN = 4$ units.



(v) Join NY.

(vi) Draw $LB \parallel NY$ and $MC \parallel NY$, cutting XY at B and C respectively.

(vii) With B as centre and radius BX, draw an arc.

(viii) With C as centre and radius CY draw another arc, cutting the previous arc at A.

(ix) Join AB and AC.

Then, $\triangle ABC$ is the required triangle.

Verification:

On measuring, we find that

$AB = 3$ cm, $BC = 4.5$ cm and $CA = 6$ cm.

$\therefore \quad AB : BC : CA = 3 : \frac{9}{2} : 6 = 6 : 9 : 12 = 2 : 3 : 4.$

RHOMBUS We know that a rhombus is a quadrilateral in which:

(i) all sides are equal.

(ii) adjacent angles are supplimentary.

(iii) opposite angles are equal.

(iv) the diagonals bisect each other at right angles.

EXAMPLE 12 *Construct a rhombus each of whose sides is of length* 3.4 cm *and one of its angles is* 45°.

STEPS OF CONSTRUCTION

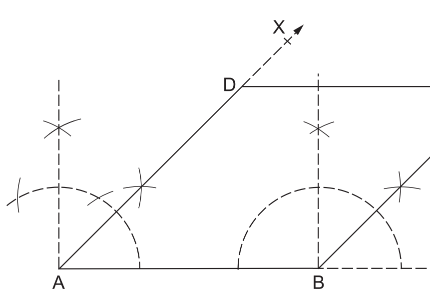
(i) Draw a line segment $AB = 3.4$ cm.

(ii) Construct $\angle BAX = 45°$.

(iii) Along AX set off $AD = 3.4$ cm.

(iv) Construct $\angle ABY = 135°$.

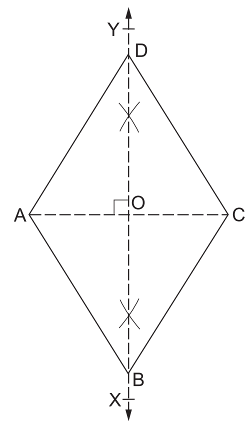
(v) Along BY set off $BC = 3.4$ cm.



(vi) Join CD.

Then, $ABCD$ is the required rhombus.

EXAMPLE 13 *Construct a rhombus whose diagonals are* 4 cm *and* 6 cm *in length. Measure each side of the rhombus.*



STEPS OF CONSTRUCTION

(i) Draw a line segment $AC = 4$ cm.

(ii) Draw the perpendicular bisector XOY of AC, cutting AC at O.

(iii) Along OX and OY, set off $OB = 3$ cm and $OD = 3$ cm respectively.

(iv) Join AB, BC, CD and AD.

Then, $ABCD$ is the required rhombus.

EXERCISE 13

1. Draw a line segment $AB = 5.6$ cm and draw its perpendicular bisector. Measure the length of each part.

2. Draw an angle of $80°$ with the help of a protractor and bisect it. Measure each part of the bisected angle.

3. Construct an angle of $90°$ using ruler and compasses and bisect it.

4. Construct each of the following angles, using ruler and compasses:

(i) $75°$ (ii) $37.5°$ (iii) $135°$ (iv) $105°$ (v) $22.5°$

5. Construct a $\triangle ABC$ in which $BC = 5$ cm, $AB = 3.8$ cm and $AC = 2.6$ cm. Bisect the largest angle of this triangle.

6. Construct a $\triangle ABC$ in which $BC = 4.8$ cm, $\angle B = 45°$ and $\angle C = 75°$. Measure $\angle A$.

7. Construct an equilateral triangle, each of whose sides measures 5 cm.

8. Construct an equilateral triangle each of whose altitudes measures 5.4 cm. Measure each of its sides.

9. Construct a right-angled triangle whose hypotenuse measures 5 cm and the length of one of whose sides containing the right angle measures 4.5 cm.

10. Construct a $\triangle ABC$ in which $BC = 4.5$ cm, $\angle B = 45°$ and $AB + AC = 8$ cm. Justify your construction.

11. Construct a $\triangle ABC$ in which $AB = 5.8$ cm, $\angle B = 60°$ and $BC + CA = 8.4$ cm. Justify your construction.

12. Construct a $\triangle ABC$ in which $BC = 6$ cm, $\angle B = 30°$ and $AB - AC = 3.5$ cm. Justify your construction.

13. Construct a $\triangle ABC$ in which base $AB = 5$ cm, $\angle A = 30°$ and $AC - BC = 2.5$ cm. Justify your construction.

14. Construct a $\triangle PQR$ whose perimeter is 12 cm and the lengths of whose sides are in the ratio 3 : 2 : 4.

15. Construct a triangle whose perimeter is 10.4 cm and the base angles are 45° and 120°.

16. Construct a $\triangle ABC$ whose perimeter is 11.6 cm and the base angles are 45° and 60°.

17. In each of the following cases, given reasons to show that the construction of $\triangle ABC$ is not possible:

(i) $AB = 6$ cm, $\angle A = 40°$ and $(BC + AC) = 5.8$ cm.

(ii) $AB = 7$ cm, $\angle A = 50°$ and $(BC - AC) = 8$ cm.

(iii) $BC = 5$ cm, $\angle B = 80°$, $\angle C = 50°$ and $\angle A = 60°$.

(iv) $AB = 4$ cm, $BC = 3$ cm and $AC = 7$ cm.

18. Construct an angle of 67.5° by using the ruler and compasses.

19. Construct a square of side 4 cm.

20. Construct a right triangle whose one side is 3.5 cm and the sum of the other side and the hypotenuse is 5.5 cm.

21. Construct a $\triangle ABC$ in which $\angle B = 45°$, $\angle C = 60°$ and the perpendicular from the vertex A to base BC is 4.5 cm.

HINTS TO SOME SELECTED QUESTIONS

18. $(67.5)^\circ = \frac{1}{2} \times (135)^\circ = \frac{1}{2} \times (90^\circ + 45^\circ)$.

20. See Example 5 on Construction of Triangles.

21. Steps of Construction

(i) Draw any line XY.

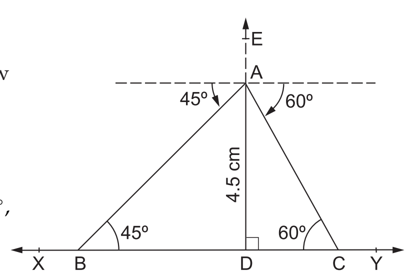
(ii) Take any point D on XY and draw $DE \perp XY$.

(iii) Cut off $DA = 4.5$ cm along DE.

(iv) Through A draw $LM \parallel XY$.

(v) Construct $\angle LAB = 45°$ and $\angle MAC = 60°$, meeting XY at B and C respectively.

Then, $\triangle ABC$ is the required triangle.

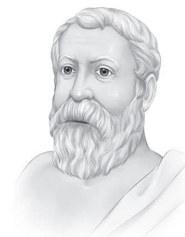


✿

14 Areas of Triangles and Quadrilaterals

INTRODUCTION

Heron (AD 10–AD 75) Heron was born in Alexandria in Egypt in AD 10. He worked on applied mathematics and wrote three books on mensuration. One of his books deals with the areas of squares, rectangles, triangles, trapezia, regular polygons, circles and surfaces of cylinders, cones and spheres, etc. He derived the famous **'Heron's formula'** for finding the area of a triangle.



Heron

FORMULAE FOR AREA OF TRIANGLES

(i) Area of a Triangle $= \left(\frac{1}{2} \times \text{base} \times \text{height}\right)$ sq units.

(ii) **HERON'S FORMULA**

Let a, b, c be the sides of a triangle. Then,

semiperimeter, $s = \frac{1}{2}(a+b+c)$;

area $= \sqrt{s(s-a)(s-b)(s-c)}$ sq units.

(iii) Let each side of an equilateral triangle be a. Then,

area $= \left(\frac{\sqrt{3}}{4} \times a^2\right)$ sq units, and height $= \left(\frac{\sqrt{3}}{2}a\right)$ units.

(iv) Consider an isosceles triangle having base $= b$ and each of equal sides $= a$. Then,

area $= \left(\frac{b}{4} \times \sqrt{4a^2 - b^2}\right)$ sq units.

SOLVED EXAMPLES

EXAMPLE 1 *Find the area of a triangle whose base is* 25 cm *long and the corresponding height is* 10.8 cm.

SOLUTION Here, base = 25 cm and height = 10.8 cm.

$$\text{Area of the triangle} = \left(\frac{1}{2} \times \text{base} \times \text{height}\right) \text{ sq units}$$

$$= \left(\frac{1}{2} \times 25 \times 10.8\right) \text{cm}^2 = 135 \text{ cm}^2.$$

Hence, the area of the given triangle is 135 cm^2.

EXAMPLE 2 *Find the perimeter and area of a triangle whose sides are of lengths* 52 cm, 56 cm *and* 60 cm *respectively.*

SOLUTION Let $a = 52$ cm, $b = 56$ cm and $c = 60$ cm.

Perimeter of the triangle $= (a + b + c)$ units

$= (52 + 56 + 60)$ cm $= 168$ cm.

$$\therefore \quad s = \frac{1}{2}(a + b + c) = \left(\frac{1}{2} \times 168\right) \text{ cm} = 84 \text{ cm}.$$

$\therefore \quad (s - a) = (84 - 52)$ cm $= 32$ cm, $(s - b) = (84 - 56)$ cm $= 28$ cm and $(s - c) = (84 - 60)$ cm $= 24$ cm.

By Heron's formula, the area of the given triangle is

$$\begin{aligned}\triangle &= \sqrt{s(s-a)(s-b)(s-c)}\\ &= \sqrt{84 \times 32 \times 28 \times 24} \text{ cm}^2\\ &= \sqrt{14 \times 6 \times 16 \times 2 \times 14 \times 2 \times 6 \times 4} \text{ cm}^2\\ &= (14 \times 6 \times 4 \times 2 \times 2) \text{ cm}^2 = 1344 \text{ cm}^2.\end{aligned}$$

Hence, the area of the given triangle is 1344 cm^2.

EXAMPLE 3 *The lengths of the sides of a triangle are in the ratio* 3 : 4 : 5 *and its perimeter is* 144 cm. *Find (i) the area of the triangle and (ii) the height corresponding to the longest side.*

SOLUTION Perimeter = 144 cm and ratio of sides = 3 : 4 : 5.

Sum of ratio terms $= 3 + 4 + 5 = 12$.

Let the lengths of the sides be a, b and c respectively.

Then, $a = \left(144 \times \frac{3}{12}\right)$ cm $= 36$ cm, $b = \left(144 \times \frac{4}{12}\right)$ cm $= 48$ cm

and $c = \left(144 \times \frac{5}{12}\right)$ cm $= 60$ cm.

$$\therefore \quad s = \frac{1}{2}(a + b + c) = \frac{1}{2}(36 + 48 + 60) \text{ cm} = 72 \text{ cm}.$$

$\therefore \quad (s - a) = (72 - 36)$ cm $= 36$ cm, $(s - b) = (72 - 48)$ cm $= 24$ cm and $(s - c) = (72 - 60)$ cm $= 12$ cm.

(i) By Heron's formula, the area of the triangle is given by

$$\begin{aligned}\triangle &= \sqrt{s(s-a)(s-b)(s-c)}\\ &= \sqrt{72 \times 36 \times 24 \times 12} \text{ cm}^2 = \sqrt{36 \times 36 \times 24 \times 24} \text{ cm}^2\\ &= (36 \times 24) \text{ cm}^2 = 864 \text{ cm}^2.\end{aligned}$$

Hence, the area of the given triangle is 864 cm^2.

(ii) Let base = longest side = 60 cm and the corresponding height = h cm.

Then, area $= \left(\frac{1}{2} \times \text{base} \times \text{height}\right)$ sq units

$= \left(\frac{1}{2} \times 60 \times h\right) \text{cm}^2 = (30h)\ \text{cm}^2.$

$\therefore \quad 30h = 864 \Rightarrow h = \left(\frac{864}{30}\right) \Rightarrow$ height = 28.8 cm.

Hence, the required height is 28.8 cm.

EXAMPLE 4 *The sides of a triangle are* 35 cm, 54 cm *and* 61 cm *respectively. Find the length of its longest altitude.*

SOLUTION Let $a = 35$ cm, $b = 54$ cm and $c = 61$ cm. Then,

$$s = \frac{1}{2}(a+b+c) = \frac{1}{2}(35+54+61)\ \text{cm} = 75\ \text{cm}.$$

$\therefore \quad (s-a) = (75-35)$ cm $= 40$ cm, $(s-b) = (75-54)$ cm $= 21$ cm and $(s-c) = (75-61)$ cm $= 14$ cm.

$$\therefore \quad \text{area of the triangle} = \sqrt{s(s-a)(s-b)(s-c)}\ \text{cm}^2$$

$$= \sqrt{75 \times 40 \times 21 \times 14}\ \text{cm}^2$$

$$= \sqrt{15 \times 15 \times 14 \times 14 \times 4 \times 5}\ \text{cm}^2$$

$$= (15 \times 14 \times 2)\sqrt{5}\ \text{cm}^2 = 420\sqrt{5}\ \text{cm}^2.$$

The longest altitude will be on smallest base.

Let the longest altitude be h cm. Then,

base = 35 cm and height = h cm.

$$\therefore \quad \frac{1}{2} \times 35 \times h = 420\sqrt{5} \Rightarrow h = \left(\frac{2 \times 420 \times \sqrt{5}}{35}\right) \text{cm} = 24\sqrt{5}\ \text{cm}.$$

Hence, the longest altitude is $24\sqrt{5}$ cm.

EXAMPLE 5 *The perimeter of an equilateral triangle is* 60 cm. *Find its (i) area and (ii) height. (Given,* $\sqrt{3} = 1.732$.)

SOLUTION Perimeter of the given equilateral triangle = 60 cm.

Length of each of its sides $= \frac{60}{3}$ cm $= 20$ cm.

(i) Area of the triangle $= \left(\frac{\sqrt{3}}{4} \times a^2\right)$ sq units

$$= \left(\frac{\sqrt{3}}{4} \times 20 \times 20\right) \text{cm}^2 = (100 \times \sqrt{3})\ \text{cm}^2$$

$$= (100 \times 1.732)\ \text{cm}^2 = 173.2\ \text{cm}^2.$$

Hence, the area of the given triangle is $173.2\ \text{cm}^2$.

(ii) Let the height of the given triangle be h cm. Then,

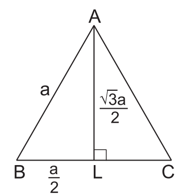
its area $= \left(\frac{1}{2} \times \text{base} \times \text{height}\right) = \left(\frac{1}{2} \times 20 \times h\right) \text{cm}^2$.

$\therefore \quad \frac{1}{2} \times 20 \times h = 173.2 \Rightarrow h = \frac{173.2}{10} \Rightarrow$ height $= 17.32$ cm.

Hence, the height of the given triangle is 17.32 cm.

EXAMPLE 6 *The height of an equilateral triangle is* 6 cm. *Find its area.*

SOLUTION Let each side of the given triangle be a cm.



Then, its height $= \left(\frac{\sqrt{3}}{2} \times a\right)$ cm.

$\therefore \quad \frac{\sqrt{3}}{2} \times a \text{ cm} = 6 \text{ cm} \Rightarrow a = \left(\frac{6 \times 2}{\sqrt{3}}\right)$

$\Rightarrow$ side $= 4\sqrt{3}$ cm.

$\therefore$ area of the given triangle

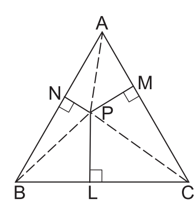
$$= \left(\frac{\sqrt{3}}{4} \times a^2\right) \text{sq units} = \left\{\frac{\sqrt{3}}{4} \times (4\sqrt{3})^2\right\} \text{cm}^2$$

$$= \left(\frac{\sqrt{3}}{4} \times 48\right) \text{cm}^2 = 12\sqrt{3} \text{ cm}^2.$$

Hence, the area of the given triangle is $12\sqrt{3}$ cm^2.

EXAMPLE 7 *From a point in the interior of an equilateral triangle, perpendiculars are drawn on the three sides. The lengths of the perpendiculars are* 14 cm, 10 cm *and* 6 cm. *Find the area of the triangle.*

SOLUTION Let $\triangle ABC$ be the given equilateral triangle having each side equal to a cm and let P be the given point in its interior.



Let $PL \perp BC$, $PM \perp AC$ and $PN \perp AB$ such that $PL = 14$ cm, $PM = 10$ cm and $PN = 6$ cm.

Clearly, we have

$$\text{ar}(\triangle ABC) = \text{ar}(\triangle PBC) + \text{ar}(\triangle PAC) + \text{ar}(\triangle PAB)$$

$$\Rightarrow \quad \frac{\sqrt{3}}{4} \times a^2 \text{ cm}^2 = \left(\frac{1}{2} \times a \text{ cm} \times PL\right) + \left(\frac{1}{2} \times a \text{ cm} \times PM\right) + \left(\frac{1}{2} \times a \text{ cm} \times PN\right)$$

$$= \left\{\left(\frac{1}{2} \times a \times 14\right) + \left(\frac{1}{2} \times a \times 10\right) + \left(\frac{1}{2} \times a \times 6\right)\right\} \text{cm}^2$$

$$= (7a + 5a + 3a) \text{ cm}^2 = (15a) \text{ cm}^2.$$

$\therefore \quad \frac{\sqrt{3}}{4} \times a^2 = 15a \Rightarrow a = \frac{15 \times 4}{\sqrt{3}} = 20\sqrt{3}.$

$\therefore$ length of each side $= a$ cm $= 20\sqrt{3}$ cm.

$\therefore$ the area of the given triangle

$$= \left(\frac{\sqrt{3}}{4} \times a^2\right) \text{cm}^2 = \left\{\frac{\sqrt{3}}{4} \times (20\sqrt{3})^2\right\} \text{cm}^2$$

$$= \left(\frac{\sqrt{3}}{4} \times 1200\right) \text{cm}^2 = 300\sqrt{3} \text{ cm}^2.$$

Hence, the area of the given triangle is $300\sqrt{3}$ cm^2.

EXAMPLE 8 *Find the area of an isosceles triangle each of whose equal sides is* 13 cm *and whose base is* 24 cm.

SOLUTION Here, $a = 13$ cm, $b = 13$ cm and $c = 24$ cm.

$\therefore \quad s = \frac{1}{2}(a + b + c) = \frac{1}{2}(13 + 13 + 24)$ cm $= 25$ cm.

$\therefore \quad (s - a) = (25 - 13)$ cm $= 12$ cm, $(s - b) = (25 - 13)$ cm $= 12$ cm

and $(s - c) = (25 - 24)$ cm $= 1$ cm.

So, by Heron's formula,

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{25 \times 12 \times 12 \times 1} \text{ cm}^2 = (5 \times 12) \text{ cm}^2 = 60 \text{ cm}^2.$$

Hence, the area of the given triangle is 60 cm^2.

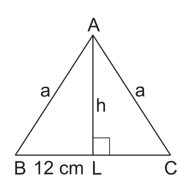
EXAMPLE 9 *The base of an isosceles triangle measures* 24 cm *and its area is* 192 cm^2. *Find its perimeter.*

SOLUTION Let ΔABC be an isosceles triangle and let $AL \perp BC$.

$\therefore \quad \frac{1}{2} \times BC \times AL = 192 \text{ cm}^2$

$\Rightarrow \quad \frac{1}{2} \times 24 \text{ cm} \times h = 192 \text{ cm}^2$

$\Rightarrow \quad h = \left(\frac{192}{12}\right)$ cm $= 16$ cm.



Now, $BL = \frac{1}{2}(BC) = \left(\frac{1}{2} \times 24\right)$ cm $= 12$ cm and $AL = 16$ cm.

$\therefore \quad a = \sqrt{BL^2 + AL^2} = \sqrt{(12)^2 + (16)^2}$ cm $= \sqrt{144 + 256}$ cm

$\Rightarrow \quad a = \sqrt{400}$ cm $= 20$ cm.

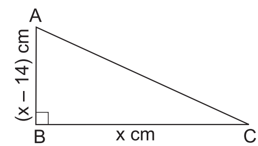
Hence, perimeter $= (20 + 20 + 24)$ cm $= 64$ cm.

EXAMPLE 10 *The difference between the sides at right angles in a right-angled triangle is* 14 cm. *The area of the triangle is* 120 cm^2. *Calculate the perimeter of the triangle.*

SOLUTION Let the sides containing the right angle be x cm and $(x-14)$ cm.

Then, the area of the triangle $= \left[\frac{1}{2} \times x \times (x-14)\right] \text{cm}^2$.

But, area = 120 cm^2 (given).



$\therefore \quad \frac{1}{2}x(x-14) = 120$

$\Rightarrow \quad x^2 - 14x - 240 = 0$

$\Rightarrow \quad x^2 - 24x + 10x - 240$

$\Rightarrow \quad x(x-24) + 10(x-24)$

$\Rightarrow \quad (x-24)(x+10) = 0$

$\Rightarrow \quad x = 24 \quad$ (neglecting $x = -10$).

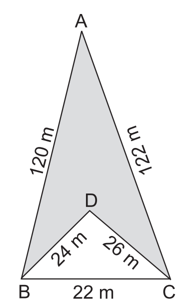
$\therefore$ one side = 24 cm, other side = (24 − 14) cm = 10 cm.

Hypotenuse $= \sqrt{(24)^2 + (10)^2}$ cm $= \sqrt{576 + 100}$ cm

$= \sqrt{676}$ cm = 26 cm.

$\therefore$ perimeter of the triangle = (24 + 10 + 26) cm = 60 cm.

EXAMPLE 11 *Calculate the area of the shaded region in the given figure.*



SOLUTION Area of the shaded region

$= \text{area}(\triangle ABC) - \text{area}(\triangle DBC)$.

For $\triangle ABC$ having sides 122 m, 120 m and 22 m, we have

$s = \frac{1}{2}(122 + 120 + 22)$ m

$= \left(\frac{1}{2} \times 264\right)$ m = 132 m.

$\therefore \quad (s-a) = (132 - 122)$ m = 10 m,

$(s-b) = (132 - 120)$ m = 12 m

and $(s-c) = (132 - 22)$ m = 110 m.

$\therefore$ area $(\triangle ABC) = \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{132 \times 10 \times 12 \times 110}$ m^2

$= (12 \times 11 \times 10)$ m^2 = 1320 m^2.

For $\triangle DBC$ having sides 26 m, 24 m and 22 m, we have

$$s = \frac{1}{2}(26+24+22)\text{ m} = \left(\frac{1}{2}\times 72\right)\text{ m} = 36\text{ m}.$$

$\therefore$ $(s-a) = (36-26)$ m = 10 m, $(s-b) = (36-24)$ m = 12 m

and $(s-c) = (36-22)$ m = 14 m.

$\therefore$ area $(\triangle DBC) = \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{36\times 10\times 12\times 14}\text{ m}^2$

$= 24\times\sqrt{105}\text{ m}^2 = (24\times 10.25)\text{ m}^2$ (approx.)

$= 246\text{ m}^2$.

$\therefore$ area of the shaded region

= area $(\triangle ABC)$ − area $(\triangle DBC)$

$= (1320-246)\text{ m}^2 = 1074\text{ m}^2$.

AREA OF QUADRILATERALS

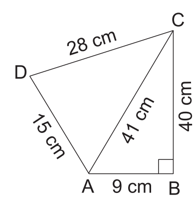
EXAMPLE 12 *Find the area of the quadrilateral ABCD in which* $AB = 9$ cm, $BC = 40$ cm, $CD = 28$ cm, $DA = 15$ cm *and* $\angle ABC = 90°$.

SOLUTION Let $ABCD$ be the given quadrilateral in which $AB = 9$ cm, $BC = 40$ cm, $DA = 15$ cm and $\angle ABC = 90°$.

In right $\triangle ABC$, by Pythagoras' theorem, we have

$$AC^2 = AB^2 + BC^2 = (81+1600)\text{ cm}^2$$
$$= 1681\text{ cm}^2$$

$\Rightarrow$ $AC = \sqrt{1681}$ cm = 41 cm.



Area of $\triangle ABC = \frac{1}{2}\times AB\times BC$

$= \left(\frac{1}{2}\times 9\times 40\right)\text{ cm}^2 = 180\text{ cm}^2$.

In $\triangle ACD$, let $a = AC = 41$ cm, $b = CD = 28$ cm and $c = AD = 15$ cm.

$$\therefore\quad s = \frac{1}{2}(41+28+15)\text{ cm} = \left(\frac{1}{2}\times 84\right)\text{ cm} = 42\text{ cm}.$$

$\therefore$ $(s-a) = (42-41) = 1$ cm, $(s-b) = (42-28)$ cm = 14 cm

and $(s-c) = (42-15)$ cm = 27 cm.

$\therefore$ area $(\triangle ACD) = \sqrt{s(s-a)(s-b)(s-c)}$

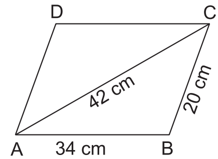
$= \sqrt{42\times 1\times 14\times 27}\text{ cm}^2$

$= (14\times 3\times 3)\text{ cm}^2 = 126\text{ cm}^2$.

$\therefore$ area of quad. $ABCD$ = area $(\triangle ABC)$ + area $(\triangle ACD)$

$= (180+126)\text{ cm}^2 = 306\text{ cm}^2$.

EXAMPLE 13 *The adjacent sides of a parallelogram ABCD are AB* = 34 cm, *BC* = 20 cm *and diagonal AC* = 42 cm. *Find the area of the parallelogram.*



SOLUTION In $\triangle ABC$, it is given that

$$a = BC = 20 \text{ cm}, b = AC = 42 \text{ cm and } c = AB = 34 \text{ cm}.$$

$\therefore$ $s = \frac{1}{2}(20 + 42 + 34)$ cm = 48 cm.

$\therefore$ $(s - a) = 28$ cm, $(s - b) = 6$ cm and $(s - c) = 14$ cm.

$\therefore$ area of $\triangle ABC = \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{48 \times 28 \times 6 \times 14}$ cm^2 = (14×24) cm^2

$= 336$ cm^2.

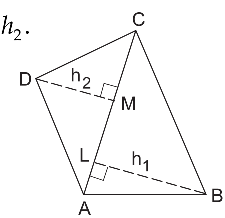
$\therefore$ area of $\|$gm $ABCD = 2 \times$ (area of $\triangle ABC$)

$= (2 \times 336)$ cm^2 = 672 cm^2.

AREA OF A QUADRILATERAL WHOSE ONE DIAGONAL AND LENGTHS OF PERPENDICULARS FROM OPPOSITE VERTICES TO THE DIAGONAL ARE GIVEN

Let *ABCD* be a quadrilateral and *AC* be its diagonal.

Let $BL \perp AC$ and $DM \perp AC$. Let $BL = h_1$ and $DM = h_2$.



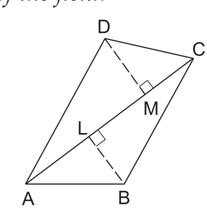
Area of quadrilateral *ABCD*

$= \text{area of } \triangle ABC + \text{area of } \triangle ACD$

$= \left(\frac{1}{2} \times AC \times h_1\right) + \left(\frac{1}{2} \times AC \times h_2\right)$

$= \left\{\frac{1}{2} \times AC \times (h_1 + h_2)\right\}$ sq units.

EXAMPLE 14 *In a four-sided field, the length of the longer diagonal is* 128 m. *The lengths of the perpendiculars from the opposite vertices upon this diagonal are* 22.7 m *and* 17.3 m. *Find the area of the field.*



SOLUTION Let *ABCD* be the field, and let *AC* be its longer diagonal.

Let $BL \perp AC$ and $DM \perp AC$.

Then, $AC = 128$ m, $BL = 22.7$ m and $DM = 17.3$ m.

$\therefore$ area of the field $= \left\{\frac{1}{2} \times AC \times (BL + DM)\right\}$ m^2

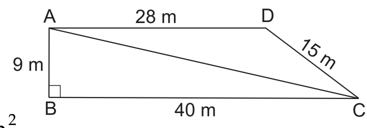
$= \left[\frac{1}{2} \times 128 \times (22.7 + 17.3)\right]$ m^2

$= (64 \times 40)$ m^2 = 2560 m^2.

SOME MORE EXAMPLES

EXAMPLE 15 *Find the area of the quadrilateral ABCD in which* $AB = 9$ m, $BC = 40$ m, $\angle ABC = 90°$, $CD = 15$ m *and* $AD = 28$ m.

SOLUTION Area of right $\triangle ABC$



$$= \left(\frac{1}{2} \times \text{base} \times \text{height}\right)$$

$$= \left(\frac{1}{2} \times 40 \times 9\right) \text{m}^2 = 180 \text{ m}^2.$$

Also, from right $\triangle ABC$, we get

$$AC^2 = AB^2 + BC^2$$
$$= [(9)^2 + (40)^2] \text{ m}^2$$
$$= (81 + 1600) \text{ m}^2 = 1681 \text{ m}^2$$

$\Rightarrow \quad AC = \sqrt{1681}$ m = 41 m.

In $\triangle ACD$, we have $AC = 41$ m, $CD = 15$ m and $AD = 28$ m.

Let $a = 41$ m, $b = 15$ m and $c = 28$ m. Then,

$$s = \frac{1}{2}(41 + 15 + 28) \text{ m} = \left(\frac{1}{2} \times 84\right) \text{m} = 42 \text{ m}.$$

$\therefore \quad (s - a) = (42 - 41)$ m = 1 m, $(s - b) = (42 - 15)$ m = 27 m and $(s - c) = (42 - 28)$ m = 14 m.

$$\therefore \quad \text{area of } \triangle ACD = \sqrt{s(s-a)(s-b)(s-c)}$$
$$= \sqrt{42 \times 1 \times 27 \times 14} \text{ m}^2$$
$$= (14 \times 9) \text{ m}^2 = 126 \text{ m}^2.$$

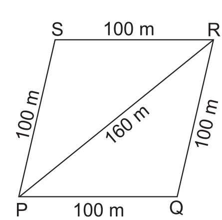
Area of the quad. $ABCD$ = area ($\triangle ABC$) + area ($\triangle ACD$)

$$= (180 + 126) \text{ m}^2 = 306 \text{ m}^2.$$

EXAMPLE 16 *A piece of land is in the shape of a rhombus whose perimeter is* 400 m *and one of its diagonals is* 160 m. *Find the area of the land.*

SOLUTION Let $PQRS$ be the field in the shape of a rhombus.

Its perimeter = 400 m.



Length of each of its sides

$$= \left(\frac{1}{4} \times 400\right) \text{m} = 100 \text{ m}.$$

Let diagonal $PR = 160$ m.

In $\triangle PQR$, we have

$PQ = 100$ m, $QR = 100$ m and $PR = 160$ m.

Let the lengths of these sides be denoted by a, b, c respectively.

Then, $a = 100$ m, $b = 100$ m and $c = 160$ m.

$\therefore \quad s = \frac{1}{2}(100 + 100 + 160)\text{ m} = \left(\frac{1}{2} \times 360\right)\text{ m} = 180\text{ m}.$

$\therefore \quad (s - a) = (180 - 100)\text{ m} = 80\text{ m}, (s - b) = (180 - 100)\text{ m} = 80\text{ m}$

and $(s - c) = (180 - 160)\text{ m} = 20\text{ m}.$

$\therefore \quad \text{area}(\triangle PQR) = \sqrt{s(s-a)(s-b)(s-c)}$

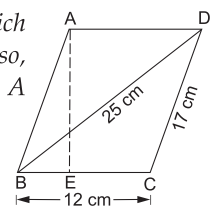
$= \sqrt{180 \times 80 \times 80 \times 20}\text{ m}^2$

$= (80 \times 60)\text{ m}^2 = 4800\text{ m}^2.$

Clearly, area $(\triangle PRS)$ = area $(\triangle PQR) = 4800\text{ m}^2$.

Hence, the area of the whole land $= (4800 + 4800)\text{ m}^2 = 9600\text{ m}^2$.

EXAMPLE 17 *Find the area of the parallelogram ABCD in which* $BC = 12$ cm, $CD = 17$ cm *and* $BD = 25$ cm. *Also, find the length of the altitude AE from vertex A on the side BC.*



SOLUTION In $\triangle BCD$, we have

$BC = 12$ cm, $CD = 17$ cm and $BD = 25$ cm.

Let these sides be denoted by a, b, c respectively.

Then, $a = 12$ cm, $b = 17$ cm and $c = 25$ cm.

$\therefore \quad s = \frac{1}{2}(12 + 17 + 25)\text{ cm} = \left(\frac{1}{2} \times 54\right)\text{ cm} = 27\text{ cm}.$

$\therefore \quad (s - a) = (27 - 12)\text{ cm} = 15\text{ cm}, (s - b) = (27 - 17)\text{ cm} = 10\text{ cm}$

and $(s - c) = (27 - 25)\text{ cm} = 2\text{ cm}.$

Now, $\text{area}(\triangle BCD) = \sqrt{s(s-a)(s-b)(s-c)}$

$= \sqrt{27 \times 15 \times 10 \times 2}\text{ cm}^2 = \sqrt{81 \times 25 \times 4}\text{ cm}^2$

$= (9 \times 5 \times 2)\text{ cm}^2 = 90\text{ cm}^2.$

Area of $\|$gm $ABCD = 2 \times (\text{area of } \triangle BCD)$

$= (2 \times 90)\text{ cm}^2 = 180\text{ cm}^2.$

Draw $AE \perp BC$. Let $AE = h$ cm. Then,

area of $\|$gm $ABCD$ = (base $\times$ altitude) $= (12 \times h)\text{ cm}^2$.

$\therefore \quad 12 \times h = 180 \Rightarrow h = \frac{180}{12} = 15 \Rightarrow AE = h\text{ cm} = 15\text{ cm}.$

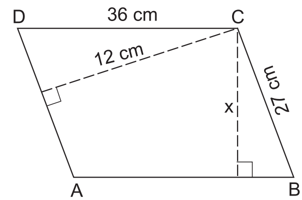
Hence, the altitude of the given parallelogram is 15 cm.

EXAMPLE 18 *The adjacent sides of a parallelogram are* 36 cm *and* 27 cm *in length. If the distance between the shorter sides is* 12 cm, *find the distance between the longer sides.*

SOLUTION Longer side = 36 cm and shorter side = 27 cm.

Distance between shorter sides = 12 cm.

Let the distance between the longer sides be x cm.



Area of the parallelogram

= (longer side $\times$ distance between the longer sides)

= (shorter side $\times$ distance between the shorter sides).

$$\therefore \quad 36 \times x = 27 \times 12 \Rightarrow x = \frac{27 \times 12}{36} = 9.$$

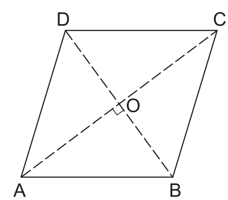
Hence, the distance between the longer sides is 9 cm.

EXAMPLE 19 *The diagonals of a rhombus are* 48 cm *and* 20 cm *long. Find (i) the area of the rhombus and (ii) the perimeter of the rhombus.*

SOLUTION (i) Area of the given rhombus

$$= \left(\frac{1}{2} \times \text{product of diagonals}\right)$$

$$= \left(\frac{1}{2} \times 48 \times 20\right) \text{cm}^2 = 480 \text{ cm}^2.$$



(ii) We know that the diagonals of a rhombus bisect each other at right angles.

$\therefore$ $OA = OC = 24$ cm, $OB = OD = 10$ cm and $\angle AOB = 90°$.

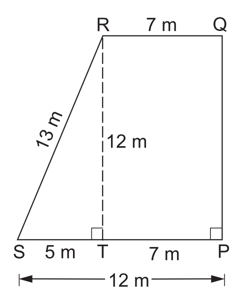
By Pythagoras' theorem, we have

$$AB = \sqrt{OA^2 + OB^2} = \sqrt{(24)^2 + (10)^2} \text{ cm}$$

$$= \sqrt{676} \text{ cm} = 26 \text{ cm}.$$

$\therefore$ perimeter of the rhombus = (4×26) cm = 104 cm.

EXAMPLE 20 *Find the area of the given trapezium PQRS in which $RQ \parallel SP$ and $PQ \perp SP$ such that $RQ = 7$ m, $RS = 13$ m and $SP = 12$ m.*



SOLUTION Draw $RT \perp SP$. Then, $TP = RQ = 7$ m.

$\therefore \quad ST = SP - TP = (12 - 7)$ m = 5 m.

From right $\triangle RTS$, we have

$$RT^2 = RS^2 - ST^2 = [(13 \text{ m})^2 - (5 \text{ m})^2]$$

$$= (169 - 25) \text{ m}^2 = 144 \text{ m}^2.$$

$\therefore \quad RT = \sqrt{144}$ m = 12 m.

Area of trap. $PQRS$

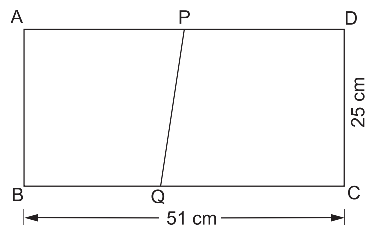
$$= \left(\frac{1}{2} \times \text{sum of parallel sides} \times \text{distance between them}\right)$$

$$= \left\{\frac{1}{2} \times (RQ + SP) \times RT\right\} \text{ sq units}$$

$$= \left\{\frac{1}{2} \times (7 + 12) \times 12\right\} \text{ m}^2 = (19 \times 6) \text{ m}^2 = 114 \text{ m}^2.$$

Hence, the area of the trapezium $PQRS$ is 114 m^2.

EXAMPLE 21 *In the given figure, ABCD is a rectangle of length* 51 cm *and breadth* 25 cm. *A trapezium PQCD with its parallel sides QC and PD in the ratio* 9 : 8 *is cut off from the rectangle, as shown in the figure. If the area of the trapezium PQCD is* $\frac{5}{6}$*th part of the area of the rectangle, find the lengths QC and PD.*



SOLUTION Length of rectangle $ABCD$ = 51 cm.

Breadth of rectangle $ABCD$ = 25 cm.

Area of rectangle $ABCD = (51 \times 25) \text{ cm}^2 = 1275 \text{ cm}^2$.

Let $QC = 9k$ and $PD = 8k$. Then,

area(trap. $PQCD$)

$$= \left(\frac{1}{2} \times \text{sum of parallel sides} \times \text{distance between them}\right)$$

$$= \left\{\frac{1}{2} \times (9k + 8k) \times 25\right\} \text{ cm}^2 = \left(\frac{17 \times 25}{2}\right) k \text{ cm}^2.$$

Now, area (trap. $PQCD$) $= \frac{5}{6} \times$ (area of rectangle $ABCD$)

$$\Rightarrow \quad \left(\frac{17 \times 25}{2}\right) k = \left(\frac{5}{6} \times 1275\right)$$

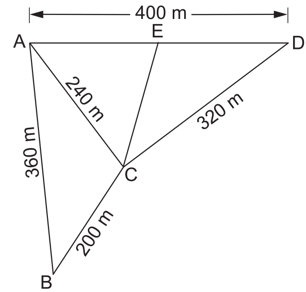
$$\Rightarrow \quad k = \left(\frac{5}{6} \times 1275 \times \frac{2}{17 \times 25}\right) \Rightarrow k = 5.$$

$\therefore \quad QC = (9k) \text{ cm} = (9 \times 5) \text{ cm} = 45 \text{ cm}$

and $PD = (8k) \text{ cm} = (8 \times 5) \text{ cm} = 40 \text{ cm}$.

Hence, $QC = 45$ cm and $PD = 40$ cm.

EXAMPLE 22 *A farmer has a triangular field with sides* 360 m, 200 m *and* 240 m, *where he grows wheat. Adjacent to this field, he has another* *triangular field with sides* 240 m, 320 m *and* 400 m, *divided into two parts by joining the midpoint of the longest side to the opposite vertex. He grows potatoes in one part and onions in the other part. How much area (in hectares) has been used for wheat, potatoes and onions?* (1 *hectare* = 10000 m^2.)



SOLUTION Let ABC be the field where wheat is grown. Let ACD be the field which has been divided into two parts by joining C to the midpoint E of AD.

For $\triangle ABC$, we have

$$a = 200 \text{ m}, b = 240 \text{ m and } c = 360 \text{ m}.$$

$$\therefore \quad s = \frac{1}{2}(200 + 240 + 360) \text{ m} = 400 \text{ m}.$$

$$\therefore \quad (s-a) = (400-200) \text{ m} = 200 \text{ m},$$

$$(s-b) = (400-240) \text{ m} = 160 \text{ m}$$

and $(s-c) = (400-360) \text{ m} = 40 \text{ m}$.

Area of $\triangle ABC = \sqrt{s(s-a)(s-b)(s-c)}$

$$= \sqrt{400 \times 200 \times 160 \times 40} \text{ m}^2 = (16000 \times \sqrt{2}) \text{ m}^2$$

$$= \left(\frac{16000 \times 1.41}{10000}\right) \text{ hectares} = 2.26 \text{ hectares}.$$

For $\triangle ACD$, we have

$$a = 240 \text{ m}, b = 320 \text{ m and } c = 400 \text{ m}.$$

$$\therefore \quad s = \frac{1}{2}(240 + 320 + 400) \text{ m} = \left(\frac{1}{2} \times 960\right) \text{ m} = 480 \text{ m}.$$

$$\therefore \quad (s-a) = (480-240) \text{ m} = 240 \text{ m},$$

$$(s-b) = (480-320) \text{ m} = 160 \text{ m}$$

and $(s-c) = (480-400) \text{ m} = 80 \text{ m}$.

Area of $\triangle ACD = \sqrt{s(s-a)(s-b)(s-c)}$

$$= \sqrt{480 \times 240 \times 160 \times 80} \text{ m}^2$$

$$= (240 \times 160) \text{ m}^2 = 38400 \text{ m}^2$$

$$= \frac{38400}{10000} \text{ hectares} = 3.84 \text{ hectares}.$$

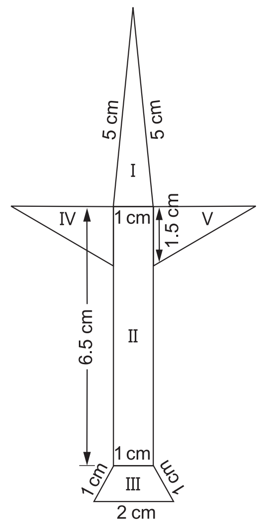
Now, $\triangle ACE$ and $\triangle DCE$ have equal bases and the same height. So, they are equal in area.

$\therefore$ area ($\triangle ACE$) = area ($\triangle DCE$) = $\dfrac{3.84}{2}$ hectares

= 1.92 hectares.

Hence, the area for wheat is 2.26 hectares, while the area for each one of the potatoes and onions is 1.92 hectares.

EXAMPLE 23 *Reenu made a picture of an aeroplane with coloured paper as shown in the figure given below. Find the total area of the paper used.*



SOLUTION The given picture has been divided into five regions marked I to V, as shown.

Area of Region I This region is a triangle in which $a = 5$ cm, $b = 5$ cm, $c = 1$ cm.

$\therefore \quad s = \frac{1}{2}(5+5+1)$ cm $= \frac{11}{2}$ cm.

$\therefore \quad (s-a) = \left(\frac{11}{2} - 5\right)$ cm $= \frac{1}{2}$ cm,

$(s-b) = \left(\frac{11}{2} - 5\right)$ cm $= \frac{1}{2}$ cm,

$(s-c) = \left(\frac{11}{2} - 1\right)$ cm $= \frac{9}{2}$ cm.

$\therefore$ area of Region I $= \sqrt{s(s-a)(s-b)(s-c)}$

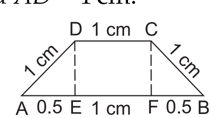
$$= \sqrt{\frac{11}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{9}{2}} \text{ cm}^2 = \sqrt{\frac{99}{16}} \text{ cm}^2$$

$$= \frac{\sqrt{99}}{4} \text{ cm}^2 = \frac{9.94}{4} \text{ cm}^2 = 2.49 \text{ cm}^2.$$

Area of Region II This region is a rectangle of length 6.5 cm and breadth 1 cm.

$\therefore$ area of Region II $= (6.5 \times 1)$ cm^2 $= 6.5$ cm^2.

Area of Region III This region is an isosceles trapezium $ABCD$ in which $AB = 2$ cm, $BC = 1$ cm, $DC = 1$ cm and $AD = 1$ cm.



Draw $DE \perp AB$ and $CF \perp AB$.

Then, $EF = DC = 1$ cm.

And, $AE = BF = 0.5$ cm.

$$DE^2 = AD^2 - AE^2 = \left\{1^2 - \left(\frac{1}{2}\right)^2\right\} \text{ cm}^2 = \left(1 - \frac{1}{4}\right) \text{ cm}^2 = \frac{3}{4} \text{ cm}^2.$$

$\therefore \quad DE = \dfrac{\sqrt{3}}{2}$ cm.

$$\text{Area of Region III} = \frac{1}{2} \times (AB + DC) \times DE$$
$$= \left\{\frac{1}{2} \times (2+1) \times \frac{\sqrt{3}}{2}\right\} \text{cm}^2 = \frac{3\sqrt{3}}{4} \text{cm}^2$$
$$= \frac{3 \times 1.732}{4} \text{cm}^2 = \frac{5.196}{4} \text{cm}^2$$
$$= 1.299 \text{ cm}^2 \approx 1.3 \text{ cm}^2.$$

Area of Region IV This region is a right triangle with base 1.5 cm and height 6 cm.

$\therefore$ area of Region IV $= \left(\frac{1}{2} \times \text{base} \times \text{height}\right)$

$$= \left(\frac{1}{2} \times \frac{3}{2} \times 6\right) \text{cm}^2 = \frac{9}{2} \text{cm}^2 = 4.5 \text{ cm}^2.$$

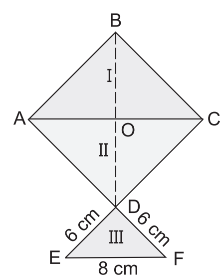
Area of Region V This region is a right triangle with base 1.5 cm and height 6 cm.

$\therefore$ area of Region V $= \left(\frac{1}{2} \times \frac{3}{2} \times 6\right) \text{cm}^2 = \frac{9}{2} \text{cm}^2 = 4.5 \text{ cm}^2.$

$\therefore$ total area of the paper used

$$= (2.49 + 6.5 + 1.3 + 4.5 + 4.5) \text{ cm}^2 = 19.29 \text{ cm}^2.$$

EXAMPLE 24 *A kite in the shape of a square with a diagonal* 32 cm *and an isosceles triangle of base* 8 cm *and sides* 6 cm *each is to be made of three different shades as shown in the figure. How much paper of each shade has been used in it?*



SOLUTION The given figure has been divided into three regions I, II and III consisting of $\triangle ABC$, $\triangle ADC$ and $\triangle DEF$ respectively.

Join BD, cutting AC at O.

We know that the diagonals of a square are equal and bisect each other at right angles.

$\therefore$ $AC = BD = 32$ cm, $OB = OD = 16$ cm,

$\angle AOB = \angle AOD = 90°$.

Area of Shade I Area $(\triangle ABC) = \left(\frac{1}{2} \times AC \times OB\right)$

$$= \left(\frac{1}{2} \times 32 \times 16\right) \text{cm}^2 = 256 \text{ cm}^2.$$

Area of Shade II Area $(\triangle ACD) = \left(\frac{1}{2} \times AC \times OD\right)$

$$= \left(\frac{1}{2} \times 32 \times 16\right) \text{ cm}^2 = 256 \text{ cm}^2.$$

Area of Shade III In $\triangle DEF$, $a = 8$ cm, $b = 6$ cm and $c = 6$ cm.

$\therefore \quad s = \frac{1}{2}(8 + 6 + 6)$ cm $= 10$ cm.

$\therefore \quad (s - a) = (10 - 8)$ cm $= 2$ cm, $(s - b) = (10 - 6)$ cm $= 4$ cm and $(s - c) = (10 - 6)$ cm $= 4$ cm.

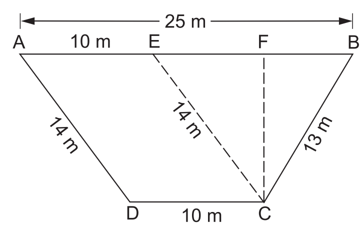
$$\therefore \quad \text{area } (\triangle DEF) = \sqrt{s(s-a)(s-b)(s-c)}$$
$$= \sqrt{10 \times 2 \times 4 \times 4} \text{ cm}^2 = 8\sqrt{5} \text{ cm}^2$$
$$= (8 \times 2.236) \text{ cm}^2 \approx (8 \times 2.24) \text{ cm}^2$$
$$= 17.92 \text{ cm}^2.$$

Hence, the areas of Shade I, Shade II and Shade III are respectively 256 cm^2, 256 cm^2 and 17.92 cm^2.

EXAMPLE 25 *A field is in the shape of a trapezium whose parallel sides are* 25 m *and* 10 m *and the nonparallel sides are* 14 m *and* 13 m. *Find the area of the field.*

SOLUTION Let $ABCD$ be the field in the form of a trapezium in which $AB \parallel CD$ such that

$AB = 25$ m, $BC = 13$ m, $CD = 10$ m and $DA = 14$ m.



Draw $CE \parallel DA$ and $CF \perp EB$.

Clearly, $ADCE$ is a parallelogram.

$\therefore \quad CE = DA = 14$ m and $AE = CD = 10$ m.

$\therefore \quad EB = AB - AE = (25 - 10)$ m $= 15$ m.

In $\triangle EBC$, we have

$EB = 15$ m, $BC = 13$ m and $CE = 14$ m.

$\therefore \quad a = 15$ m, $b = 13$ m and $c = 14$ m.

$\therefore \quad s = \frac{1}{2}(15 + 13 + 14)$ m $= 21$ m.

$\therefore \quad (s - a) = (21 - 15)$ m $= 6$ m, $(s - b) = (21 - 13)$ m $= 8$ m and $(s - c) = (21 - 14)$ m $= 7$ m.

$$\therefore \quad \text{area } (\triangle EBC) = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{21 \times 6 \times 8 \times 7} \text{ m}^2$$
$$= \sqrt{7 \times 3 \times 3 \times 2 \times 2 \times 4 \times 7} \text{ m}^2$$
$$= (7 \times 3 \times 2 \times 2) \text{ m}^2 = 84 \text{ m}^2.$$

Also, area $(\triangle EBC) = \left(\frac{1}{2} \times EB \times CF\right) = \left(\frac{1}{2} \times 15 \text{ m} \times CF\right)$

$\therefore \quad \frac{1}{2} \times 15 \text{ m} \times CF = 84 \text{ m}^2 \Rightarrow CF = \frac{84 \times 2}{15} \text{ m} = \frac{56}{5} \text{ m} = 11.2 \text{ m}.$

$\therefore \quad CF = 11.2 \text{ m}.$

$$\begin{aligned}\text{Area (trap. } ABCD) &= \frac{1}{2} \times (AB + CD) \times CF \\ &= \left\{\frac{1}{2} \times (25 + 10) \times 11.2\right\} \text{m}^2 \\ &= (35 \times 5.6) \text{ m}^2 = 196 \text{ m}^2.\end{aligned}$$

Hence, the area of the trapezium $ABCD$ is 196 m^2.

EXAMPLE 26 *If each side of a triangle is doubled then find the ratio of the area of the new triangle thus formed and the given triangle.*

SOLUTION Let the sides of the given triangle be a, b, c and let s be its semiperimeter. Then, the area of this triangle is given by $\triangle = \sqrt{s(s-a)(s-b)(s-c)}$ sq units.

When each side of this triangle is doubled then we get a new triangle having sides $2a$, $2b$ and $2c$.

Let S be the semiperimeter of the new triangle. Then,

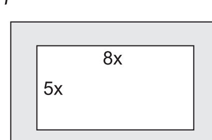
$$S = \frac{1}{2}(2a + 2b + 2c) = (a + b + c) = 2s.$$

$$\begin{aligned}\text{Area of the new triangle} &= \sqrt{S(S-2a)(S-2b)(S-2c)} \\ &= \sqrt{2s(2s-2a)(2s-2b)(2s-2c)} \text{ sq units} \\ &= \sqrt{16s(s-a)(s-b)(s-c)} \text{ sq units} \\ &= 4 \times \sqrt{s(s-a)(s-b)(s-c)} \text{ sq units} \\ &= 4\triangle.\end{aligned}$$

$\therefore$ (area of the new triangle) : (area of the given triangle)
$= 4\triangle : \triangle = 4 : 1.$

EXAMPLE 27 *The length and breadth of a rectangular park are in the ratio* 8 : 5. *A path,* 1.5 m *wide, running all around the outside of the park has an area of* 594 m^2. *Find the dimensions of the park.*

SOLUTION Let the length and breadth of the park be $8x$ metres and $5x$ metres respectively.



Area of the park $= (8x \times 5x) \text{ m}^2$
$= (40x^2) \text{ m}^2.$

Length of the park including the path $= (8x + 3)$ m.

Breadth of the park including the path $= (5x + 3)$ m.

Area of the park including the path = $(8x+3)(5x+3)\ \text{m}^2$.

Area of the path = $\{(8x+3)(5x+3)-40x^2\}\ \text{m}^2 = (39x+9)\ \text{m}^2$.

But, area of the path = $594\ \text{m}^2$ (given)

$\therefore \quad 39x+9=594 \Rightarrow 39x=585$

$$\Rightarrow x=\frac{585}{39}=15.$$

$\therefore \quad$ length of the park = (8×15) m = 120 m

and breadth of the park = (5×15) m = 75 m.

EXERCISE 14

1. Find the area of the triangle whose base measures 24 cm and the corresponding height measures 14.5 cm.

2. The base of a triangular field is three times its altitude. If the cost of sowing the field at ₹ 58 per hectare is ₹ 783, find its base and height.

3. Find the area of the triangle whose sides are 42 cm, 34 cm and 20 cm in length. Hence, find the height corresponding to the longest side.

4. Calculate the area of the triangle whose sides are 18 cm, 24 cm and 30 cm in length. Also, find the length of the altitude corresponding to the smallest side.

5. Find the area of a triangular field whose sides are 91 m, 98 m and 105 m in length. Find the height corresponding to the longest side.

6. The sides of a triangle are in the ratio 5 : 12 : 13 and its perimeter is 150 m. Find the area of the triangle.

7. The perimeter of a triangular field is 540 m and its sides are in the ratio 25 : 17 : 12. Find the area of the field. Also, find the cost of ploughing the field at ₹ 5 per m^2.

8. Two sides of a triangular field are 85 m and 154 m in length and its perimeter is 324 m. Find (i) the area of the field and (ii) the length of the perpendicular from the opposite vertex on the side measuring 154 m.

9. Find the area of an isosceles triangle each of whose equal sides measures 13 cm and whose base measures 20 cm.

10. The base of an isosceles triangle measures 80 cm and its area is $360\ \text{cm}^2$. Find the perimeter of the triangle.

11. The perimeter of an isosceles triangle is 32 cm. The ratio of the equal side to its base is 3 : 2. Find the area of the triangle.

HINT Ratio of sides = 3 : 3 : 2.

12. The perimeter of a triangle is 50 cm. One side of the triangle is 4 cm longer than the smallest side and the third side is 6 cm less than twice the smallest side. Find the area of the triangle.

13. The triangular side walls of a flyover have been used for advertisements. The sides of the walls are 13 m, 14 m, 15 m. The advertisements yield an earning of ₹ 2000 per m^2 a year. A company hired one of its walls for 6 months. How much rent did it pay?

14. The perimeter of an isosceles triangle is 42 cm and its base is $1\frac{1}{2}$ times each of the equal sides. Find (i) the length of each side of the triangle, (ii) the area of the triangle, and (iii) the height of the triangle. (Given, $\sqrt{7} = 2.64$.)

15. If the area of an equilateral triangle is $36\sqrt{3}\ \text{cm}^2$, find its perimeter.

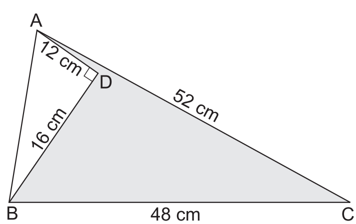
16. If the area of an equilateral triangle is $81\sqrt{3}\ \text{cm}^2$, find its height.

17. Each side of an equilateral triangle measures 8 cm. Find (i) the area of the triangle, correct to 2 places of decimal and (ii) the height of the triangle, correct to 2 places of decimal. (Take $\sqrt{3} = 1.732$.)

18. The height of an equilateral triangle measures 9 cm. Find its area, correct to 2 places of decimal. (Take $\sqrt{3} = 1.732$.)

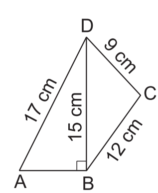
19. The base of a right-angled triangle measures 48 cm and its hypotenuse measures 50 cm. Find the area of the triangle.

20. Find the area of the shaded region in the figure given below.

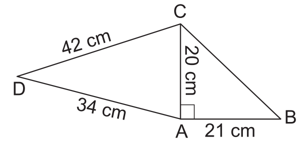


21. The sides of a quadrilateral *ABCD* taken in order are 6 cm, 8 cm, 12 cm and 14 cm respectively and the angle between the first two sides is a right angle. Find its area. (Given, $\sqrt{6} = 2.45$.)

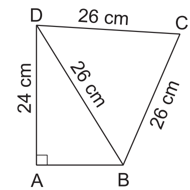
22. Find the perimeter and area of a quadrilateral *ABCD* in which $BC = 12$ cm, $CD = 9$ cm, $BD = 15$ cm, $DA = 17$ cm and $\angle ABD = 90°$.



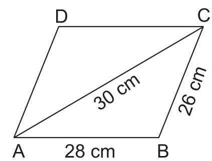
23. Find the perimeter and area of the quadrilateral $ABCD$ in which $AB = 21$ cm, $\angle BAC = 90^\circ$, $AC = 20$ cm, $CD = 42$ cm and $AD = 34$ cm.



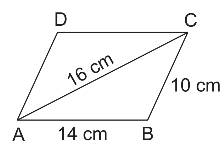
24. Find the area of the quadrilateral $ABCD$ in which BCD is an equilateral triangle, each of whose sides is 26 cm, $AD = 24$ cm and $\angle BAD = 90^\circ$. Also, find the perimeter of the quadrilateral. (Given, $\sqrt{3} = 1.73$.)



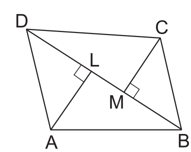
25. Find the area of a parallelogram $ABCD$ in which $AB = 28$ cm, $BC = 26$ cm and diagonal $AC = 30$ cm.



26. Find the area of a parallelogram $ABCD$ in which $AB = 14$ cm, $BC = 10$ cm and $AC = 16$ cm. (Given, $\sqrt{3} = 1.73$.)

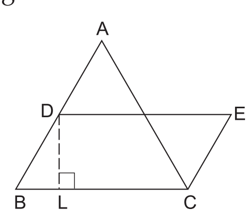


27. In the given figure, $ABCD$ is a quadrilateral in which diagonal $BD = 64$ cm, $AL \perp BD$ and $CM \perp BD$ such that $AL = 16.8$ cm and $CM = 13.2$ cm. Calculate the area of the quadrilateral $ABCD$.



28. The area of a trapezium is 475 cm^2 and its height is 19 cm. Find the lengths of its two parallel sides if one side is 4 cm greater than the other.

29. In the given figure, a $\triangle ABC$ has been given in which $AB = 7.5$ cm, $AC = 6.5$ cm and $BC = 7$ cm. On base BC, a parallelogram $DBCE$ of the same area as that of $\triangle ABC$ is constructed. Find the height DL of the parallelogram.



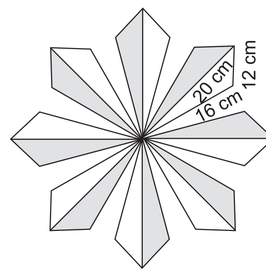
30. A field is in the shape of a trapezium having parallel sides 90 m and 30 m. These sides meet the third side at right angles. The length of the fourth side is 100 m. If it costs ₹5 to plough 1 m^2 of the field, find the total cost of ploughing the field.

31. A rectangular plot is given for constructing a house, having a measurement of 40 m long and 15 m in the front. According to the laws, a minimum of 3-m-wide space should be left in the front and back each and 2 m wide space on each of the other sides. Find the largest area where house can be constructed.

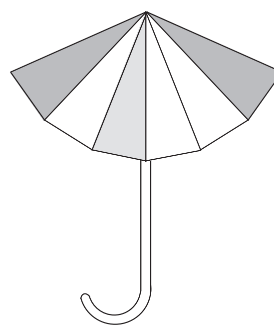
32. A rhombus-shaped sheet with perimeter 40 cm and one diagonal 12 cm, is painted on both sides at the rate of ₹5 per cm^2. Find the cost of painting.

33. The difference between the semiperimeter and the sides of a $\triangle ABC$ are 8 cm, 7 cm and 5 cm respectively. Find the area of the triangle.

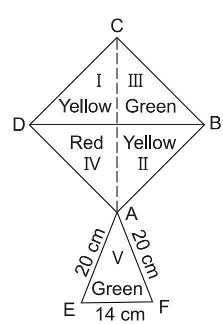
34. A floral design on a floor is made up of 16 tiles, each triangular in shape having sides 16 cm, 12 cm and 20 cm. Find the cost of polishing the tiles at ₹ 1 per sq cm.



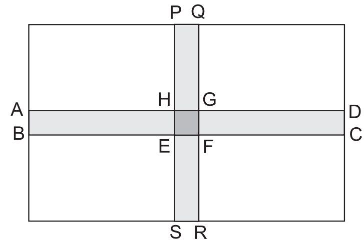
35. An umbrella is made by stitching 12 triangular pieces of cloth, each measuring (50 cm × 20 cm × 50 cm). Find the area of the cloth used in it.



36. In the given figure, *ABCD* is a square with diagonal 44 cm. How much paper of each shade is needed to make a kite given in the figure?



37. A rectangular lawn, 75 m by 60 m, has two roads, each road 4 m wide, running through the middle of the lawn, one parallel to length and the other parallel to breadth, as shown in the figure. Find the cost of gravelling the roads at ₹ 50 per m^2.



38. The shape of the cross section of a canal is a trapezium. If the canal is 10 m wide at the top, 6 m wide at the bottom and the area of its cross section is 640 m^2, find the depth of the canal.

39. Find the area of a trapezium whose parallel sides are 11 m and 25 m long, and the nonparallel sides are 15 m and 13 m long.

40. The difference between the lengths of the parallel sides of a trapezium is 8 cm, the perpendicular distance between these sides is 24 cm and the area of the trapezium is 312 cm^2. Find the length of each of the parallel sides.

41. A parallelogram and a rhombus are equal in area. The diagonals of the rhombus measure 120 m and 44 m. If one of the sides of the parallelogram measures 66 m, find its corresponding altitude.

42. A parallelogram and a square have the same area. If the sides of the square measure 40 m and altitude of the parallelogram measures 25 m, find the length of the corresponding base of the parallelogram.

43. Find the area of a rhombus one side of which measures 20 cm and one of whose diagonals is 24 cm.

44. The area of a rhombus is 480 cm^2, and one of its diagonals measures 48 cm. Find (i) the length of the other diagonal, (ii) the length of each of its sides, and (iii) its perimeter.

ANSWERS (EXERCISE 14)

1. 174 cm^2 **2.** 900 m, 300 m **3.** 336 cm^2, 16 cm
4. 216 cm^2, 24 cm **5.** 4116 m^2, 78.4 m **6.** 750 m^2
7. 9000 m^2, ₹ 45000 **8.** (i) 2772 m^2 (ii) 36 m **9.** 83.06 cm^2
10. 162 cm **11.** $32\sqrt{3}$ cm^2 **12.** $20\sqrt{30}$ cm^2 **13.** ₹ 84000
14. (i) 12 cm, 12 cm, 18 cm (ii) 71.28 cm^2 (iii) 7.92 cm **15.** 36 cm
16. $9\sqrt{3}$ cm **17.** (i) 27.71 cm^2 (ii) 6.93 cm **18.** 46.76 cm^2
19. 336 cm^2 **20.** 384 cm^2 **21.** 82.8 cm^2 **22.** 46 cm, 114 cm^2
23. 126 cm, 546 cm^2 **24.** 412.37 cm^2, 86 cm **25.** 672 cm^2
26. 138.4 cm^2 **27.** 960 cm^2 **28.** 23 cm, 27 cm **29.** 3 cm

30. ₹ 24000 **31.** 374 m^2 **32.** ₹ 960 **33.** $20\sqrt{14}$ cm^2
34. ₹ 1536 **35.** 5880 cm^2
36. Yellow: 484 cm^2, Red: 242 cm^2, Green: 373.11 cm^2 **37.** ₹ 26200
38. 80 m **39.** 216 m^2 **40.** 17 cm, 9 cm **41.** 40 m
42. 64 m **43.** 384 cm^2 **44.** (i) 20 cm (ii) 26 cm (iii) 104 cm

HINTS TO SOME SELECTED QUESTIONS

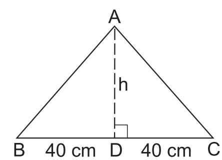
2. Let the altitude be x metres. Then, base = $3x$ metres.

$\therefore$ area $= \left(\frac{1}{2} \times \text{base} \times \text{altitude}\right) = \left(\frac{1}{2} \times 3x \times x\right) \text{m}^2 = \frac{3x^2}{2} \text{m}^2$.

But, area $= \frac{\text{total cost of sowing}}{\text{rate per hectare}} = \frac{783}{58}$ hectares $= \left(\frac{27}{2} \times 10000\right) \text{m}^2 = 135000 \text{ m}^2$.

$\therefore$ $\frac{3x^2}{2} = 135000 \Rightarrow x^2 = 90000 \Rightarrow x = 300$.

10. $\frac{1}{2} \times 80 \text{ cm} \times h = 360 \text{ cm}^2 \Rightarrow h = \frac{360}{40} \text{ cm} = 9 \text{ cm}$.



$AB^2 = (40 \text{ cm})^2 + h^2 = (40 \text{ cm})^2 + (9 \text{ cm})^2 = 1681 \text{ cm}^2$

$\Rightarrow$ $AB = \sqrt{1681}$ cm = 41 cm.

$\therefore$ $AC = AB = 41$ cm.

Hence, perimeter = (41 + 41 + 80) cm = 162 cm.

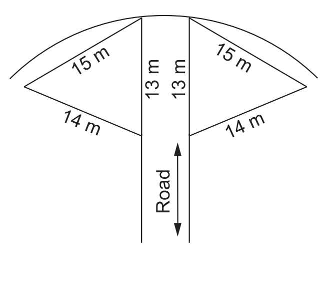
12. Let the smallest side be x cm.

Then, the other sides are $(x+4)$ cm and $(2x-6)$ cm.

$\therefore$ $x + (x+4) + (2x-6) = 50 \Rightarrow 4x = 52 \Rightarrow x = 13$.

Thus, the sides are 13 cm, 17 cm and 20 cm.

13. Sides of one wall are 13 m, 14 m, 15 m.



$\therefore$ area of one wall $= \sqrt{21 \times 8 \times 7 \times 6} \text{ m}^2$
$= 84 \text{ m}^2$.

Rent of one wall for 6 months = ₹ (84×1000)
= ₹ 84000.

14. (i) $x + x + \frac{3}{2}x = 42 \Rightarrow \frac{7x}{2} = 42 \Rightarrow x = \frac{84}{7} = 12$.

So, the sides are 12 cm, 12 cm, 18 cm.

(ii) Area of the triangle $= \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{21 \times 9 \times 9 \times 3} \text{ cm}^2$
$= (27 \times \sqrt{7}) \text{ cm}^2 = (27 \times 2.64) \text{ cm}^2 = 71.28 \text{ cm}^2$.

(iii) $\frac{1}{2} \times 18 \text{ cm} \times h = 71.28 \text{ cm}^2 \Rightarrow h = \frac{71.28}{9} \text{ cm} = 7.92 \text{ cm}$.

15. $\frac{\sqrt{3}}{4} \times a^2 = 36\sqrt{3} \Rightarrow a^2 = 144 \Rightarrow a = 12$.

$\therefore$ perimeter = (3×12) cm = 36 cm.

16. $\frac{\sqrt{3}}{4} \times a^2 = 81\sqrt{3}\ \text{cm}^2 \Rightarrow a^2 = (81 \times 4)\ \text{cm}^2 \Rightarrow a = (9 \times 2)\ \text{cm} = 18\ \text{cm}.$

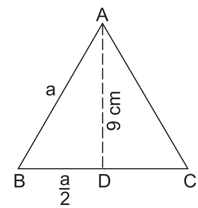
$\frac{1}{2} \times 18\ \text{cm} \times h = 81\sqrt{3}\ \text{cm}^2 \Rightarrow h = \frac{81\sqrt{3}}{9}\ \text{cm} = 9\sqrt{3}\ \text{cm}.$

17. (i) Area $= \frac{\sqrt{3}}{4}a^2 = \left(\frac{\sqrt{3}}{4} \times 64\right)\text{cm}^2 = (16\sqrt{3})\ \text{cm}^2$

$= (16 \times 1.732)\ \text{cm}^2 = 27.71\ \text{cm}^2.$

(ii) $\frac{1}{2} \times 8\ \text{cm} \times h = 27.71\ \text{cm}^2 \Rightarrow h = \frac{27.71}{4}\ \text{cm} = 6.93\ \text{cm}.$

18. $(9\ \text{cm})^2 + \left(\frac{a}{2}\right)^2 = a^2 \Rightarrow a^2 - \frac{a^2}{4} = 81\ \text{cm}^2$



$\Rightarrow \frac{3a^2}{4} = 81\ \text{cm}^2 \Rightarrow a^2 = \frac{81 \times 4}{3}\ \text{cm}^2 = 108\ \text{cm}^2.$

Area $= \frac{\sqrt{3}}{4} \times a^2 = \left(\frac{\sqrt{3}}{4} \times 108\right)\text{cm}^2 = (27\sqrt{3})\ \text{cm}^2$

$= (27 \times 1.732)\ \text{cm}^2 = 46.76\ \text{cm}^2.$

20. $AB^2 = \{(16)^2 + (12)^2\}\ \text{cm}^2 = (256 + 144)\ \text{cm}^2 = 400\ \text{cm}^2 \Rightarrow AB = \sqrt{400}\ \text{cm} = 20\ \text{cm}.$

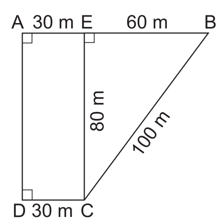
The sides of $\triangle ABC$ are 20 cm, 48 cm, 52 cm. Find its area.

Area of the shaded region = area$(\triangle ABC)$ − area$(\triangle ADB)$.

28. Let the parallel sides be x cm and $(x + 4)$ cm. Then,

$\frac{1}{2} \times [x + (x + 4)] \times 19 = 475 \Rightarrow 2x + 4 = \frac{475 \times 2}{19} \Rightarrow 2x = 46 \Rightarrow x = 23.$

30. $CE^2 = (CB^2 - EB^2)$



$= [(100)^2 - (60)^2]\ \text{m}^2 = (100 + 60)(100 - 60)\ \text{m}^2$

$= (160 \times 40)\ \text{m}^2 = 6400\ \text{m}^2.$

$\therefore\quad CE = \sqrt{6400}\ \text{m} = 80\ \text{m}.$

Area of the field

= area (rect. $ADCE$) + area ($\triangle BEC$)

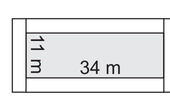
$= \left[(80 \times 30) + \left(\frac{1}{2} \times 60 \times 80\right)\right]\text{m}^2$

$= (2400 + 2400)\ \text{m}^2 = 4800\ \text{m}^2.$

Total cost of ploughing = ₹ (4800×5) = ₹ 24000.

31. Required area $= (40 - 6)\ \text{m} \times (15 - 4)\ \text{m}$

$= (34 \times 11)\ \text{m}^2 = 374\ \text{m}^2.$



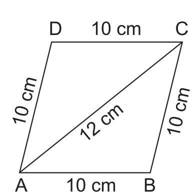
32. Required area

$= 2 \times$ area of rhombus $ABCD$

$= 2 \times (2 \times \text{area}\ \triangle ABC)$

$= 4 \times \sqrt{16 \times 4 \times 6 \times 6}\ \text{cm}^2$

$= (4 \times 48)\ \text{cm}^2 = 192\ \text{cm}^2.$



33. $(s-a)+(s-b)+(s-c)=(8+7+5) \Rightarrow 3s-(a+b+c)=20$

$\therefore \quad 3s-2s=20 \Rightarrow s=20.$

$\therefore \quad$ area of the triangle $=\sqrt{s(s-a)(s-b)(s-c)}$

$$=\sqrt{20\times 8\times 7\times 5}=\sqrt{400\times 14}=20\sqrt{14}\text{ cm}^2.$$

36. Area of the square sheet $=\frac{1}{2}\times(\text{diagonal})^2=\left(\frac{1}{2}\times 44\times 44\right)\text{cm}^2=968\text{ cm}^2.$

Area of yellow sheet = area of region I + area of region II

$$=\frac{1}{2}\times\text{area of square sheet}=\left(\frac{1}{2}\times 968\right)\text{cm}^2=484\text{ cm}^2.$$

Area of red sheet = area of region IV

$$=\frac{1}{4}\times\text{ area of square sheet}=\left(\frac{1}{4}\times 968\right)\text{cm}^2=242\text{ cm}^2.$$

Area of green sheet = area of region III + area of region V

$$=\left(\frac{1}{4}\times\text{area of square sheet}\right)+(\text{area of }\triangle AEF).$$

37. Area of roads = area $ABCD$ + area $PQRS$ – area $EFGH$

$$=(75\times 4)\text{ m}^2+(60\times 4)\text{ m}^2-(4\times 4)\text{ m}^2=(300+240-16)\text{ m}^2=524\text{ m}^2.$$

38. $\frac{1}{2}(10+6)\times d=640.$ Find d.

MULTIPLE-CHOICE QUESTIONS (MCQ)

Choose the correct answer in each of the following.

1. In a $\triangle ABC$, it is given that base = 12 cm and height = 5 cm. Its area is

(a) 60 cm^2 (b) 30 cm^2 (c) $15\sqrt{3}$ cm^2 (d) 45 cm^2

2. The lengths of three sides of a triangle are 20 cm, 16 cm and 12 cm. The area of the triangle is

(a) 96 cm^2 (b) 120 cm^2 (c) 144 cm^2 (d) 160 cm^2

3. Each side of an equilateral triangle measures 8 cm. The area of the triangle is

(a) $8\sqrt{3}$ cm^2 (b) $16\sqrt{3}$ cm^2 (c) $32\sqrt{3}$ cm^2 (d) 48 cm^2

4. The base of an isosceles triangle is 8 cm long and each of its equal sides measures 6 cm. The area of the triangle is

(a) $16\sqrt{5}$ cm^2 (b) $8\sqrt{5}$ cm^2 (c) $16\sqrt{3}$ cm^2 (d) $8\sqrt{3}$ cm^2

5. The base of an isosceles triangle is 6 cm and each of its equal sides is 5 cm. The height of the triangle is

(a) 8 cm (b) $\sqrt{30}$ cm (c) 4 cm (d) $\sqrt{11}$ cm

6. Each of the two equal sides of an isosceles right triangle is 10 cm long. Its area is

(a) $5\sqrt{10}$ cm^2 (b) 50 cm^2 (c) $10\sqrt{3}$ cm^2 (d) 75 cm^2

7. Each side of an equilateral triangle is 10 cm long. The height of the triangle is

(a) $10\sqrt{3}$ cm (b) $5\sqrt{3}$ cm (c) $10\sqrt{2}$ cm (d) 5 cm

8. The height of an equilateral triangle is 6 cm. Its area is

(a) $12\sqrt{3}\text{ cm}^2$ (b) $6\sqrt{3}\text{ cm}^2$ (c) $12\sqrt{2}\text{ cm}^2$ (d) 18 cm^2

9. The lengths of the three sides of a triangular field are 40 m, 24 m and 32 m respectively. The area of the triangle is

(a) 480 m^2 (b) 320 m^2 (c) 384 m^2 (d) 360 m^2

10. The sides of a triangle are in the ratio 5 : 12 : 13 and its perimeter is 150 cm. The area of the triangle is

(a) 375 cm^2 (b) 750 cm^2 (c) 250 cm^2 (d) 500 cm^2

11. The lengths of the three sides of a triangle are 30 cm, 24 cm and 18 cm respectively. The length of the altitude of the triangle corresponding to the smallest side is

(a) 24 cm (b) 18 cm (c) 30 cm (d) 12 cm

12. The base of an isosceles triangle is 16 cm and its area is 48 cm^2. The perimeter of the triangle is

(a) 41 cm (b) 36 cm (c) 48 cm (d) 324 cm

13. The area of an equilateral triangle is $36\sqrt{3}\text{ cm}^2$. Its perimeter is

(a) 36 cm (b) $12\sqrt{3}$ cm (c) 24 cm (d) 30 cm

14. Each of the equal sides of an isosceles triangle is 13 cm and its base is 24 cm. The area of the triangle is

(a) 156 cm^2 (b) 78 cm^2 (c) 60 cm^2 (d) 120 cm^2

15. The base of a right triangle is 48 cm and its hypotenuse is 50 cm long. The area of the triangle is

(a) 168 cm^2 (b) 252 cm^2 (c) 336 cm^2 (d) 504 cm^2

16. The area of an equilateral triangle is $81\sqrt{3}\text{ cm}^2$. Its height is

(a) $9\sqrt{3}$ cm (b) $6\sqrt{3}$ cm (c) $18\sqrt{3}$ cm (d) 9 cm

ANSWERS (MCQ)

1. (b) **2.** (a) **3.** (b) **4.** (b) **5.** (c) **6.** (b) **7.** (b) **8.** (a)
9. (c) **10.** (b) **11.** (a) **12.** (b) **13.** (a) **14.** (c) **15.** (c) **16.** (a)

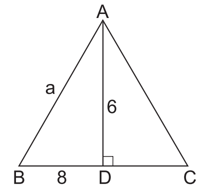
HINTS TO SOME SELECTED QUESTIONS

8. $a^2 - \left(\frac{a}{2}\right)^2 = 36 \Rightarrow a^2 - \frac{a^2}{4} = 36$

$\Rightarrow \dfrac{3a^2}{4} = 36 \Rightarrow a^2 = 48.$

$\text{Area} = \dfrac{\sqrt{3}}{4} \times a^2 = \left(\dfrac{\sqrt{3}}{4} \times 48\right) \text{cm}^2 = 12\sqrt{3} \text{ cm}^2.$

12. $\dfrac{1}{2} \times 16 \times h = 48 \Rightarrow h = 6$ cm.



$a^2 = [(8)^2 + (6)^2] \text{ cm}^2 = (64 + 36) \text{ cm}^2 = 100 \text{ cm}^2 \Rightarrow a = 10$ cm.

$\therefore$ perimeter $= (10 + 10 + 16)$ cm $= 36$ cm.

16. $\dfrac{\sqrt{3}}{4} a^2 = 81\sqrt{3} \text{ cm}^2 \Rightarrow a^2 = (81 \times 4) \text{ cm}^2 \Rightarrow a = 18$ cm.

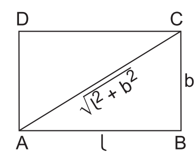
$\text{Height} = \dfrac{\sqrt{3}a}{2} = \left(\dfrac{\sqrt{3}}{2} \times 18\right) \text{cm} = 9\sqrt{3}$ cm.

REVIEW OF FACTS AND FORMULAE

1. For a Rectangle, we have

(i) Area = (length $\times$ breadth) sq units.

(ii) $\text{Length} = \dfrac{\text{area}}{\text{breadth}}$ and $\text{breadth} = \dfrac{\text{area}}{\text{length}}$.



(iii) $\text{Diagonal} = \sqrt{(\text{length})^2 + (\text{breadth})^2}$.

(iv) Perimeter = $2 \times$ (length + breadth).

(v) Area of 4 walls of a room

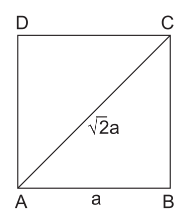
= [2 (length + breadth) $\times$ height] sq units.

2. For a Square, we have

(i) $\text{Area} = (\text{side})^2 = \dfrac{1}{2} \times (\text{diagonal})^2$.

(ii) Perimeter = ($4 \times$ side).

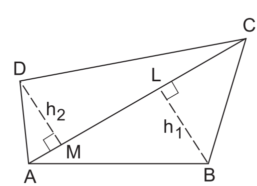
(iii) Diagonal = ($\sqrt{2} \times$ side).



3. Area of a Quadrilateral

Let $ABCD$ be a quadrilateral having diagonal AC. Let $BL \perp AC$ and $DM \perp AC$.

Then, we have



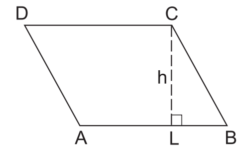
area (quad. $ABCD$)

$= \left\{\dfrac{1}{2} \times AC \times (BL + DM)\right\}$ sq units.

$= \left\{\dfrac{1}{2} \times AC \times (h_1 + h_2)\right\}$ sq units.

4. Area of a parallelogram = (base $\times$ height)

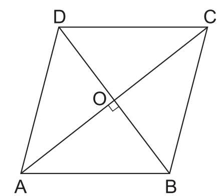
$$= (AB \times h) \text{ sq units.}$$



5. (i) Area of a Rhombus

$$= \frac{1}{2} \times \text{(product of diagonals).}$$

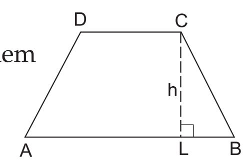
(ii) The diagonals of a rhombus bisect each other at right angles.



6. Area of a trapezium

$$= \frac{1}{2} \times \text{sum of parallel sides} \times \text{distance between them}$$

$$= \left\{\frac{1}{2} \times (AB + CD) \times h\right\} \text{ sq units.}$$



✿

15 Volume and Surface Area of Solids

SOLIDS *The bodies occupying space are called solids.*

The solid bodies occur in various shapes such as:

a cuboid, a cube, a cylinder, a cone, a sphere, etc.

VOLUME OF A SOLID *The space occupied by a solid body is called its volume.*

The units of volume are cubic centimetres (written as cm^3) or *cubic metres* (written as m^3).

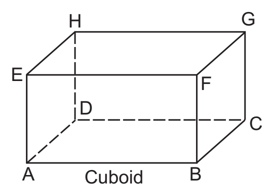
CUBOID *A solid bounded by six rectangular faces is called a cuboid.*

A matchbox, a chalkbox, a brick, a tile, a book, etc., are all examples of a cuboid.

A cuboid has 6 *rectangular faces,* 12 *edges and* 8 *vertices.*

In the given figure, *ABCDEFGH* is a cuboid whose

(i) 6 *faces are*:
ABCD, EFGH, ADHE, BCGF, ABFE and *DCGH*.
Out of these, the four faces, namely *ABFE, DCGH, BCGF* and *ADHE* are called *lateral faces* of the cuboid.



Cuboid

(ii) 12 *edges are*:
AB, BC, CD, DA, EF, FG, GH, HE, CG, BF, AE and *DH*.

(iii) 8 *vertices are*: *A, B, C, D, E, F, G, H*.

REMARK A rectangular room is in the form of a cuboid and its 4 walls are its lateral faces.

CUBE *A cuboid whose length, breadth and height are all equal is called a cube.*

Dice, ice cubes, sugar cubes, etc., are all examples of a cube.

Each edge of a cube is called its side.

VOLUME AND SURFACE AREA OF A CUBOID

Formulae:

Let us consider a cuboid of length $= l$ units, breadth $= b$ units and height $= h$ units. Then, we have

(i) *Volume of the cuboid* $= (l \times b \times h)$ *cu units.*

(ii) *Diagonal of the cuboid* $= \sqrt{l^2 + b^2 + h^2}$ *units.*

(iii) *Total surface area of the cuboid* $= 2(lb + bh + lh)$ *sq units.*

(iv) *Lateral surface area of the cuboid* $= [2(l + b) \times h]$ *sq units.*

(v) *Area of four walls of a room* $= [2(l + b) \times h]$ *sq units.*

VOLUME AND SURFACE AREA OF A CUBE

Formulae:

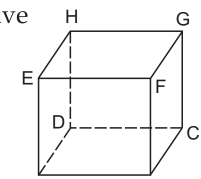
Let us consider a cube of edge $= a$ units. Then, we have

(i) *Volume of the cube* $= a^3$ *cu units.*

(ii) *Diagonal of the cube* $= \sqrt{3}\,a$ *units.*

(iii) *Total surface area of the cube* $= 6a^2$ *sq units.*

(iv) *Lateral surface area of the cube* $= 4a^2$ *sq units.*



Cube

SOLVED EXAMPLES

EXAMPLE 1 *Find the volume, the total surface area and the lateral surface area of a cuboid which is* 15 m *long,* 12 m *wide and* 4.5 m *high.*

SOLUTION Here, $l = 15$ m, $b = 12$ m and $h = 4.5$ m.

Volume of the cuboid $= (l \times b \times h)$ cubic units

$$= (15 \times 12 \times 4.5)\ \text{m}^3 = 810\ \text{m}^3.$$

Total surface area of the cuboid

$$= 2(lb + bh + lh) \text{ sq units}$$

$$= 2(15 \times 12 + 12 \times 4.5 + 15 \times 4.5)\ \text{m}^2 = 603\ \text{m}^2.$$

Lateral surface area of the cuboid

$$= [2(l + b) \times h] \text{ sq units}$$

$$= [2(15 + 12) \times 4.5]\ \text{m}^2 = 243\ \text{m}^2.$$

EXAMPLE 2 *How many bricks will be required to construct a wall* 13.5 m *long,* 6 m *high and* 22.5 cm *thick? It is being given that each brick measures* (27 cm × 12.5 cm × 9 cm)?

SOLUTION Length of the wall $= (13.5 \times 100)$ cm $= 1350$ cm.

Breadth of the wall $= 22.5$ cm.

Height of the wall $= (6 \times 100)$ cm $= 600$ cm.

Volume of the wall $= (1350 \times 22.5 \times 600)\ \text{cm}^3$.

Volume of each brick $= (27 \times 12.5 \times 9)\ \text{cm}^3$.

$$\text{Number of bricks required} = \left(\frac{\text{volume of the wall}}{\text{volume of 1 brick}}\right)$$

$$= \left(\frac{1350 \times 22.5 \times 600}{27 \times 12.5 \times 9}\right) = 6000.$$

EXAMPLE 3 *A room is* 16 m *long,* 9 m *wide and* 3 m *high. It has two doors, each of dimensions* (2 m × 1.5 m) *and three windows, each of dimensions* (1.6 m × 75 cm). *Find the cost of distempering the walls of the room from inside at the rate of* ₹ 50 *per square metre.*

SOLUTION Area of 4 walls of the room $= [2(l+b) \times h]$ sq units

$$= [2(16+9) \times 3]\ \text{m}^2 = 150\ \text{m}^2.$$

$$\text{Area of 2 doors} = \left[2 \times \left(2 \times \frac{3}{2}\right)\right]\ \text{m}^2 = 6\ \text{m}^2.$$

$$\text{Area of 3 windows} = \left[3 \times \left(1.6 \times \frac{75}{100}\right)\right]\ \text{m}^2 = \frac{18}{5}\ \text{m}^2 = 3.6\ \text{m}^2.$$

Area not to be distempered $= (6+3.6)\ \text{m}^2 = 9.6\ \text{m}^2$.

Area to be distempered $= (150-9.6)\ \text{m}^2 = 140.4\ \text{m}^2$.

Cost of distempering the walls = ₹ (140.4×50) = ₹ 7020.

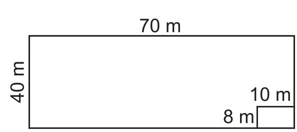
EXAMPLE 4 *A field is* 70 m *long and* 40 m *broad. In one corner of the field, a pit which is* 10 m *long,* 8 m *broad and* 5 m *deep, has been dug out. The earth taken out of it is evenly spread over the remaining part of the field. Find the rise in the level of the field.*

SOLUTION Area of the field $= (70 \times 40)\ \text{m}^2 = 2800\ \text{m}^2$.

Area of the pit $= (10 \times 8)\ \text{m}^2 = 80\ \text{m}^2$.

Area over which the earth is spread over

$$= (2800-80)\ \text{m}^2 = 2720\ \text{m}^2.$$



Volume of the earth dug out $= (10 \times 8 \times 5)\ \text{m}^3 = 400\ \text{m}^3$.

$$\text{Rise in level of the field} = \left(\frac{\text{volume of the earth dug out}}{\text{area on which the earth is spread}}\right)$$

$$= \left(\frac{400}{2720}\right)\ \text{m} = \left(\frac{400 \times 100}{2720}\right)\ \text{cm}$$

$$= \frac{250}{17}\ \text{cm} = 14.70\ \text{cm}.$$

EXAMPLE 5 *An open wooden box* 80 cm *long,* 65 cm *wide and* 45 cm *high, is made of* 2.5 cm *thick wood. Find (i) the capacity of the box, (ii) volume of wood used and (iii) weight of the box, it being given that* 100 cm^3 *of wood weighs* 8 g.

SOLUTION External length of the box = 80 cm.

External breadth of the box = 65 cm.

External height of the box = 45 cm.

External volume of the box = $(80 \times 65 \times 45)$ cu cm
$= 234000 \text{ cm}^3$.

Internal length of the box = $[80 - (2.5 \times 2)]$ cm = $(80 - 5)$ cm
$= 75$ cm.

Internal breadth of the box = $[65 - (2.5 \times 2)]$ cm = $(65 - 5)$ cm
$= 60$ cm.

Internal height of the box = $(45 - 2.5)$ cm = 42.5 cm.

(i) Capacity of the box = internal volume of the box
$= (75 \times 60 \times 42.5) \text{ cm}^3 = 191250 \text{ cm}^3$.

(ii) Volume of wood used
= (external volume) – (internal volume)
$= (234000 - 191250) \text{ cm}^3 = 42750 \text{ cm}^3$.

(iii) Weight of 100 cm^3 of wood = 8 g.

Weight of 42750 cm^3 of wood $= \left(\frac{8}{100} \times 42750\right)$ g = 3420 g
= 3 kg 420 g.

EXAMPLE 6 *Find the volume, total surface area, lateral surface area and the length of diagonal of a cube, each of whose edges measures* 20 cm.

SOLUTION Here, $a = 20$ cm.

Volume of the cube = a^3 cubic units
$= (20 \times 20 \times 20) \text{ cm}^3 = 8000 \text{ cm}^3$.

Total surface area of the cube = $(6a^2)$ sq units
$= (6 \times 20 \times 20) \text{ cm}^2 = 2400 \text{ cm}^2$.

Lateral surface area of the cube = $(4a^2)$ sq units
$= (4 \times 20 \times 20) \text{ cm}^2 = 1600 \text{ cm}^2$.

Diagonal of the cube = $(\sqrt{3}\, a)$ units
$= (\sqrt{3} \times 20)$ cm = (1.732×20) cm
$= 34.64$ cm.

EXAMPLE 7 *The total surface area of a cube is* $216\ \text{cm}^2$. *Find its volume.*

SOLUTION Let each side of the cube be a cm.

Then, the total surface area of the cube = $(6a^2)\ \text{cm}^2$.

$\therefore\quad 6a^2 = 216 \Rightarrow a^2 = 36 \Rightarrow a = \sqrt{36} = 6.$

Volume of the cube = $a^3\ \text{cm}^3$

$= (6 \times 6 \times 6)\ \text{cm}^3 = 216\ \text{cm}^3.$

EXAMPLE 8 *The lateral surface area of a cube is* $324\ \text{cm}^2$. *Find its volume and the total surface area.*

SOLUTION Let each side of the cube be a cm.

Then, the lateral surface area of the cube = $(4a^2)\ \text{cm}^2$.

$\therefore\quad 4a^2 = 324 \Rightarrow a^2 = 81 \Rightarrow a = \sqrt{81} = 9.$

Volume of the cube = $a^3\ \text{cm}^3$

$= (9 \times 9 \times 9)\ \text{cm}^3 = 729\ \text{cm}^3.$

Total surface area of the cube = $(6a^2)$ sq units

$= (6 \times 9 \times 9)\ \text{cm}^2 = 486\ \text{cm}^2.$

SOME MORE EXAMPLES

EXAMPLE 9 *How many square sheets of coloured paper of side* 30 cm *would be required to cover a wooden box having length, breadth and height as* 90 cm, 60 cm *and* 30 cm *respectively?*

SOLUTION Clearly, the quantity of paper required would be equal to the total surface area of the wooden box, which is in the shape of a cuboid.

The dimensions of the box are

length = 90 cm, breadth = 60 cm and height = 30 cm.

The total surface area of the box

$= 2(lb + bh + lh)$

$= 2 \times [(90 \times 60) + (60 \times 30) + (90 \times 30)]\ \text{cm}^2$

$= 2 \times (5400 + 1800 + 2700)\ \text{cm}^2$

$= (2 \times 9900)\ \text{cm}^2 = 19800\ \text{cm}^2.$

The area of each sheet of paper = $(30 \times 30)\ \text{cm}^2 = 900\ \text{cm}^2$.

Number of sheets required

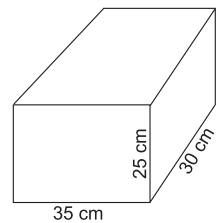
$$= \frac{\text{total surface area of the box}}{\text{area of one sheet of paper}} = \frac{19800}{900} = 22.$$

EXAMPLE 10 *A small indoor greenhouse (herbarium) is made entirely of glass panes (including base) held together with tape. It is* 35 cm *long,* 30 cm *wide and* 25 cm *high.*

(i) What is the area of the glass?

(ii) What length of tape is needed for all the 12 *edges?*

SOLUTION The indoor greenhouse is in the shape of a cuboid having length = 35 cm, breadth = 30 cm and height = 25 cm.



(i) Area of the glass

$= \text{the total surface area of the cuboid}$

$= 2(lb + bh + lh)$

$= 2 \times [(35 \times 30) + (30 \times 25) + (35 \times 25)] \text{ cm}^2$

$= 2(1050 + 750 + 875) \text{ cm}^2 = (2 \times 2675) \text{ cm}^2 = 5350 \text{ cm}^2.$

(ii) Length of tape needed for all 12 edges

= (perimeter of top) + (perimeter of bottom) + (4 vertical edges)

$= \{2(l + b) + 2(l + b) + 4h\} = 4(l + b + h)$

$= 4(35 + 30 + 25) \text{ cm} = 360 \text{ cm} = 3 \text{ m } 60 \text{ cm}.$

EXAMPLE 11 *A sweet seller placed an order for making cardboard boxes of two sizes, namely* $(25 \text{ cm} \times 20 \text{ cm} \times 5 \text{ cm})$ *and* $(15 \text{ cm} \times 12 \text{ cm} \times 5 \text{ cm})$. *If* 5% *of the total surface area is required extra for all overlaps and the cost of the cardboard is* ₹ 5 *for* 1000 cm^2, *find the cost of cardboard required for* 200 *boxes of each kind.*

SOLUTION For each bigger box, we have $l = 25$ cm, $b = 20$ cm and $h = 5$ cm.

Surface area of the cardboard for each bigger box

$= 2(lb + bh + lh)$

$= 2(25 \times 20 + 20 \times 5 + 25 \times 5) \text{ cm}^2$

$= 2(500 + 100 + 125) \text{ cm}^2 = (2 \times 725) \text{ cm}^2 = 1450 \text{ cm}^2.$

Total surface area of the cardboard required for 200 bigger boxes

$= (1450 \times 200 + 5\% \text{ of } 1450 \times 200) \text{ cm}^2$

$= \left(290000 + \frac{5}{100} \times 1450 \times 200\right) \text{ cm}^2$

$= (290000 + 14500) \text{ cm}^2 = 304500 \text{ cm}^2.$

For each smaller box, we have $l = 15$ cm, $b = 12$ cm and $h = 5$ cm.

Surface area of the cardboard for each smaller box

$= 2(lb + bh + lh)$

$= 2(15 \times 12 + 12 \times 5 + 15 \times 5)\ \text{cm}^2$

$= 2(180 + 60 + 75)\ \text{cm}^2 = (2 \times 315)\ \text{cm}^2 = 630\ \text{cm}^2.$

Total surface area of the cardboard required for 200 smaller boxes

$= (630 \times 200 + 5\% \text{ of } 630 \times 200)\ \text{cm}^2$

$= \left(126000 + \frac{5}{100} \times 630 \times 200\right)\ \text{cm}^2$

$= (126000 + 6300)\ \text{cm}^2 = 132300\ \text{cm}^2.$

Total surface area of the cardboard required for 200 boxes of each size $= (304500 + 132300)\ \text{cm}^2 = 436800\ \text{cm}^2.$

Cost of cardboard for $1000\ \text{cm}^2 =$ ₹ 5.

Cost of cardboard for $436800\ \text{cm}^2 =$ ₹ $\left(\frac{5}{1000} \times 436800\right) =$ ₹ 2184.

EXAMPLE 12 *A man wants to make a temporary shelter for his car by making a box-like structure with tarpaulin that covers all the four sides and the top of the car. If the height of the shelter is* 2.5 m *and its base dimensions are* 4 m *by* 3 m, *how much tarpaulin would be required?*

SOLUTION Area of tarpaulin required

= (area of four sides) + (area of the top)

$= \{2(l + b) \times h\} + (l \times b)$

$= \{2 \times (4 + 3) \times 2.5\}\ \text{m}^2 + (4 \times 3)\ \text{m}^2 = (35 + 12)\ \text{m}^2 = 47\ \text{m}^2.$

EXAMPLE 13 *A cubical box has each edge* 10 cm *and another cuboidal box is* 12.5 cm *long,* 10 cm *wide and* 8 cm *high.*

(i) Which box has the greater lateral surface area and by how much?

(ii) Which box has the smaller total surface area and by how much?

SOLUTION (i) Lateral surface area of the cubical box

$= 4(\text{edge})^2 = (4 \times 10 \times 10)\ \text{cm}^2 = 400\ \text{cm}^2.$

Lateral surface area of the cuboidal box

$= 2(\text{length} + \text{breadth}) \times \text{height}$

$= \{2(12.5 + 10) \times 8\}\ \text{cm}^2 = (2 \times 22.5 \times 8)\ \text{cm}^2 = 360\ \text{cm}^2.$

Difference in lateral surface areas

$= (400 - 360)\ \text{cm}^2 = 40\ \text{cm}^2.$

Hence, the lateral surface area of the cubical box is larger than that of the cuboidal box by $40\ \text{cm}^2$.

(ii) Total surface area of the cubical box

$= 6(\text{edge})^2 = (6 \times 10 \times 10)\ \text{cm}^2 = 600\ \text{cm}^2.$

Total surface area of the cuboidal box

$= 2(lb + bh + lh)$

$= 2(12.5 \times 10 + 10 \times 8 + 12.5 \times 8)\ \text{cm}^2$

$= 2(125 + 80 + 100)\ \text{cm}^2 = (2 \times 305)\ \text{cm}^2 = 610\ \text{cm}^2.$

Difference in total surface areas $= (610 - 600)\ \text{cm}^2 = 10\ \text{cm}^2.$

Hence, the total surface area of the cubical box is smaller than that of the cuboidal box by $10\ \text{cm}^2$.

EXAMPLE 14 *Kunal built a cubical water tank with lid for his house, with each outer edge* 1.5 m *long. He gets the outer surface of the tank excluding the base covered with square tiles of side* 25 cm. *Find how much he would spend for the tiles, if the cost of the tiles is* ₹ 540 *per dozen.*

SOLUTION Leaving aside the base, the remaining five faces of the tank are to be covered with tiles.

Length of each edge of the tank $= 1.5\ \text{m} = (1.5 \times 100)\ \text{cm} = 150\ \text{cm}.$

Surface area of each face $= (150 \times 150)\ \text{cm}^2.$

Total surface area of five faces to be covered with tiles

$= (5 \times 150 \times 150)\ \text{cm}^2.$

Area of each square tile $= (25 \times 25)\ \text{cm}^2.$

$$\text{Number of tiles required} = \frac{\text{surface area of five faces}}{\text{area of each tile}} = \left(\frac{5 \times 150 \times 150}{25 \times 25}\right) = 180.$$

Cost of 12 tiles = ₹ 540.

Cost of 1 tile = ₹ $\left(\frac{540}{12}\right)$.

Cost of 180 tiles = ₹ $\left(\frac{540}{12} \times 180\right)$ = ₹ 8100.

EXAMPLE 15 *A solid cube of side* 12 cm *is cut into* 8 *cubes of equal volume. What is the side of the new cube? Also, find the ratio between their surface areas.*

SOLUTION Volume of the big cube $= (12 \times 12 \times 12)\ \text{cm}^3 = 1728\ \text{cm}^3.$

Volume of a small (new) cube $= \frac{1728}{8}\ \text{cm}^3 = 216\ \text{cm}^3.$

Let each side of the new cube be a cm.

Then, its volume = a^3 cm^3.

$\therefore \quad a^3 = 216 = (6)^3 \Rightarrow a = 6.$

Thus, the side of the new cube is 6 cm.

Surface area of the big cube

$= \{6 \times (12)^2\}$ cm^2 = (6×144) cm^2 = 864 cm^2.

Surface area of the smaller cube

$= \{6 \times (6)^2\}$ cm^2 = (6×36) cm^2 = 216 cm^2.

Ratio between the surface areas of big cube and small cube

$$= 864 : 216 = \frac{864}{216} = \frac{4}{1} = 4 : 1.$$

EXAMPLE 16 *A river* 3 m *deep and* 40 m *wide is flowing into the sea at the rate of* 2 km *per hour. How much water will fall into the sea in a minute?*

SOLUTION Width of the river = 40 m, depth of the river = 3 m.

Length of water that falls into the sea in a minute

$$= \left(\frac{2000}{60}\right) \text{m} = \frac{100}{3} \text{ m}.$$

Volume of water that falls into the sea in a minute

= (length $\times$ width $\times$ depth)

$$= \left(\frac{100}{3} \times 40 \times 3\right) \text{m}^3 = 4000 \text{ m}^3.$$

EXAMPLE 17 *The length of a cold storage is double its breadth and its height is* 3 *metres. If the area of its four walls be* 108 m^2, *find its volume.*

SOLUTION Let the breadth of the cold storage be x metres.

Then, its length = $2x$ metres and height = 3 metres.

Area of the four walls of the cold storage

$= \{2(l + b) \times h\}$ sq units

$= \{2(2x + x) \times 3\}$ m^2 = $(18x)$ m^2.

But, area of 4 walls = 108 m^2 (given).

$$\therefore \quad 18x = 108 \Rightarrow x = \frac{108}{18} = 6.$$

So, breadth = 6 m and length = 12 m.

Volume of the cold storage = $(l \times b \times h)$ cubic units

$= (12 \times 6 \times 3)$ m^3 = 216 m^3.

EXAMPLE 18 *A village having a population of* 4000, *requires* 150 *litres of water per head per day. It has a tank measuring* $20 \text{ m} \times 15 \text{ m} \times 6 \text{ m}$. *For how many days will the water of this tank last?*

SOLUTION Volume of the water tank $= (20 \times 15 \times 6) \text{ m}^3 = 1800 \text{ m}^3$.

Consumption of water per head per day

$$= 150 \text{ litres}$$

$$= \left(150 \times \frac{1}{1000}\right) \text{m}^3 = \frac{3}{20} \text{ m}^3 \quad \left[\because 1 \text{ litre} = \frac{1}{1000} \text{ m}^3\right].$$

Total consumption of water per day for 4000 persons

$$= \left(\frac{3}{20} \times 4000\right) \text{m}^3 = 600 \text{ m}^3.$$

Required no. of days for which water of the tank will last

$$= \frac{\text{volume of water tank}}{\text{consumption per day}} = \frac{1800}{600} \text{ days} = 3 \text{ days}.$$

EXAMPLE 19 *A room is half as long again as it is broad. The cost of carpeting the room at* ₹ 13 *per* m^2 *is* ₹ 702 *and the cost of papering the walls at* ₹ 7 *per* m^2 *is* ₹ 1204. *If* 1 *door and* 2 *windows occupy* 8 m^2, *find the dimensions of the room.*

SOLUTION Let the breadth of the room be x metres.

Then, its length $= \left(x + \frac{x}{2}\right)$ metres $= \frac{3x}{2}$ metres.

Area of the floor $= (l \times b) = \left(\frac{3x}{2} \times x\right) \text{m}^2 = \frac{3x^2}{2} \text{ m}^2$.

Also, area of the floor $= \dfrac{\text{total cost of carpeting}}{\text{rate per m}^2}$

$$= \left(\frac{702}{13}\right) \text{m}^2 = 54 \text{ m}^2.$$

$$\therefore \quad \frac{3x^2}{2} = 54 \Rightarrow x^2 = \left(54 \times \frac{2}{3}\right) = 36 = (6)^2 \Rightarrow x = 6.$$

$\therefore$ breadth $= 6$ m and length $= \left(\frac{3}{2} \times 6\right) \text{m} = 9$ m.

Area of papered walls $= \dfrac{\text{total cost of papering}}{\text{rate per m}^2}$

$$= \frac{1204}{7} \text{ m}^2 = 172 \text{ m}^2.$$

Area of 1 door and 2 windows $= 8 \text{ m}^2$.

Area of 4 walls $= (172 + 8)\ \text{m}^2 = 180\ \text{m}^2$.

Let the height of the room be h metres.

Then, area of 4 walls $= \{2(l + b) \times h\} = \{2(9 + 6) \times h\}\ \text{m}^2$

$= (30h)\ \text{m}^2$.

$\therefore \quad 30h = 180 \Rightarrow h = \dfrac{180}{30} = 6\ \text{m}$.

Hence, length = 9 m, breadth = 6 m and height = 6 m.

EXAMPLE 20 *Water flows in a tank* 60 m × 40 m *at the base through a pipe whose cross section is* 2 dm *by* 1.5 dm *at the speed of* 8 km *per hour. In what time will the water be* 5 *metres deep?*

SOLUTION Let in x hours the water be 5 m deep in the tank.

Volume of water required in the tank $= (60 \times 40 \times 5)\ \text{m}^3$.

Volume of water flown in the tank in x hours

$$= \left(\frac{2}{10} \times \frac{1.5}{10} \times 8000 \times x\right)\ \text{m}^3.$$

$$\therefore \quad \frac{2}{10} \times \frac{1.5}{10} \times 8000 \times x = 60 \times 40 \times 5$$

$$\Rightarrow \quad 240x = 60 \times 40 \times 5 \Rightarrow x = \frac{60 \times 40 \times 5}{240} \Rightarrow x = 50.$$

Hence, the required time is 50 hours.

EXAMPLE 21 *A storage tank is in the form of a cube. When it is full of water, the volume of water is* $15.625\ \text{m}^3$. *If the present depth of water is* 1.3 m, *find the volume of water used from the tank.*

SOLUTION Let each edge of the tank be a metres.

Volume of water in the tank when it is full $= (a^3)\ \text{m}^3$.

But, volume of water in full tank $= 15.625\ \text{m}^3$.

$$\therefore \quad a^3 = 15.625 = \frac{15625}{1000} = \frac{125}{8} = \left(\frac{5}{2}\right)^3$$

$$\Rightarrow \quad a = \frac{5}{2} = 2.5.$$

Thus, each edge of the tank is 2.5 m.

Present depth of water = 1.3 m.

Present volume of water in the tank

$$= (2.5 \times 2.5 \times 1.3)\ \text{m}^3$$

$$= \left(\frac{5}{2} \times \frac{5}{2} \times \frac{13}{10}\right)\ \text{m}^3 = \frac{65}{8}\ \text{m}^3 = 8.125\ \text{m}^3.$$

Volume of water used from the tank

$$= (15.625 - 8.125)\ m^3 = 7.5\ m^3.$$

EXERCISE 15A

1. Find the volume, the lateral surface area and the total surface area of the cuboid whose dimensions are:
   (i) length = 12 cm, breadth = 8 cm and height = 4.5 cm
   (ii) length = 26 m, breadth = 14 m and height = 6.5 m
   (iii) length = 15 m, breadth = 6 m and height = 5 dm
   (iv) length = 24 m, breadth = 25 cm and height = 6 m
2. A matchbox measures $4\ cm \times 2.5\ cm \times 1.5\ cm$. What is the volume of a packet containing 12 such matchboxes?
3. A cuboidal water tank is 6 m long, 5 m wide and 4.5 m deep. How many litres of water can it hold? (Given, $1\ m^3 = 1000$ litres.)
4. The capacity of a cuboidal tank is 50000 litres of water. Find the breadth of the tank if its length and depth are respectively 10 m and 2.5 m. (Given, 1000 litres $= 1\ m^3$.)
5. A godown measures $40\ m \times 25\ m \times 15\ m$. Find the maximum number of wooden crates, each measuring $1.5\ m \times 1.25\ m \times 0.5\ m$, that can be stored in the godown.
6. How many planks of dimensions ($5\ m \times 25\ cm \times 10\ cm$) can be stored in a pit which is 20 m long, 6 m wide and 80 cm deep?
7. How many bricks will be required to construct a wall 8 m long, 6 m high and 22.5 cm thick if each brick measures ($25\ cm \times 11.25\ cm \times 6\ cm$)?
8. Find the capacity of a closed rectangular cistern whose length is 8 m, breadth 6 m and depth 2.5 m. Also, find the area of the iron sheet required to make the cistern.
9. The dimensions of a room are $(9\ m \times 8\ m \times 6.5\ m)$. It has one door of dimensions ($2\ m \times 1.5\ m$) and two windows, each of dimensions ($1.5\ m \times 1\ m$). Find the cost of whitewashing the walls at ₹ 25 per square metre.
10. A wall 15 m long, 30 cm wide and 4 m high is made of bricks, each measuring $(22\ cm \times 12.5\ cm \times 7.5\ cm)$. If $\frac{1}{12}$ of the total volume of the wall consists of mortar, how many bricks are there in the wall?
11. How many cubic centimetres of iron are there in an open box whose external dimensions are 36 cm, 25 cm and 16.5 cm, the iron being 1.5 cm thick throughout? If $1\ cm^3$ of iron weighs 15 g, find the weight of the empty box in kilograms.

12. A box made of sheet metal costs ₹ 6480 at ₹ 120 per square metre. If the box is 5 m long and 3 m wide, find its height.

13. The volume of a cuboid is 1536 m^3. Its length is 16 m, and its breadth and height are in the ratio 3 : 2. Find the breadth and height of the cuboid.

14. How many persons can be accommodated in a dining hall of dimensions $(20 \text{ m} \times 16 \text{ m} \times 4.5 \text{ m})$, assuming that each person requires 5 cubic metres of air?

15. A classroom is 10 m long, 6.4 m wide and 5 m high. If each student be given 1.6 m^2 of the floor area, how many students can be accommodated in the room? How many cubic metres of air would each student get?

16. The surface area of a cuboid is 758 cm^2. Its length and breadth are 14 cm and 11 cm respectively. Find its height.

17. In a shower, 5 cm of rain falls. Find the volume of water that falls on 2 hectares of ground.

18. Find the volume, the lateral surface area, the total surface area and the diagonal of a cube, each of whose edges measures 9 m. (Take $\sqrt{3} = 1.73$.)

19. The total surface area of a cube is 1176 cm^2. Find its volume.

20. The lateral surface area of a cube is 900 cm^2. Find its volume.

21. The volume of a cube is 512 cm^3. Find its surface area.

22. Three cubes of metal with edges 3 cm, 4 cm and 5 cm respectively are melted to form a single cube. Find the lateral surface area of the new cube formed.

23. Find the length of the longest pole that can be put in a room of dimensions $(10 \text{ m} \times 10 \text{ m} \times 5 \text{ m})$.

24. The sum of length, breadth and depth of a cuboid is 19 cm and the length of its diagonal is 11 cm. Find the surface area of the cuboid.

25. Each edge of a cube is increased by 50%. Find the percentage increase in the surface area of the cube.

26. If V is the volume of a cuboid of dimensions a, b, c and S is its surface area then prove that $\frac{1}{V} = \frac{2}{S}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$.

27. Water in a canal, 30 dm wide and 12 dm deep, is flowing with a velocity of 20 km per hour. How much area will it irrigate, if 9 cm of standing water is desired?

28. A solid metallic cuboid of dimensions $(9 \text{ m} \times 8 \text{ m} \times 2 \text{ m})$ is melted and recast into solid cubes of edge 2 m. Find the number of cubes so formed.
[CBSE 2017]

ANSWERS (EXERCISE 15A)

1. (i) 432 cm^3, 180 cm^2, 372 cm^2 (ii) 2366 m^3, 520 m^2, 1248 m^2
(iii) 45 m^3, 21 m^2, 201 m^2 (iv) 36 m^3, 291 m^2, 303 m^2

2. 180 cm^3 **3.** 135000 litres **4.** 2 m **5.** 16000
6. 768 **7.** 6400 **8.** 120 m^3, 166 m^2 **9.** ₹ 5375
10. 8000 **11.** 3960 cm^3, 59.4 kg **12.** 1.5 m **13.** 12 m, 8 m
14. 288 **15.** 40, 8 m^3 **16.** 9 cm **17.** 1000 m^3
18. 729 m^3, 324 m^2, 486 m^2, 15.57 m **19.** 2744 cm^3 **20.** 3375 cm^3
21. 384 cm^2 **22.** 144 cm^2 **23.** 15 m **24.** 240 cm^2
25. 125% **27.** 400000 m^2 **28.** 18

HINTS TO SOME SELECTED QUESTIONS

4. Volume of the tank = 50000 litres = $\frac{50000}{1000}$ m^3 = 50 m^3.
$\therefore$ $10 \times x \times 2.5 = 50$. Find x.

5. Number of crates = $\dfrac{\text{volume of godown}}{\text{volume of 1 crate}} = \dfrac{40 \times 25 \times 15}{1.5 \times 1.25 \times 0.5}$.

6. Volume of pit = $\left(20 \times 6 \times \frac{80}{100}\right)$ m^3 = 96 m^3.

Volume of one plank = $\left(5 \times \frac{25}{100} \times \frac{10}{100}\right)$ $m^3 = \frac{1}{8}$ m^3.

Number of planks = $\dfrac{\text{volume of pit}}{\text{volume of 1 plank}}$.

7. Required number of bricks = $\dfrac{\text{volume of the wall in cm}^3}{\text{volume of 1 brick in cm}^3}$
$= \dfrac{800 \times 600 \times 22.5}{25 \times 11.25 \times 6}$.

8. Capacity = $(l \times b \times h)$ m^3.
Area of iron sheet = total surface area of the cistern
$= 2(lb + bh + lh)$ m^2.

9. Area of 4 walls = $[2(9+8) \times 6.5]$ m^2 = 221 m^2.
Area not to be whitewashed = (area of 1 door) + (area of 2 windows)
$= (2 \times 1.5)$ m^2 + $(2 \times 1.5 \times 1)$ m^2 = 6 m^2.
Area to be whitewashed = $(221 - 6)$ m^2 = 215 m^2.

10. Volume of the wall = $\left(15 \times \frac{30}{100} \times 4\right)$ m^3 = 18 m^3.
Volume of mortar = $\left(\frac{1}{12} \times 18\right)$ m^3 = 1.5 m^3.
Volume of bricks = $(18 - 1.5)$ m^3 = 16.5 m^3.

Volume of 1 brick $= \left(\frac{22}{100} \times \frac{12.5}{100} \times \frac{7.5}{100}\right) = \left(\frac{33}{16000}\right) \text{m}^3$.

Number of bricks $= \frac{\text{volume of bricks}}{\text{volume of 1 brick}} = \left(\frac{33}{2} \times \frac{16000}{33}\right) = 8000$.

11. External volume of the box $= \left(36 \times 25 \times \frac{33}{2}\right) \text{cm}^3 = 14850 \text{ cm}^3$.

Internal volume of the box $= (33 \times 22 \times 15) \text{ cm}^3 = 10890 \text{ cm}^3$.

Volume of iron in the box $= (14850 - 10890) \text{ cm}^3 = 3960 \text{ cm}^3$.

Weight of the empty box $= \left(3960 \times 15 \times \frac{1}{1000}\right) \text{kg} = 59.4 \text{ kg}$.

12. Area of sheet metal $= \frac{\text{total cost}}{\text{cost per m}^2} = \frac{6480}{120} \text{ m}^2 = 54 \text{ m}^2$.

$2(lb + bh + lh) = 54 \Rightarrow 2(5 \times 3 + 3h + 5h) = 54$

$\Rightarrow 16h = (54 - 30) = 24 \Rightarrow h = \frac{24}{16} \text{ m} = \frac{3}{2} \text{ m} = 1.5 \text{ m}$.

13. Let the breadth be $3x$ metres and height $2x$ metres. Then,

$16 \times 3x \times 2x = 1536 \Rightarrow x^2 = \frac{1536}{96} = 16 \Rightarrow x = 4$.

14. Required number of persons $= \left(\frac{20 \times 16 \times 4.5}{5}\right) = 288$.

15. Required number of students $= \frac{10 \times 6.4}{1.6} = 40$.

Air needed for each student $= \left(\frac{10 \times 6.4 \times 5}{40}\right) \text{m}^3 = 8 \text{ m}^3$.

16. $2(lb + bh + lh) = 758$. Here, $l = 14$ cm, $b = 11$ cm. Find h.

17. Volume of water = (area $\times$ depth) $= \left(2 \times 10000 \times \frac{5}{100}\right) \text{m}^3$.

19. $6a^2 = 1176 \Rightarrow a^2 = 196 = (14)^2$.

22. Volume of the new cube $= (3^3 + 4^3 + 5^3) \text{ cm}^3 = 216 \text{ cm}^3$.

Edge of this cube = a cm, where $a^3 = 216$.

$\therefore$ $a = 6$ and the lateral surface area of the new cube is $4a^2 \text{ cm}^2$.

23. Length of the longest pole = length of diagonal $= \sqrt{l^2 + b^2 + h^2}$ metres.

24. $(l + b + h) = 19$ cm and $\sqrt{l^2 + b^2 + h^2} = 11$ cm

$\Rightarrow$ $(l + b + h)^2 = 361 \text{ cm}^2$ and $(l^2 + b^2 + h^2) = 121 \text{ cm}^2$

$\Rightarrow$ $(l^2 + b^2 + h^2) + 2(lb + bh + lh) = 361 \text{ cm}^2$ and $l^2 + b^2 + h^2 = 121 \text{ cm}^2$

$\Rightarrow$ $121 \text{ cm}^2 + 2(lb + bh + lh) = 361 \text{ cm}^2$

$\Rightarrow$ $2(lb + bh + lh) = (361 - 121) \text{cm}^2 = 240 \text{ cm}^2$.

25. Consider a cube having edge = a cm.

Then, its surface area $= (6a^2) \text{ cm}^2$.

New edge = (150% of a) cm $= \left(\frac{150a}{100}\right) \text{cm} = \frac{3a}{2}$ cm.

New surface area $= \left\{6 \times \left(\frac{3a}{2}\right)^2\right\} \text{cm}^2 = \frac{27a^2}{2} \text{ cm}^2$.

Surface area increased $= \left(\frac{27a^2}{2} - 6a^2\right) \text{cm}^2 = \left(\frac{15a^2}{2}\right) \text{cm}^2$.

Percentage increase in surface area $= \left(\frac{15a^2}{2} \times \frac{1}{6a^2} \times 100\right)\% = 125\%$.

26. We have, $V = abc$ and $S = 2(ab + bc + ca)$.

$\therefore \quad \frac{2}{S}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = \frac{2}{S} \times \frac{(bc + ca + ab)}{abc} = \frac{2}{S} \times \frac{S}{2V} = \frac{1}{V}$.

27. Distance covered by water in 30 minutes

= velocity of water in m/hr $\times$ time in hours

$= \left(20000 \times \frac{30}{60}\right)$ m = 10000 m.

Volume of water flown in 30 min $= \left(10000 \times \frac{30}{10} \times \frac{12}{10}\right) \text{m}^3 = 36000 \text{ m}^3$.

Let the area irrigated be x m^2.

Then, $x \times \frac{9}{100} = 36000 \Rightarrow x = \left(36000 \times \frac{100}{9}\right) \text{m}^2 = 400000 \text{ m}^2$.

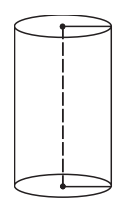
28. Required number of cubes $= \left(\frac{9 \times 8 \times 2}{2 \times 2 \times 2}\right) = 18$.

CYLINDERS

CYLINDER Solids like circular pillars, circular pipes, circular pencils, measuring jars, road rollers and gas cylinders, etc., are said to be in cylindrical shapes.

Formulae:

Let us consider a cylinder whose height is h units and the radius of whose base is r units. Then, we have:



(i) *Volume of the cylinder* = $(\pi r^2 h)$ *cubic units.*

(ii) *Curved surface area of the cylinder* = $(2\pi rh)$ *sq units.*

(iii) *Total surface area of the cylinder*

= (*area of curved surface*) + 2(*base area*)

= $(2\pi rh + 2\pi r^2)$ *sq units.*

HOLLOW CYLINDERS Solids like iron pipes, rubber tubes, etc., are in the shape of hollow cylinders.

Formulae:

For a hollow cylinder with external radius R units, internal radius r units and height (or length) h units, we have:

(i) *Volume of material* = (*exterior volume*) – (*interior volume*)

$= (\pi R^2 h - \pi r^2 h)$ cubic units.

(ii) *Curved surface area* = (*external surface*) + (*internal surface*)

$$= (2\pi Rh + 2\pi rh) \text{ } sq \text{ } units.$$

(iii) *Total surface area* = (*curve surface*) + 2(*area of base ring*)

$$= [(2\pi Rh + 2\pi rh) + 2(\pi R^2 - \pi r^2)] \text{ } sq \text{ } units.$$

SOLVED EXAMPLES

EXAMPLE 1 *Find the (i) volume, (ii) area of curved surface and (iii) total surface area of a cylinder having radius of the base* 14 cm *and height* 30 cm.

SOLUTION Here, $r = 14$ cm and $h = 30$ cm.

(i) Volume of the cylinder = $(\pi r^2 h)$ cubic units

$$= \left(\frac{22}{7} \times 14 \times 14 \times 30\right) \text{cm}^3 = 18480 \text{ cm}^3.$$

(ii) Curved surface area of the cylinder = $(2\pi rh)$ sq units

$$= \left(2 \times \frac{22}{7} \times 14 \times 30\right) \text{cm}^2 = 2640 \text{ cm}^2.$$

(iii) Total surface area of the cylinder

$$= (2\pi rh + 2\pi r^2) \text{ sq units} = 2\pi r(r + h) \text{ sq units}$$

$$= \left\{2 \times \frac{22}{7} \times 14 \times (14 + 30)\right\} \text{cm}^2 = 3872 \text{ cm}^2.$$

EXAMPLE 2 *The total surface area of a cylinder of radius* 7 cm *is* 880 cm^2. *Find the height and the volume of the cylinder.*

SOLUTION Given, $r = 7$ cm. Let the height of the cylinder be h cm.

Then, total surface area = $(2\pi r^2 + 2\pi rh)$ sq units

$$= 2\pi r(r + h) \text{ sq units}$$

$$= 2 \times \frac{22}{7} \times 7 \times (7 + h) \text{ cm}^2$$

$$= 44(7 + h) \text{ cm}^2.$$

$\therefore \quad 44(7 + h) = 880 \Rightarrow (7 + h) = 20 \Rightarrow h = (20 - 7) = 13.$

Hence, the height of the cylinder is 13 cm.

Volume of the cylinder = $(\pi r^2 h)$ cubic units

$$= \left(\frac{22}{7} \times 7 \times 7 \times 13\right) \text{cm}^3 = 2002 \text{ cm}^3.$$

Hence, the volume of the cylinder is 2002 cm^3.

EXAMPLE 3 *The curved surface area and the volume of a pillar are* 264 m^2 *and* 396 m^3 *respectively. Find the diameter and the height of the pillar.*

SOLUTION Let the radius of the pillar be r metres and its height be h metres. Then,

$$2\pi rh = 264 \quad \ldots \text{(i)}$$

and $\pi r^2 h = 396$ … (ii)

On dividing (ii) by (i), we get

$$\frac{\pi r^2 h}{2\pi rh} = \frac{396}{264} \Rightarrow r = \left(\frac{2 \times 396}{264}\right) = 3 \Rightarrow 2r = 6.$$

Hence, the diameter of the pillar is 6 m.

Putting $r = 3$ in (i), we get

$$2 \times \frac{22}{7} \times 3 \times h = 264 \Rightarrow h = \left(264 \times \frac{7}{132}\right) = 14.$$

Hence, the height of the pillar is 14 m.

EXAMPLE 4 *How many cubic metres of earth must be dug out to sink a well* 14 m *deep and having a radius of* 4 m*? If the earth taken out is spread over a plot of dimensions* (25 m × 16 m)*, what is the height of the platform so formed?*

SOLUTION Clearly, we have $r = 4$ m and $h = 14$ m.

Volume of the earth dug out of the well

$$= (\pi r^2 h) \text{ cubic units} = \left(\frac{22}{7} \times 4 \times 4 \times 14\right) \text{m}^3 = 704 \text{ m}^3.$$

Area of the given plot $= (25 \times 16)\ \text{m}^2 = 400\ \text{m}^2$.

Volume of the platform formed = volume of the earth dug out

$= 704\ \text{m}^3$.

$$\text{Height of the platform} = \left(\frac{\text{volume in m}^3}{\text{area in m}^2}\right)$$

$$= \left(\frac{704}{400}\right) \text{m} = \frac{176}{100} \text{ m} = 1.76 \text{ m}.$$

Hence, the height of the platform so formed = 1.76 m.

EXAMPLE 5 *A well of inner diameter* 14 m *is dug to a depth of* 15 m. *Earth taken out of it has been evenly spread all around it to a width of* 7 m *to form an embankment. Find the height of the embankment so formed.*

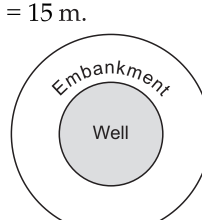
SOLUTION Radius of the well, $r = 7$ m, and its depth, $h = 15$ m.

Volume of the earth dug out

= volume of the well

$= (\pi r^2 h)$ cubic units

$$= \left(\frac{22}{7} \times 7 \times 7 \times 15\right) \text{m}^3 = 2310 \text{ m}^3.$$



Width of the embankment = 7 m.

External radius of the embankment = (7 + 7) m = 14 m.

Internal radius of the embankment = 7 m.

Area of the embankment $= \pi \times [(14)^2 - 7^2]\ \text{m}^2$

$$= \left[\frac{22}{7} \times (14+7) \times (14-7)\right] \text{m}^2$$

$$= \left(\frac{22}{7} \times 21 \times 7\right) \text{m}^2 = 462\ \text{m}^2.$$

Volume of the embankment = volume of the earth dug out

$$= 2310\ \text{m}^3.$$

Height of the embankment

$$= \left(\frac{\text{volume of the embankment in m}^3}{\text{area of the embankment in m}^2}\right) = \left(\frac{2310}{462}\right) \text{m} = 5\ \text{m}.$$

Hence, the height of the embankment formed = 5 m.

EXAMPLE 6 *The diameter of a roller,* 120 cm *long, is* 84 cm. *If it takes* 500 *complete revolutions to level a playground, find the cost of levelling it at* ₹ 5 *per square metre.*

SOLUTION Radius of the roller, $r = 42$ cm, and its length, $h = 120$ cm.

Area covered by the roller in 1 revolution

= curved surface area of the roller

$= (2\pi rh)$ sq units

$$= \left(2 \times \frac{22}{7} \times 42 \times 120\right) \text{cm}^2 = 31680\ \text{cm}^2.$$

Area covered by the roller in 500 revolutions

$$= (31680 \times 500)\ \text{cm}^2 = \left(\frac{31680 \times 500}{100 \times 100}\right) \text{m}^2 = 1584\ \text{m}^2.$$

$\therefore$ area of the playground $= 1584\ \text{m}^2$.

Cost of levelling the playground = ₹ (1584 × 5) = ₹ 7920.

EXAMPLE 7 *Find the number of coins* 1.5 cm *in diameter and* 0.2 cm *thick to be melted to form a right circular cylinder of height* 5 cm *and diameter* 4.5 cm.

SOLUTION Each coin is cylindrical in shape.

Radius of each coin, $r = \frac{1.5}{2}$ cm = 0.75 cm.

Thickness of each coin, $h = 0.2$ cm.

Volume of each coin $= (\pi r^2 h)$ cubic units

$= (\pi \times 0.75 \times 0.75 \times 0.2)\ \text{cm}^3$.

Radius of the new cylinder formed, $R = \frac{4.5}{2}$ cm = 2.25 cm.

Height of the new cylinder formed, $H = 5$ cm.

Volume of the new cylinder formed $= \pi R^2 H$

$= (\pi \times 2.25 \times 2.25 \times 5)\ \text{cm}^3$.

$$\text{Number of coins} = \left(\frac{\text{volume of new cylinder}}{\text{volume of 1 coin}}\right)$$

$$= \left(\frac{\pi \times 2.25 \times 2.25 \times 5}{\pi \times 0.75 \times 0.75 \times 0.2}\right)$$

$$= \left(\frac{225 \times 225 \times 5 \times 10}{75 \times 75 \times 2}\right) = 225.$$

Hence, the number of coins required = 225.

EXAMPLE 8 2.2 cu dm *of brass is to be drawn into a cylindrical wire of diameter* 0.50 cm. *Find the length of the wire.*

SOLUTION Volume of brass $= (2.2 \times 10 \times 10 \times 10)\ \text{cm}^3 = 2200\ \text{cm}^3$.

Radius of the wire = 0.25 cm.

Let the required length of the wire be x cm.

Then, volume of the wire $= (\pi r^2 x)$ cu units

$$= \left(\frac{22}{7} \times 0.25 \times 0.25 \times x\right) \text{cm}^3.$$

$$\therefore \quad \frac{22}{7} \times 0.25 \times 0.25 \times x = 2200$$

$$\Rightarrow \quad x = \left(\frac{2200 \times 7}{22 \times 0.25 \times 0.25}\right) = 11200 \text{ cm} = 112 \text{ m}.$$

Hence, the length of the wire is 112 m.

EXAMPLE 9 *The external diameter of a lead pipe is* 2.4 cm *and the thickness of the lead is* 2 mm. *Find the weight of a pipe of length* 7 m, *it being given that* 1 cu cm *of lead weighs* 10 g.

SOLUTION Length of the pipe, $h = (7 \times 100)$ cm = 700 cm.

External radius of the pipe, $R = 1.2$ cm.

Internal radius of the pipe, $r = (1.2 - 0.2)$ cm = 1 cm.

External volume of the pipe

$$= (\pi R^2 h) \text{ cu units} = \left(\frac{22}{7} \times 1.2 \times 1.2 \times 700\right) \text{cm}^3 = 3168 \text{ cm}^3.$$

Internal volume of the pipe

$$= (\pi r^2 h) \text{ cu units} = \left(\frac{22}{7} \times 1 \times 1 \times 700\right) \text{cm}^3 = 2200 \text{ cm}^3.$$

Volume of lead = (external volume) – (internal volume)

$$= (3168 - 2200) \text{ cm}^3 = 968 \text{ cm}^3.$$

Weight of the pipe $= \left(\frac{968 \times 10}{1000}\right)$ kg = 9.68 kg.

EXAMPLE 10 *The external diameter of an iron pipe is* 25 cm *and its length is* 20 cm. *If the thickness of the pipe is* 1 cm, *find the total surface area of the pipe.*

SOLUTION Length of the pipe, $h = 20$ cm.

External radius of the pipe, $R = \frac{25}{2}$ cm = 12.5 cm.

Thickness of the pipe = 1 cm.

Internal radius of the pipe, $r = (12.5 - 1)$ cm = 11.5 cm.

Total surface area of the pipe

= (external curved surface area) + (internal curved surface area) + 2(area of the base ring)

$$= [2\pi Rh + 2\pi rh + 2(\pi R^2 - \pi r^2)] \text{ sq units}$$

$$= [2\pi h(R + r) + 2\pi(R^2 - r^2)] \text{ sq units}$$

$$= 2\pi(R + r)[h + (R - r)] \text{ sq units}$$

$$= 2 \times \frac{22}{7} \times (12.5 + 11.5)(20 + 12.5 - 11.5) \text{ cm}^2$$

$$= \left(2 \times \frac{22}{7} \times 24 \times 21\right) \text{cm}^2 = 3168 \text{ cm}^2.$$

Hence, the total surface area of the pipe is 3168 cm^2.

EXAMPLE 11 *A cylindrical metallic pipe is* 14 cm *long. The difference between the outside and inside surfaces is* 44 cm^2. *If the pipe is made up of* 99 cu cm *of metal, find the outer and inner radii of the pipe.*

SOLUTION Let the outer and inner radii be R cm and r cm respectively.

Also, it is given that $h = 14$ cm. Then,

outside surface area = $(2\pi Rh)$ sq units

$$= \left(2 \times \frac{22}{7} \times R \times 14\right) \text{cm}^2$$

$$= (88R) \text{ cm}^2.$$

Inside surface area = $(2\pi rh)$ sq units

$$= \left(2 \times \frac{22}{7} \times r \times 14\right) \text{cm}^2 = (88r) \text{ cm}^2.$$

$$\therefore \quad (88R - 88r) = 44 \Rightarrow 88(R - r) = 44 \Rightarrow (R - r) = \frac{1}{2} \qquad \ldots \text{(i)}$$

External volume = $(\pi R^2 h)$

$$= \left(\frac{22}{7} \times R^2 \times 14\right) \text{cm}^3 = (44R^2) \text{ cm}^3.$$

Internal volume = $(\pi r^2 h)$ cubic units

$$= \left(\frac{22}{7} \times r^2 \times 14\right) \text{cm}^3 = (44r^2) \text{ cm}^3.$$

Volume of metal = (external volume) – (internal volume)

$$= (44R^2 - 44r^2) \text{ cm}^3 = 44(R^2 - r^2) \text{ cm}^3.$$

$$\therefore \quad 44(R^2 - r^2) = 99 \Rightarrow (R^2 - r^2) = \frac{99}{44} \Rightarrow (R^2 - r^2) = \frac{9}{4} \qquad \ldots \text{(ii)}$$

On dividing (ii) by (i), we get $(R + r) = \frac{9}{2}$ … (iii)

On solving (i) and (iii), we get $R = \frac{5}{2}$ and $r = 2$.

Hence, the outer radius = 2.5 cm and the inner radius = 2 cm.

EXAMPLE 12 *A solid iron rectangular block of dimensions* $(2.2 \text{ m} \times 1.2 \text{ m} \times 1 \text{ m})$ *is cast into a hollow cylindrical pipe of internal radius* 35 cm *and thickness* 5 cm. *Find the length of the pipe.*

SOLUTION Volume of the iron block = $(220 \times 120 \times 100) \text{ cm}^3$.

Volume of iron in hollow pipe formed

$$= (220 \times 120 \times 100) \text{ cm}^3.$$

Internal radius of the pipe, $r = 35$ cm.

External radius of the pipe, $R = (35 + 5)$ cm = 40 cm.

Let the length of the pipe be x cm.

Volume of iron in hollow pipe = $(\pi R^2 h - \pi r^2 h)$ cu units

$$= \pi h(R^2 - r^2) \text{ cu units}$$

$$= \pi h(R + r)(R - r) \text{ cu units}$$

$$= \frac{22}{7} \times x \times (40 + 35)(40 - 35) \text{ cm}^3$$

$$= \left(\frac{22}{7} \times x \times 75 \times 5\right) \text{cm}^3.$$

$$\therefore \quad \frac{22}{7} \times x \times 75 \times 5 = 220 \times 120 \times 100$$

$$\Rightarrow \quad x = \left(\frac{220 \times 120 \times 100 \times 7}{75 \times 5 \times 22}\right) = 2240.$$

Hence, the length of the pipe is 2240 cm = 22.4 m.

EXAMPLE 13 *A copper wire of diameter* 6 mm *is evenly wrapped on a cylinder of length* 15 cm *and diameter* 49 cm *to cover its whole surface. Find the length and volume of the wire. If the specific gravity of copper be* 9 g *per* cu cm, *find the weight of the wire.*

SOLUTION Length of the cylinder = 15 cm and its radius = 24.5 cm.

Diameter of the wire = 0.6 cm.

$$\text{Number of turns} = \frac{\text{length of the cylinder}}{\text{diameter of the wire}} = \frac{15}{0.6} = 25.$$

Length of wire in 1 turn = circumference of the base of cylinder

$$= \left(2 \times \frac{22}{7} \times 24.5\right) \text{ cm} = 154 \text{ cm}.$$

Length of the wire in 25 turns = (154×25) cm = 3850 cm

= 38.5 m.

Volume of the wire = $\pi r^2 h$

$$= \left(\frac{22}{7} \times 0.3 \times 0.3 \times 3850\right) \text{ cu cm}$$

= 1089 cu cm.

$$\text{Weight of the wire} = \left(\frac{1089 \times 9}{1000}\right) \text{ kg} = 9.801 \text{ kg}.$$

SOME MORE EXAMPLES

EXAMPLE 14 *The frame of a lampshade has base diameter* 20 cm *and height* 30 cm. *It is to be covered with a decorative cloth leaving a margin of* 2.5 cm *for folding it over the top as well as for bottom of the frame. Find how much cloth is required for covering the lampshade.*

SOLUTION Clearly, the decorative cloth forms a cylinder of base diameter 20 cm and height = $(30 + 2.5 + 2.5)$ cm = 35 cm.

Thus, $r = 10$ cm and $h = 35$ cm.

Area of the decorative cloth

= lateral surface area of a cylinder having $r = 10$ cm and $h = 35$ cm.

= $(2\pi rh)$ cm^2

$$= \left(2 \times \frac{22}{7} \times 10 \times 35\right) \text{ cm}^2 = 2200 \text{ cm}^2.$$

EXAMPLE 15 *The sum of radius of the base and height of a cylinder is* 37 m. *If the total surface area of the cylinder is* 1628 m^2, *find the curved surface area and volume of the cylinder.*

SOLUTION Let the radius of the base of the given cylinder be r metres and its height be h metres.

Then, its total surface area

$$= (2\pi rh + 2\pi r^2)\ m^2$$
$$= 2\pi r(h + r)\ m^2$$
$$= (2\pi r \times 37)\ m^2 \qquad [\because\ (h + r) = 37 \text{ (given)}].$$

But, total surface area = 1628 m^2.

$$\therefore\quad 2\pi r \times 37 = 1628 \Rightarrow 2 \times \frac{22}{7} \times r \times 37 = 1628$$
$$\Rightarrow r = \left(1628 \times \frac{7}{44} \times \frac{1}{37}\right)\ m = 7\ m.$$

Now, $h + r = 37 \Rightarrow h + 7 = 37 \Rightarrow h = 30$.

$\therefore$ curved surface area of the cylinder

$$= (2\pi rh)\ m^2$$
$$= \left(2 \times \frac{22}{7} \times 7 \times 30\right)\ m^2 = 1320\ m^2.$$

Volume of the cylinder

$$= (\pi r^2 h)\ m^3$$
$$= \left(\frac{22}{7} \times 7 \times 7 \times 30\right)\ m^3 = 4620\ m^3.$$

EXAMPLE 16 *A metal pipe is* 77 cm *long. The inner diameter of a cross section is* 4 cm, *the outer diameter being* 4.4 cm. *Find its*

(i) inner curved surface area
(ii) outer curved surface area
(iii) total surface area.

SOLUTION Let us denote the inner radius by r cm, outer radius by R cm and length of the metal pipe by h cm. Then,

$r = 2$, $R = 2.2$ and $h = 77$.

(i) Inner curved surface area

$$= (2\pi rh)\ cm^2$$
$$= \left(2 \times \frac{22}{7} \times 2 \times 77\right)\ cm^2 = 968\ cm^2.$$

(ii) Outer curved surface area

$$= (2\pi Rh)\ \text{cm}^2$$

$$= \left(2 \times \frac{22}{7} \times 2.2 \times 77\right) \text{cm}^2 = 1064.8\ \text{cm}^2.$$

(iii) Total surface area

= (inner curved surface area) + (outer curved surface area) + (area of two bases)

$$= \{968 + 1064.8 + 2(\pi R^2 - \pi r^2)\}\ \text{cm}^2$$

$$= \{2032.8 + 2\pi(R^2 - r^2)\}\ \text{cm}^2$$

$$= \left\{2032.8 + 2 \times \frac{22}{7} \times (R + r)(R - r)\right\} \text{cm}^2$$

$$= \left\{2032.8 + \frac{44}{7} \times (2.2 + 2) \times (2.2 - 2)\right\} \text{cm}^2$$

$$= \left\{2032.8 + \frac{44}{7} \times 4.2 \times 0.2\right\} \text{cm}^2$$

$$= \left(2032.8 + \frac{44}{7} \times \frac{42}{10} \times \frac{2}{10}\right) \text{cm}^2$$

$$= \left(2032.8 + \frac{132}{25}\right) \text{cm}^2 = (2032.8 + 5.28)\ \text{cm}^2 = 2038.08\ \text{cm}^2.$$

EXAMPLE 17 *Rainwater, which falls on a flat rectangular rooftop of dimensions* 22 m *by* 10 m, *is transferred into a cylindrical vessel of internal radius* 50 cm *through a circular pipe. A certain day recorded a rainfall of* 2.5 cm. *Find the (i) volume and (ii) height of the water, filled in the cylindrical vessel.*

SOLUTION (i) Volume of water filled in the cylindrical vessel

= volume of water collected on the rooftop

$$= \left(22 \times 10 \times \frac{2.5}{100}\right) \text{m}^3 = \frac{11}{2}\ \text{m}^3 = 5.5\ \text{m}^3.$$

(ii) Let the required height of water in the vessel be h metres.

Radius of the vessel, $r = \frac{50}{100}$ m $= \frac{1}{2}$ m.

Volume of water in the cylindrical vessel

$$= (\pi r^2 h)\ \text{m}^3$$

$$= \left(\frac{22}{7} \times \frac{1}{2} \times \frac{1}{2} \times h\right) \text{m}^3 = \frac{11h}{14}\ \text{m}^3.$$

$$\therefore \quad \frac{11h}{14} = \frac{11}{2} \Rightarrow h = \left(\frac{11}{2} \times \frac{14}{11}\right) \text{m} = 7\ \text{m}.$$

Hence, the height of water in the cylindrical vessel is 7 m.

EXAMPLE 18 *(i) Find the lateral, or curved, surface area of a closed cylindrical storage tank which is* 4.2 m *in diameter and* 4.5 m *high.*

(ii) How much steel was actually used in it, if $\frac{1}{12}$ *of the steel actually used was wasted in making the tank?*

SOLUTION (i) Radius of the cylinder, $r = \frac{4.2}{2}$ m = 2.1 m.

Height of the cylinder, $h = 4.5$ m.

Curved surface area of the cylinderical tank

$$= (2\pi rh)\ \text{m}^2 = \left(2 \times \frac{22}{7} \times 2.1 \times 4.5\right) \text{m}^2$$

$$= \left(2 \times \frac{22}{7} \times \frac{21}{10} \times \frac{45}{10}\right) \text{m}^2 = \frac{297}{5}\ \text{m}^2 = 59.4\ \text{m}^2.$$

(ii) Total surface area of the tank

$$= (2\pi rh + 2\pi r^2)\ \text{m}^2 = 2\pi r(h + r)\ \text{m}^2$$

$$= \left\{2 \times \frac{22}{7} \times 2.1 \times (4.5 + 2.1)\right\} \text{m}^2 = \left(\frac{44}{7} \times \frac{21}{10} \times \frac{66}{10}\right) \text{m}^2$$

$$= \frac{8712}{100}\ \text{m}^2 = 87.12\ \text{m}^2.$$

Let the actual area of steel used be x m^2.

Then, wasted steel = $\frac{x}{12}$ m^2.

Area of steel used in the tank = $\left(x - \frac{x}{12}\right) \text{m}^2 = \frac{11x}{12}\ \text{m}^2$.

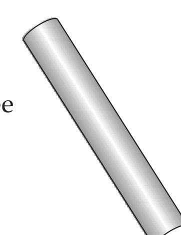
$$\therefore \quad \frac{11x}{12} = 87.12 \Rightarrow x = \left(\frac{87.12 \times 12}{11}\right) \text{m}^2 = (7.92 \times 12)\ \text{m}^2$$

$$= 95.04\ \text{m}^2.$$

Hence, 95.04 m^2 of steel was actually used in the tank.

EXAMPLE 19 *Find the area of the chart paper used to make the curved surface of a model of cylindrical kaleidoscope of length* 25 cm *and radius* 3.5 cm.

SOLUTION For the given cylindrical kaleidoscope, we have radius of the base, $r = 3.5$ cm and height, $h = 25$ cm.



Area of the chart paper required

= curved surface area of the kaleidoscope

$$= (2\pi rh)\ \text{cm}^2$$

$$= \left(2 \times \frac{22}{7} \times 3.5 \times 25\right) \text{cm}^2$$

$$= \left(2 \times \frac{22}{7} \times \frac{35}{10} \times 25\right) \text{cm}^2 = 550\ \text{cm}^2.$$

EXAMPLE 20 *How much cardboard is required to make* 35 *penholders in the shape of cylinders, each of radius* 3 cm *and height* 10.5 cm?

SOLUTION Cardboard required to make 1 penholder

$$= (\text{curved surface area} + \text{area of the base})$$

$$= (2\pi rh + \pi r^2)\ \text{cm}^2, \text{ where } r = 3 \text{ cm and } h = \frac{21}{2} \text{ cm}$$

$$= \pi r(2h + r)\ \text{cm}^2 = \left\{\frac{22}{7} \times 3 \times \left(2 \times \frac{21}{2} + 3\right)\right\} \text{cm}^2$$

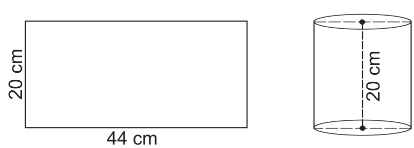
$$= \left(\frac{22}{7} \times 3 \times 24\right) \text{cm}^2.$$

Cardboard required to make 35 penholders

$$= \left(\frac{22}{7} \times 3 \times 24 \times 35\right) \text{cm}^2 = 7920 \text{ cm}^2.$$

EXAMPLE 21 *A rectangular sheet of paper* 44 cm $\times$ 20 cm *is rolled along its length and a cylinder is formed. Find the volume of the cylinder formed. (Take* $\pi = \frac{22}{7}$ *.)*

SOLUTION When a rectangular sheet is rolled along its length then the length of the sheet becomes the circumferene of the base of the cylinder and breadth of the sheet becomes the height of the cylinder.



Clearly, the height of the cylinder is 20 cm.

Let the radius of its base be r cm.

$\therefore$ circumference of the base of the cylinder
= length of the sheet

$\Rightarrow$ $2\pi r = 44$

$\Rightarrow$ $2 \times \frac{22}{7} \times r = 44 \Rightarrow r = \left(44 \times \frac{7}{44}\right) = 7.$

Thus, $r = 7$ cm and $h = 20$ cm.

$\therefore$ volume of the cylinder formed

$$= (\pi r^2 h)\ \text{cm}^3$$

$$= \left(\frac{22}{7} \times 7 \times 7 \times 20\right) \text{cm}^3 = 3080 \text{ cm}^3.$$

EXAMPLE 22 *The radii of two cylinders are in the ratio* $2:3$ *and their heights are in the ratio* $5:3$. *Calculate the ratio of their volumes and the ratio of their curved surfaces.*

SOLUTION Let their radii be $2R$, $3R$ and their heights be $5H$, $3H$.

Then, $\frac{V_1}{V_2} = \frac{\pi \times (2R)^2 \times 5H}{\pi \times (3R)^2 \times 3H} = \frac{20}{27} \Rightarrow V_1 : V_2 = 20 : 27.$

and $\frac{S_1}{S_2} = \frac{2\pi (2R)(5H)}{2\pi (3R)(3H)} = \frac{10}{9} \Rightarrow S_1 : S_2 = 10 : 9.$

Hence, their volumes are in the ratio 20 : 27 and their surface areas are in the ratio 10 : 9.

EXAMPLE 23 *The volume of a metallic cylindrical pipe is* 748 cm^3. *Its length is* 14 cm *and its external radius is* 9 cm. *Find its thickness.*

SOLUTION Volume of the pipe, $V = 748 \text{ cm}^3$. Length of the pipe, $h = 14$ cm.
External radius of the pipe, $R = 9$ cm.
Let the internal radius of the pipe be r cm. Then,

$$\begin{aligned} \text{volume of the pipe} &= (\pi R^2 h - \pi r^2 h) \\ &= \{\pi (R^2 - r^2) h\} \text{ cm}^3 \\ &= \left\{\frac{22}{7} \times (81 - r^2) \times 14\right\} \text{ cm}^3. \end{aligned}$$

But, volume of the pipe $= 748 \text{ cm}^3$.

$\therefore \quad \frac{22}{7} \times (81 - r^2) \times 14 = 748$

$\Rightarrow \quad 44 \times (81 - r^2) = 748$

$\Rightarrow \quad (81 - r^2) = \frac{748}{44} = 17 \Rightarrow r^2 = (81 - 17) = 64 = 8^2$

$\Rightarrow \quad r = 8$ cm.

$\therefore \quad$ thickness $= (R - r)$ cm $= (9 - 8)$ cm $= 1$ cm.

EXAMPLE 24 *A lead pencil consists of a cylinder of wood with a solid cylinder of graphite filled into it. The diameter of the pencil is* 7 mm, *the diameter of the graphite is* 1 mm *and the length of the pencil is* 14 cm.

Find the (i) volume of the graphite, (ii) volume of the wood, (iii) weight of the whole pencil, if the specific gravity of wood is 0.7 g/cm^3 *and that of the graphite is* 2.1 g/cm^3.

SOLUTION (i) Diameter of the graphite cylinder $= 1 \text{ mm} = \frac{1}{10}$ cm.

Radius of the graphite cylinder $= \left(\frac{1}{2} \times \frac{1}{10}\right)$ cm $= \frac{1}{20}$ cm.

Length of the graphite cylinder = 14 cm.

Volume of the graphite cylinder

$$= (\pi r^2 h)\ \text{cm}^3 = \left(\frac{22}{7} \times \frac{1}{20} \times \frac{1}{20} \times 14\right) \text{cm}^3$$

$$= \frac{11}{100}\ \text{cm}^3 = 0.11\ \text{cm}^3.$$

(ii) Diameter of the pencil $= 7\ \text{mm} = \frac{7}{10}$ cm.

Radius of the pencil $= \left(\frac{1}{2} \times \frac{7}{10}\right)$ cm $= \frac{7}{20}$ cm.

Length of the pencil = 14 cm.

Volume of the pencil

$$= (\pi R^2 h)\ \text{cm}^3 = \left(\frac{22}{7} \times \frac{7}{20} \times \frac{7}{20} \times 14\right) \text{cm}^3$$

$$= \frac{539}{100}\ \text{cm}^3 = 5.39\ \text{cm}^3.$$

$\therefore$ volume of wood in the pencil

= (volume of the pencil) – (volume of graphite)

$= (5.39 - 0.11)\ \text{cm}^3 = 5.28\ \text{cm}^3.$

(iii) Weight of the pencil

= (volume of wood) × (specific gravity of wood)

+ (volume of graphite) × (specific gravity of graphite)

$= (5.28 \times 0.7)\ \text{g} + (0.11 \times 2.1)\ \text{g}$

$= (3.696 + 0.231)\ \text{g} = 3.927\ \text{g}.$

EXAMPLE 25 30 *circular plates, each of radius* 14 cm *and thickness* 3 cm, *are placed one above the other to form a cylindrical solid.*
Find (i) the total surface area,
(ii) volume of the cylinder so formed.

SOLUTION Clearly, the cylinder formed has base radius, $r = 14$ cm and height, $h = (3 \times 30)$ cm $= 90$ cm.

(i) Total surface area of the cylinder formed

$$= (2\pi rh + 2\pi r^2) = 2\pi r(h + r)\ \text{cm}^2$$

$$= \left\{2 \times \frac{22}{7} \times 14 \times (90 + 14)\right\} \text{cm}^2$$

$$= (88 \times 104)\ \text{cm}^2 = 9152\ \text{cm}^2.$$

(ii) Volume of the cylinder formed

$$= \pi r^2 h = \left(\frac{22}{7} \times 14 \times 14 \times 90\right) \text{cm}^3$$

$$= 55440\ \text{cm}^3.$$

EXERCISE 15B

1. The diameter of a cylinder is 28 cm and its height is 40 cm. Find the curved surface area, total surface area and the volume of the cylinder.
2. A patient in a hospital is given soup daily in a cylindrical bowl of diameter 7 cm. If the bowl is filled with soup to a height of 4 cm, how much soup the hospital has to prepare daily to serve 250 patients?
3. The pillars of a temple are cylindrically shaped. Each pillar has a circular base of radius 20 cm and height 10 m. How much concrete mixture would be required to build 14 such pillars?
4. A soft drink is available in two packs: (i) a tin can with a rectangular base of length 5 cm, breadth 4 cm and height 15 cm, and (ii) a plastic cylinder with circular base of diameter 7 cm and height 10 cm. Which container has greater capacity and by how much?
5. There are 20 cylindrical pillars in a building, each having a diameter of 50 cm and height 4 m. Find the cost of cleaning them at ₹ 14 per m^2.
6. The curved surface area of a right circular cylinder is 4.4 m^2. If the radius of its base is 0.7 m, find its (i) height and (ii) volume.
7. The lateral surface area of a cylinder is 94.2 cm^2 and its height is 5 cm. Find (i) the radius of its base and (ii) its volume. (Take $\pi = 3.14$.)
8. The capacity of a closed cylindrical vessel of height 1 m is 15.4 litres. Find the area of the metal sheet needed to make it.
9. The inner diameter of a cylindrical wooden pipe is 24 cm and its outer diameter is 28 cm. The length of the pipe is 35 cm. Find the mass of the pipe, if 1 cm^3 of wood has a mass of 0.6 g.
10. In a water heating system, there is a cylindrical pipe of length 28 m and diameter 5 cm. Find the total radiating surface in the system.
11. Find the weight of a solid cylinder of radius 10.5 cm and height 60 cm if the material of the cylinder weighs 5 g per cm^3.
12. The curved surface area of a cylinder is 1210 cm^2 and its diameter is 20 cm. Find its height and volume.
13. The curved surface area of a cylinder is 4400 cm^2 and the circumference of its base is 110 cm. Find the height and the volume of the cylinder.
14. The radius of the base and the height of a cylinder are in the ratio 2 : 3. If its volume is 1617 cm^3, find the total surface area of the cylinder.
15. The total surface area of a cylinder is 462 cm^2. Its curved surface area is one third of its total surface area. Find the volume of the cylinder.
16. The total surface area of a solid cylinder is 231 cm^2 and its curved surface area is $\frac{2}{3}$ of the total surface area. Find the volume of the cylinder.

17. The ratio between the curved surface area and the total surface area of a right circular cylinder is 1 : 2. Find the volume of the cylinder if its total surface area is 616 cm^2.

18. A cylindrical bucket, 28 cm in diameter and 72 cm high, is full of water. The water is emptied into a rectangular tank, 66 cm long and 28 cm wide. Find the height of the water level in the tank.

19. The barrel of a fountain pen, cylindrical in shape, is 7 cm long and 5 mm in diameter. A full barrel of ink in the pen will be used up on writing 330 words on an average. How many words would use up a bottle of ink containing one fifth of a litre?

20. 1 cm^3 of gold is drawn into a wire 0.1 mm in diameter. Find the length of the wire.

21. If 1 cm^3 of cast iron weighs 21 g, find the weight of a cast iron pipe of length 1 m with a bore of 3 cm in which the thickness of the metal is 1 cm.

22. A cylindrical tube, open at both ends, is made of metal. The internal diameter of the tube is 10.4 cm and its length is 25 cm. The thickness of the metal is 8 mm everywhere. Calculate the volume of the metal.

23. It is required to make a closed cylindrical tank of height 1 m and base diameter 140 cm from a metal sheet. How many square metres of the sheet are required for the same?

24. A juiceseller has a large cylindrical vessel of base radius 15 cm filled up to a height of 32 cm with orange juice. The juice is filled in small cylindrical glasses of radius 3 cm up to a height of 8 cm, and sold for ₹ 15 each. How much money does he receive by selling the juice completely?

25. A well with inside diameter 10 m is dug 8.4 m deep. Earth taken out of it is spread all around it to a width of 7.5 m to form an embankment. Find the height of the embankment.

26. How many litres of water flows out of a pipe having an area of cross section of 5 cm^2 in 1 minute, if the speed of water in the pipe is 30 cm/sec?

27. A cylindrical water tank of diameter 1.4 m and height 2.1 m is being fed by a pipe of diameter 3.5 cm through which water flows at the rate of 2 m per second. In how much time will the tank be filled?

28. A cylindrical container with diameter of base 56 cm contains sufficient water to submerge a rectangular solid of iron with dimensions $(32\text{ cm} \times 22\text{ cm} \times 14\text{ cm})$. Find the rise in the level of water when the solid is completely submerged.

29. Find the cost of sinking a tube-well 280 m deep, having a diameter 3 m at the rate of ₹ 15 per cubic metre. Find also the cost of cementing its inner curved surface at ₹ 10 per square metre.

30. Find the length of 13.2 kg of copper wire of diameter 4 mm, when 1 cubic centimetre of copper weighs 8.4 g.

31. It costs ₹ 3300 to paint the inner curved surface of a cylindrical vessel 10 m deep at the rate of ₹ 30 per m^2. Find the
(i) inner curved surface area of the vessel,
(ii) inner radius of the base, and
(iii) capacity of the vessel.

32. The difference between inside and outside surfaces of a cylindrical tube 14 cm long, is 88 cm^2. If the volume of the tube is 176 cm^3, find the inner and outer radii of the tube.

33. A rectangular sheet of paper 30 cm $\times$ 18 cm can be transformed into the curved surface of a right circular cylinder in two ways namely, either by rolling the paper along its length or by rolling it along its breadth. Find the ratio of the volumes of the two cylinders, thus formed.

ANSWERS (EXERCISE 15B)

1. 3520 cm^2, 4752 cm^2, 24640 cm^3 **2.** 38.5 litres **3.** 17.6 m^3
4. Plastic cylinder has more capacity, 85 cm^3 **5.** ₹ 1760
6. (i) 1 m (ii) 1.54 m^3 **7.** (i) 3 cm (ii) 141.3 cm^3 **8.** 4708 cm^2
9. 3.432 kg **10.** 44000 cm^2 **11.** 103.95 kg **12.** 19.25 cm, 6050 cm^3
13. 40 cm, 38500 cm^3 **14.** 770 cm^2 **15.** 539 cm^3 **16.** 269.5 cm^3
17. 1078 cm^3 **18.** 24 cm **19.** 48000 **20.** 127.27 m
21. 26.4 kg **22.** 704 cm^3 **23.** 7.48 m^2 **24.** ₹ 1500 **25.** 1.6 m
26. 9 litres **27.** 28 minutes **28.** 4 cm **29.** ₹ 29700, ₹ 26400
30. 125 m **31.** (i) 1.75 m (ii) 110 m^2 (iii) 96.25 m^3
32. 1.5 cm, 2.5 cm **33.** 5 : 3

HINTS TO SOME SELECTED QUESTIONS

3. Volume of concrete mixture in 14 pillars

$$= \left(\frac{22}{7} \times \frac{20}{100} \times \frac{20}{100} \times 10 \times 14\right) m^3 = 17.6 \ m^3.$$

5. Lateral surface area to be cleaned

$$= \left(2 \times \frac{22}{7} \times \frac{25}{100} \times 4 \times 20\right) m^2 = \left(\frac{880}{7}\right) m^2.$$

Cost of cleaning $= ₹ \left(\frac{880}{7} \times 14\right) = ₹\ 1760.$

8. $h = 100$ cm, $V = (15.4 \times 1000)\text{ cm}^3 = 15400\text{ cm}^3$.

$V = \pi r^2 h \Rightarrow \frac{22}{7} \times r^2 \times 100 = 15400 \Rightarrow r^2 = 49 \Rightarrow r = 7$ cm.

Area of the metal sheet $= (2\pi rh + 2\pi r^2) = 2\pi r(h + r)$.

9. Volume of the pipe $= \{\pi(R^2 - r^2) \times h\}$

$$= \left[\frac{22}{7} \times \{(14)^2 - (12)^2\} \times 35\right]\text{cm}^3 = 5720\text{ cm}^3.$$

Mass of the pipe $= \left(5720 \times \frac{6}{10} \times \frac{1}{1000}\right)$ kg $= 3.432$ kg.

10. Given, $r = 2.5$ cm and $h = 2800$ cm.

Total radiating surface area $= (2\pi rh)\text{ cm}^2$.

14. Suppose that radius $= 2x$ cm and height $= 3x$ cm.

Volume $= \pi r^2 h = \left(\frac{22}{7} \times 4x^2 \times 3x\right) = \left(\frac{264x^3}{7}\right)\text{cm}^3$.

$$\therefore \quad \frac{264x^3}{7} = 1617 \Rightarrow x^3 = \left(\frac{1617 \times 7}{264}\right) = \left(\frac{7}{2}\right)^3 \Rightarrow x = \frac{7}{2}.$$

Thus, $r = 7$ cm and $h = 10.5$ cm.

Total surface area $= (2\pi rh + 2\pi r^2) = 2\pi r(h + r)$.

15. Curved surface area $= \left(\frac{1}{3} \times 462\right)\text{cm}^2 = 154\text{ cm}^2$.

$2\pi rh = 154$ ... (i) and $2\pi r(h + r) = 462$ ... (ii).

$$\therefore \quad \frac{2\pi rh}{2\pi r(h + r)} = \frac{154}{462} = \frac{1}{3} \Rightarrow \frac{h}{h + r} = \frac{1}{3}$$

$$\Rightarrow 3h = h + r \Rightarrow r = 2h.$$

Putting $r = 2h$ in (i), we get $h^2 = \frac{49}{4} = \left(\frac{7}{2}\right)^2 \Rightarrow h = \frac{7}{2}$.

Thus, $r = 7$ and $h = 3.5$. Now, find the volume.

19. We have, $r = \frac{5}{20}$ cm $= \frac{1}{4}$ cm and $h = 7$ cm.

Volume of the full barrel $= \pi r^2 h = \left(\frac{22}{7} \times \frac{1}{16} \times 7\right)\text{cm}^3 = \frac{11}{8}\text{ cm}^3$.

Given, volume of ink $= \left(\frac{1}{5} \times 1000\right)\text{cm}^3 = 200\text{ cm}^3$.

$\frac{11}{8}\text{ cm}^3$ is sufficient to write 330 words.

200 cm^3 will be sufficient to write $\left(330 \times \frac{8}{11} \times 200\right) = 48000$ words.

20. Radius of wire, $r = \frac{1}{20}$ mm $= \frac{1}{200}$ cm.

$\pi r^2 h = 1 \Rightarrow \frac{22}{7} \times \frac{1}{200} \times \frac{1}{200} \times h = 1$.

$$\therefore \quad h = \left(\frac{200 \times 200 \times 7}{22}\right)\text{cm} = \left(\frac{200 \times 200 \times 7}{22 \times 100}\right)\text{m} = 127.27\text{ m}.$$

21. Internal radius = 1.5 cm and external radius = 2.5 cm.

Volume of cast iron $= \left[\frac{22}{7} \times \{(2.5)^2 - (1.5)^2\} \times 100\right] \text{cm}^3$

$= \left(\frac{22}{7} \times 4 \times 1 \times 100\right) \text{cm}^3.$

Weight $= \left(\frac{8800}{7} \times \frac{21}{1000}\right)$ kg = 26.4 kg.

22. Internal radius = 5.2 cm, external radius = (5.2 + 0.8) cm = 6 cm.

Also, h = 25 cm.

Volume of metal = $\{\pi (R^2 - r^2) \times h\}$, where R = 6 cm and r = 5.2 cm.

25. Volume of earth dug out = volume of earth in embankment.

$\pi \times 5 \times 5 \times 8.4 = \pi \times \{(12.5)^2 - 5^2\} \times h$. Find h.

26. Speed of water = 30 m/s.

Volume of water that flows out of the pipe in 1 s

= area of cross section × length of water flown in 1 s

$= (5 \times 30) \text{ cm}^3 = 150 \text{ cm}^3$.

Volume of water that flows in 1 minute

$= (150 \times 60) \text{ cm}^3 = \left(\frac{150 \times 60}{1000}\right)$ litres = 9 litres.

27. Volume of water tank $= \{\pi \times (0.7)^2 \times 2.1\} \text{ m}^2 = \left(\frac{1029\pi}{1000}\right) \text{m}^2.$

Volume of water that flows through the pipe in 1 s.

$= \left(\pi \times \frac{3.5}{200} \times \frac{3.5}{200} \times 2\right) \text{m}^3 = \left(\frac{49\pi}{80000}\right) \text{m}^3.$

Let the tank be filled in x seconds. Then,

$\frac{49\pi}{80000} \times x = \frac{1029\pi}{1000} \Rightarrow x = \left(\frac{1029 \times 80}{49}\right)$ seconds

= 1680 s = 28 min.

28. Let the rise in level be x cm. Then,

volume of cylinder with base radius 28 cm and height x cm

= volume of iron solid

$\therefore \quad \frac{22}{7} \times 28 \times 28 \times x = 32 \times 22 \times 14 \Rightarrow x = 4$ cm.

29. Here, r = 1.5 m and h = 280 m.

Cost of sinking the tube-well = ₹ $\left\{\frac{22}{7} \times \frac{15}{10} \times \frac{15}{10} \times 280 \times 15\right\}$ = ₹ 29700.

Cost of cementing its inner curved surface

= ₹ $\left(2 \times \frac{22}{7} \times \frac{15}{10} \times 280 \times 10\right)$ = ₹ 26400.

30. Let the required length of wire be x cm.

Then, its volume $= \left(\frac{22}{7} \times \frac{2}{10} \times \frac{2}{10} \times x\right) \text{cm}^3 = \left(\frac{22x}{175}\right) \text{cm}^3.$

$\therefore$ its weight $= \left(\frac{22x}{175} \times 8.4\right) \text{g} = \left(\frac{22x \times 8.4}{175 \times 1000}\right)$ kg

$\therefore \quad \frac{22x \times 8.4}{175 \times 1000} = 13.2$

$\Rightarrow \quad x = \left(\frac{13.2 \times 175 \times 1000}{22 \times 8.4}\right) \text{cm} = \left(\frac{132 \times 175 \times 1000}{22 \times 84 \times 100}\right) \text{m} = 125 \text{ m}.$

31. Let the inner radius of the base be r metres.

(i) Area of inner curved surface $= \frac{3300}{30} \text{ m}^2 = 110 \text{ m}^2$.

(ii) $2\pi rh = 110 \Rightarrow 2 \times \frac{22}{7} \times r \times 10 = 110 \Rightarrow r = \frac{7}{4} \text{ m} = 1.75 \text{ m}$.

(iii) Capacity of the vessel $= \pi r^2 h = \left(\frac{22}{7} \times \frac{7}{4} \times \frac{7}{4} \times 10\right) \text{m}^3 = 96.25 \text{ m}^3$.

32. Let the inner radius be r cm and outer radius R cm.

$$\therefore \quad (2\pi Rh - 2\pi rh) = 88 \Rightarrow 2\pi h(R-r) = 88$$
$$\Rightarrow 2 \times \frac{22}{7} \times 14 \times (R-r) = 88$$
$$\Rightarrow R - r = \left(88 \times \frac{7}{44 \times 14}\right) = 1.$$

$$\pi(R^2 - r^2) \times h = 176 \Rightarrow \frac{22}{7} \times (R+r) \times (R-r) \times 14 = 176$$

$$\Rightarrow \quad R + r = \frac{176}{44} = 4 \qquad [\because R - r = 1].$$

On solving $R + r = 4$, $R - r = 1$, we get

$R = 2.5$ cm and $r = 1.5$ cm.

33. **Case I** *When the sheet is folded along its length:*

In this case, it forms a cylinder having height $h_1 = 18$ cm and the circumference of its base equal to 30 cm.

Let the radius of its base be r_1. Then,

$$2\pi r_1 = 30 \Rightarrow r_1 = \frac{15}{\pi}.$$

$$\therefore \quad V_1 = \pi r_1^2 h_1 = \left\{\pi \times \left(\frac{15}{\pi}\right)^2 \times 18\right\} \text{cm}^3 = \frac{4050}{\pi} \text{ cm}^3.$$

Case II *When the sheet is folded along its breadth:*

In this case, we get a cylinder having height $h_2 = 30$ cm and the circumference of its base equal to 18 cm.

Let the radius of its base be r_2. Then,

$$2\pi r_2 = 18 \Rightarrow r_2 = \frac{9}{\pi}.$$

$$\therefore \quad V_2 = \pi r_2^2 h_2 = \left\{\pi \times \left(\frac{9}{11}\right)^2 \times 30\right\} \text{cm}^3 = \frac{2430}{\pi} \text{ cm}^3.$$

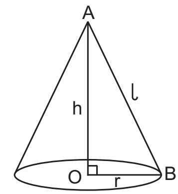
$$\therefore \quad \frac{V_1}{V_2} = \left(\frac{4050}{\pi} \times \frac{\pi}{2430}\right) = \frac{4050}{2430} = \frac{5}{3}.$$

RIGHT CIRCULAR CONE

We see around us many objects such as an *ice-cream cone, a conical vessel, a clown's cap,* etc. These objects are said to have the shape of a right circular cone. In geometry, we define it as under.

RIGHT CIRCULAR CONE *The solid generated by the rotation of a right-angled triangle about one of the sides containing the right angle is called a right circular cone.*

Thus, on rotating a right-angled triangular lamina *AOB* about *OA*, it generates a cone.



Cone

The point *A* is the *vertex* of the cone.

Its *base* is a circle with centre *O* and radius *OB*.

The length *OA* is the *height* of the cone and the length *AB* is called its *slant height*.

Clearly, $\angle AOB = 90°$.

If radius of the base = r units, height = h units and slant height = l units then

$$l^2 = (h^2 + r^2) \Rightarrow l = \sqrt{h^2 + r^2}.$$

Formulae:

For a right circular cone of radius = r units, height = h units and slant height = l units, we have:

(i) *Slant height of the cone* $(l) = \sqrt{h^2 + r^2}$ *units.*

(ii) *Volume of the cone* $= \frac{1}{3}\pi r^2 h$ *cubic units.*

(iii) *Area of curved surface* $= (\pi rl)$ *sq units* $= (\pi r\sqrt{h^2 + r^2})$ *sq units.*

(iv) *Total surface area* = (*area of the curved surface*) + (*area of the base*)

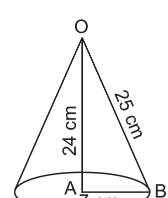
$= (\pi rl + \pi r^2)$ *sq units* $= \pi r(l + r)$ *sq units.*

NOTE Take $x = \frac{22}{7}$, unless stated otherwise.

SOLVED EXAMPLES

EXAMPLE 1 *The height of a cone is* 24 cm *and the diameter of its base is* 14 cm. *Find the slant height, volume, area of curved surface and the total surface area of the cone.*

SOLUTION Here, $h = 24$ cm and $r = 7$ cm.



Let the slant height be l cm. Then,

$$l = \sqrt{h^2 + r^2} = \sqrt{(24)^2 + (7)^2} = \sqrt{625} = 25.$$

$\therefore$ slant height = l cm = 25 cm.

Volume of the cone $= \frac{1}{3}\pi r^2 h$

$= \left(\frac{1}{3} \times \frac{22}{7} \times 7 \times 7 \times 24\right) \text{cm}^3$

$= 1232 \text{ cm}^3$.

Curved surface area of the cone $= \pi r l$

$= \left(\frac{22}{7} \times 7 \times 25\right) \text{cm}^2 = 550 \text{ cm}^2$.

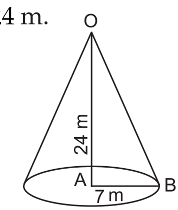
Total surface area of the cone $= \pi r(l + r)$

$= \left[\frac{22}{7} \times 7 \times (25 + 7)\right] \text{cm}^2$

$= 704 \text{ cm}^2$.

EXAMPLE 2 *How many metres of cloth,* 5 m *wide, will be required to make a conical tent, the radius of whose base is* 7 m *and height is* 24 m*?*

SOLUTION Radius of the tent, $r = 7$ m and its height, $h = 24$ m.



$\therefore$ slant height, $l = \sqrt{r^2 + h^2} = \sqrt{(7)^2 + (24)^2}$ m

$= \sqrt{625}$ m $= 25$ m.

Area of the curved surface $= (\pi r l)$ sq units

$= \left(\frac{22}{7} \times 7 \times 25\right) \text{m}^2$

$= 550 \text{ m}^2$.

Thus, the area of the cloth $= 550 \text{ m}^2$.

Length of the cloth required $= \left(\frac{\text{area}}{\text{width}}\right) = \left(\frac{550}{5}\right)$ m $= 110$ m.

Hence, length of the cloth required is 110 m.

EXAMPLE 3 *The height and the slant height of a cone are* 21 cm *and* 28 cm *respectively. Find the volume of the cone.*

SOLUTION We have, $h = 21$ cm and $l = 28$ cm.

$\therefore \quad l^2 = r^2 + h^2 \Rightarrow r = \sqrt{l^2 - h^2} = \sqrt{(28)^2 - (21)^2}$ cm

$\Rightarrow r = \sqrt{(28 + 21) \times (28 - 21)}$ cm $= \sqrt{49 \times 7}$ cm

$= 7\sqrt{7}$ cm.

Volume of the cone $= \frac{1}{3}\pi r^2 h$

$= \left\{\frac{1}{3} \times \frac{22}{7} \times (7\sqrt{7})^2 \times 21\right\} \text{cm}^3$

$= (22 \times 343) \text{ cm}^3 = 7546 \text{ cm}^3$.

EXAMPLE 4 *The volume of a right circular cone is* 9856 cm^3. *If the diameter of the base is* 28 cm, *find*

(i) *height of the cone,* (ii) *slant height of the cone,*

(iii) *curved surface area of the cone.*

SOLUTION We have, $V = 9856 \text{ cm}^3$ and $r = 14$ cm.

(i) Let the height of the cone be h cm.

Then, volume $= \frac{1}{3}\pi r^2 h$

$$\Rightarrow \quad 9856 = \frac{1}{3} \times \frac{22}{7} \times 14 \times 14 \times h$$

$$\Rightarrow \quad h = \left(\frac{9856 \times 3 \times 7}{22 \times 14 \times 14}\right) = 48.$$

$\therefore$ height of the cone is 48 cm.

(ii) Let the slant height of the cone be l cm. Then,

$$l^2 = r^2 + h^2$$

$$\Rightarrow \quad l^2 = (14)^2 + (48)^2 = 196 + 2304 = 2500$$

$$\Rightarrow \quad l = \sqrt{2500} = 50.$$

$\therefore$ slant height of the cone is 50 cm.

(iii) Curved surface area of the cone

$$= \pi r l$$

$$= \left(\frac{22}{7} \times 14 \times 50\right) \text{cm}^2 = 2200 \text{ cm}^2.$$

$\therefore$ curved surface area of the cone is 2200 cm^2.

EXAMPLE 5 *The radius and the height of a right circular cone are in the ratio of* 5 : 12 *and its volume is* 2512 cu cm. *Find the curved surface area and the total surface area of the cone. (Use* $\pi = 3.14$.*)*

SOLUTION Let radius $= 5x$ cm and height $= 12x$ cm. Then,

$$\text{volume} = \left[\frac{1}{3} \times 3.14 \times (5x)^2 \times (12x)\right] \text{cm}^3 = (314x^3) \text{ cm}^3.$$

$$\therefore \quad 314x^3 = 2512 \Rightarrow x^3 = \left(\frac{2512}{314}\right) = 8 \Rightarrow x = 2.$$

$\therefore$ radius = 10 cm and height = 24 cm.

$$\therefore \quad \text{slant height} = \sqrt{r^2 + h^2} = \sqrt{(10)^2 + (24)^2} \text{ cm}$$

$$= \sqrt{676} \text{ cm} = 26 \text{ cm}.$$

So, area of the curved surface $= \pi rl$

$$= (3.14 \times 10 \times 26)\ \text{cm}^2$$
$$= 816.4\ \text{cm}^2.$$

Total surface area = (curved surface area + base area)

$$= (816.4 + 3.14 \times 10 \times 10)\ \text{cm}^2$$
$$= (816.4 + 314)\ \text{cm}^2 = 1130.4\ \text{cm}^2.$$

EXAMPLE 6 *A cone of height* 8 m *has a curved surface area* 188.4 *square metres. Find its volume. (Take* $\pi = 3.14$.*)*

SOLUTION Let the base radius be r metres and slant height be l metres. Then,

$$\text{curved surface area} = \pi rl = \pi r\sqrt{r^2 + h^2}$$
$$= (3.14 \times r \times \sqrt{r^2 + 64})\ \text{m}^2.$$

$\therefore \quad 3.14 \times r \times \sqrt{r^2 + 64} = 188.4$

$\Rightarrow \quad r\sqrt{r^2 + 64} = \dfrac{188.4}{3.14} = 60$

$\Rightarrow \quad r^2(r^2 + 64) = 3600$

$\Rightarrow \quad r^4 + 64r^2 - 3600 = 0 \qquad \left[\begin{array}{l}\because\ x^2 + 64x - 3600 = 0, \text{ where } r^2 = x \\ \Rightarrow\ (x + 100)(x - 36) = 0\end{array}\right]$

$\Rightarrow \quad (r^2 + 100)(r^2 - 36) = 0$

$\Rightarrow \quad r^2 = 36 \quad [\because\ r^2 = -100 \text{ gives imaginary value of } r]$

$\Rightarrow \quad r = 6.$

So, the radius of the base = 6 m.

$$\text{Volume} = \left(\frac{1}{3}\pi r^2 h\right) \text{cubic units}$$
$$= \left(\frac{1}{3} \times 3.14 \times 6 \times 6 \times 8\right) \text{m}^3 = 301.44\ \text{m}^3.$$

Hence, the volume of the cone is $301.44\ \text{m}^3$.

EXAMPLE 7 *What length of tarpaulin* 3 m *wide will be required to make a conical tent of height* 8 m *and base radius* 6 m*? Assume that the extra length of material required for stitching margins and wastage in cutting is* 20 cm. *(Use* $\pi = 3.14$.*)*

SOLUTION Base radius of the tent, $r = 6$ m and its height, $h = 8$ m.

$$\therefore \quad l = \sqrt{r^2 + h^2} = \sqrt{6^2 + 8^2}\ \text{m} = \sqrt{36 + 64}\ \text{m}$$
$$= \sqrt{100}\ \text{m} = 10\ \text{m}.$$

Curved surface area of the conical tent

$$= \pi rl = (3.14 \times 6 \times 10)\ \text{m}^2 = 188.4\ \text{m}^2.$$

Area of tarpaulin required = 188.4 m^2.

Width of tarpaulin = 3 m.

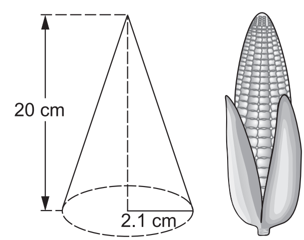
Length of tarpaulin = $\frac{\text{area}}{\text{width}} = \frac{188.4}{3}$ m = 62.8 m.

Extra length required = 20 cm = $\frac{20}{100}$ m = 0.2 m.

Total length of tarpaulin required = (62.8 + 0.2) m = 63 m.

EXAMPLE 8 *A corn cob, shaped somewhat like a cone, has the radius of its broadest end as* 2.1 cm *and length as* 20 cm. *If each* 1 cm^2 *of the surface of the cob carries an average of four grains, find how many grains you would find on the entire cob.*

SOLUTION We know that the grains of corn are found on the curved surface of the corn cob.



$\therefore$ number of grains on the corn cob

= (curved surface area of the corn cob) $\times$ (number of grains of corn on 1 cm^2).

So, we shall find the curved surface area of the corn cob.

We have, r = 2.1 cm and h = 20 cm.

$\therefore$ $l^2 = r^2 + h^2 = (2.1)^2 + (20)^2 = 404.41$.

So, $l = \sqrt{404.41}$ cm = 20.109 cm $\approx$ 20.11 cm.

Curved surface area of the corn cob

$$= \pi r l = \left(\frac{22}{7} \times 2.1 \times 20.11\right) \text{cm}^2$$

$$= 132.726 \text{ cm}^2 \approx 132.73 \text{ cm}^2.$$

Number of grains on 1 cm^2 = 4.

$\therefore$ number of grains on entire corn cob

$$= (132.73 \times 4) = 530.92 \approx 531.$$

So, there would be approximately 531 grains on the corn cob.

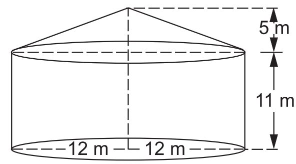
EXAMPLE 9 *A tent is in the form of a right circular cylinder, surmounted by a cone. The diameter of the cylinder is* 24 m. *The height of the cylindrical portion is* 11 m, *while the vertex of the cone is* 16 m *above the ground. Find the area of the canvas required for the tent.*

SOLUTION Radius of the cylinder, R = 12 m and its height, H = 11 m.

Area of the curved surface of the cylindrical portion

$= 2\pi RH$ sq units

$= (2\pi \times 12 \times 11)\ \text{m}^2 = (264\pi)\ \text{m}^2$.



Radius of the cone, $r = 12$ m

and its height, $h = (16 - 11)$ m $= 5$ m.

Slant height of the cone, $l = \sqrt{r^2 + h^2}$

$\therefore \quad l = \sqrt{(12)^2 + (5)^2} = \sqrt{169} = 13$ m.

Area of curved surface of the cone

$= (\pi rl)$ sq units

$= (\pi \times 12 \times 13)\ \text{m}^2 = (156\pi)\ \text{m}^2$.

Area of canvas = (curved surface area of the cylindrical part) + (curved surface area of the conical part)

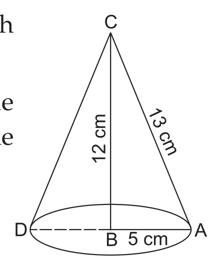
$= (264\pi + 156\pi)\ \text{m}^2 = (420\pi)\ \text{m}^2$

$= \left(420 \times \frac{22}{7}\right) \text{m}^2 = 1320\ \text{m}^2$.

Hence, the area of canvas required is $1320\ \text{m}^2$.

EXAMPLE 10 *A right* $\triangle ABC$ *with sides* 5 cm, 12 cm *and* 13 cm *is revolved about the side* 12 cm. *Find the volume of the solid so formed. If* $\triangle ABC$ *is revolved about the side* 5 cm *then find the volume of the solid so formed. Find also the ratio of the volumes of the two solids obtained.*

SOLUTION Let $\triangle ABC$ be right angled at B with $AB = 5$ cm, $BC = 12$ cm and $AC = 13$ cm.



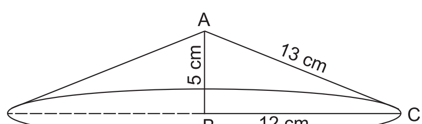
On revolving $\triangle ABC$ about the side $BC = 12$ cm, we obtain a right circular cone of radius 5 cm and height 12 cm.

Volume of the cone so formed

$= \left\{\frac{1}{3}\pi \times (5)^2 \times 12\right\} \text{cm}^3$

$= (100\pi)\ \text{cm}^3 = V_1$ (say).

On revolving $\triangle ABC$ about the side $AB = 5$ cm, we obtain a right circular cone of radius 12 cm and height 5 cm.



Volume of the cone so formed

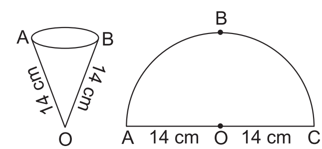
$$= \left\{\frac{1}{3}\pi \times (12)^2 \times 5\right\} \text{cm}^3$$

$$= (240\pi)\ \text{cm}^3 = V_2 \text{ (say)}.$$

$$\therefore \quad \frac{V_1}{V_2} = \frac{100\pi}{240\pi} = \frac{5}{12} \Rightarrow V_1 : V_2 = 5 : 12.$$

EXAMPLE 11 *A semicircular sheet of diameter* 28 cm *is bent to form an open conical cup. Find the capacity of the cup. (Use* $\sqrt{3} = 1.732$.*)*

SOLUTION When the semicircular sheet is bent into an open conical cup, the radius of the sheet becomes the slant height of the conical cup.



$\therefore \quad l = 14$ cm.

Circumference of the base of the cone

$$= \text{length of arc } ABC$$

$$= (\pi \times 14)\ \text{cm} = \left(\frac{22}{7} \times 14\right) \text{cm} = 44 \text{ cm}.$$

Let the radius of the cone be r cm. Then,

$$2\pi r = 44 \Rightarrow 2 \times \frac{22}{7} \times r = 44 \Rightarrow r = 7 \text{ cm}.$$

Let the height of the cone be h cm. Then,

$$h^2 = l^2 - r^2 = (14)^2 - (7)^2 = 196 - 49 = 147.$$

$$\therefore \quad h = \sqrt{147} = \sqrt{7 \times 7 \times 3} = 7\sqrt{3} \text{ cm}$$

$$= (7 \times 1.732) \text{ cm} = 12.12 \text{ cm}.$$

Capacity of the conical cup

$$= \text{volume of the conical cup}$$

$$= \frac{1}{3}\pi r^2 h$$

$$= \left(\frac{1}{3} \times \frac{22}{7} \times 7 \times 7 \times 12.12\right) \text{cm}^3$$

$$= (154 \times 4.04)\ \text{cm}^3 = 622.16 \text{ cm}^3.$$

Hence, the capacity of the conical cup is 622.16 cm^3.

EXAMPLE 12 *A conical tent is to accommodate* 11 *persons. Each person must have* 4 m^2 *of the space on the ground and* 20 m^3 *of air to breath. Find the height of the tent.*

SOLUTION Let the radius of the base of the conical tent be r metres and its height be h metres.

Area of the base $= (4 \times 11)\ \text{m}^2 = 44\ \text{m}^2 \Rightarrow \pi r^2 = 44\ \text{m}^2$. ... (i)

Volume of the cone $= (20 \times 11)\ \text{m}^3 = 220\ \text{m}^3$

$\Rightarrow \frac{1}{3}\pi r^2 h = 220\ \text{m}^3 \Rightarrow \pi r^2 h = 660\ \text{m}^3$. ... (ii)

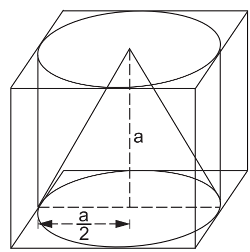
On dividing (ii) by (i), we get

$$\frac{\pi r^2 h}{\pi r^2} = \frac{660\ \text{m}^3}{44\ \text{m}^2} \Rightarrow h = 15\ \text{m}.$$

Hence, the height of the tent is 15 metres.

EXAMPLE 13 *A cylinder lies within a cube touching all its vertical faces and a cone lies inside the cylinder. If their heights are same with the same base, find the ratio of their volumes.*

SOLUTION Let the length of each edge of the given cube be a units.



Then, volume of the cube

$= a^3$ cubic units $= V_1$ (say).

It is given that the cylinder lies within the cube and touches all its vertical faces.

So, the radius of the base of the cylinder $= \frac{a}{2}$ units and height of the cylinder $= a$ units.

Volume of the cylinder $= \pi r^2 h$

$$= \left\{\frac{22}{7} \times \left(\frac{a}{2}\right)^2 \times a\right\} \text{ cubic units}$$

$$= \left(\frac{11a^3}{14}\right) \text{ cubic units} = V_2 \text{ (say)}.$$

A cone is drawn inside the cylinder such that both have the same base and same height.

$\therefore$ radius of the base of the cone $= \frac{a}{2}$ units

and height of the cone $= a$ units.

$\therefore$ volume of the cone $= \frac{1}{3}\pi r^2 h$

$$= \left(\frac{1}{3} \times \frac{22}{7} \times \frac{a^2}{4} \times a\right) \text{ cubic units}$$

$$= \frac{11a^3}{42} \text{ cubic units} = V_3 \text{ (say)}.$$

$\therefore$ ratio of their volumes is given as

$$V_1 : V_2 : V_3 = a^3 : \frac{11a^3}{14} : \frac{11a^3}{42} = 42 : 33 : 11.$$

EXAMPLE 14 *Find the volume of the largest right circular cone that can be cut out of a cube whose edge is* 21 cm.

SOLUTION The base of the largest right circular cone will be the circle inscribed in a face of the cube and its height will be equal to an edge of the cube.

$\therefore$ radius of the base of the cone, $r = \frac{21}{2}$ cm

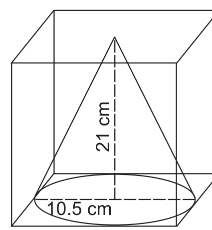
and height of the cone, $h = 21$ cm.

$\therefore$ volume of the required cone

$$= \frac{1}{3}\pi r^2 h$$

$$= \left(\frac{1}{3} \times \frac{22}{7} \times \frac{21}{2} \times \frac{21}{2} \times 21\right) \text{cm}^3$$

$$= \frac{4851}{2}\ \text{cm}^3 = 2425.5\ \text{cm}^3.$$



EXAMPLE 15 *The height of a cone is* 30 cm. *A small cone is cut off at the top by a plane parallel to the base. If its volume be* $\frac{1}{27}$ *of the volume of the given cone, at what height above the base, the section has been made?*

SOLUTION Let the smaller cone have radius = r cm and height = h cm.

And, let the radius of the given original cone be R cm.

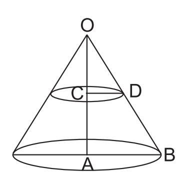
Then, $\frac{r}{R} = \frac{h}{30} \Rightarrow r = \frac{hR}{30}$ ... (i) [$\because \triangle OCD \sim \triangle OAB$].

Volume of smaller cone $= \frac{1}{27} \times$ (volume of larger cone)

$\therefore$ $\frac{1}{3}\pi r^2 h = \frac{1}{27} \times \left(\frac{1}{3}\pi R^2 \times 30\right)$

$\Rightarrow$ $r^2 h = \frac{10}{9} R^2$

$\Rightarrow$ $\left(\frac{hR}{30}\right)^2 \times h = \frac{10}{9} R^2$ [using (i)]



$\Rightarrow$ $h^3 = \left(\frac{10}{9} \times 900\right) = 1000 = (10)^3 \Rightarrow h = 10$ cm.

$\therefore$ $AC = OA - OC = (30 - 10)$ cm = 20 cm.

Hence, the section has been made at a height of 20 cm from the base.

EXERCISE 15C

NOTE Use $\pi = \frac{22}{7}$, unless stated otherwise.

1. Find the curved surface area of a cone with base radius 5.25 cm and slant height 10 cm.
2. Find the total surface area of a cone, if its slant height is 21 m and diameter of its base is 24 m.
3. A joker's cap is in the form of a right circular cone of base radius 7 cm and height 24 cm. Find the area of the sheet required to make 10 such caps.
4. The curved surface area of a cone is 308 cm^2 and its slant height is 14 cm. Find the radius of the base and total surface area of the cone.
5. The slant height and base diameter of a conical tomb are 25 m and 14 m respectively. Find the cost of whitewashing its curved surface at the rate of ₹ 12 per m^2.
6. A conical tent is 10 m high and the radius of its base is 24 m. Find the slant height of the tent. If the cost of 1 m^2 canvas is ₹ 70, find the cost of canvas required to make the tent.
7. A bus stop is barricaded from the remaining part of the road by using 50 hollow cones made of recycled cardboard. Each one has a base diameter of 40 cm and height 1 m. If the outer side of each of the cones is to be painted and the cost of painting is ₹ 25 per m^2, what will be the cost of painting all these cones? (Use $\pi = 3.14$ and $\sqrt{1.04} = 1.02$.)
8. Find the volume, curved surface area and the total surface area of a cone having base radius 35 cm and height 12 cm.
9. Find the volume, curved surface area and the total surface area of a cone whose height is 6 cm and slant height 10 cm. (Take $\pi = 3.14$.)
10. A conical pit of diameter 3.5 m is 12 m deep. What is its capacity in kilolitres?

    HINT $1 \text{ m}^3 = 1$ kilolitre.
11. A heap of wheat is in the form of a cone of diameter 9 m and height 3.5 m. Find its volume. How much canvas cloth is required to just cover the heap? (Use $\pi = 3.14$.)
12. A man uses a piece of canvas having an area of 551 m^2, to make a conical tent of base radius 7 m. Assuming that all the stitching margins and wastage incurred while cutting, amount to approximately 1 m^2, find the volume of the tent that can be made with it.

13. How many metres of cloth, 2.5 m wide, will be required to make a conical tent whose base radius is 7 m and height 24 m?

14. Two cones have their heights in the ratio 1 : 3 and the radii of their bases in the ratio 3 : 1. Show that their volumes are in the ratio 3 : 1.

15. A cylinder and a cone have equal radii of their bases and equal heights. If their curved surface areas are in the ratio 8 : 5, show that the radius and height of each has the ratio 3 : 4.

16. A right circular cone is 3.6 cm high and the radius of its base is 1.6 cm. It is melted and recast into a right circular cone having base radius 1.2 cm. Find its height.

17. A circus tent is cylindrical to a height of 3 metres and conical above it. If its diameter is 105 m and the slant height of the conical portion is 53 m, calculate the length of the canvas 5 m wide to make the required tent.

18. An iron pillar consists of a cylindrical portion 2.8 m high and 20 cm in diameter and a cone 42 cm high is surmounting it. Find the weight of the pillar, given that 1 cm^3 of iron weighs 7.5 g.

19. From a solid right circular cylinder with height 10 cm and radius of the base 6 cm, a right circular cone of the same height and base is removed. Find the volume of the remaining solid. (Take $\pi = 3.14$.)

20. Water flows at the rate of 10 metres per minute through a cylindrical pipe 5 mm in diameter. How long would it take to fill a conical vessel whose diameter at the surface is 40 cm and depth 24 cm?

21. A cloth having an area of 165 m^2 is shaped into the form of a conical tent of radius 5 m. (i) How many students can sit in the tent if a student, on an average, occupies $\frac{5}{7}$ m^2 on the ground? (ii) Find the volume of the cone.

ANSWERS (EXERCISE 15C)

1. 165 cm^2 **2.** 1244.57 cm^2 **3.** 5500 cm^2 **4.** 7 cm, 462 cm^2

5. ₹ 6600 **6.** 26 m, ₹ 137280 **7.** ₹ 800.70

8. 15400 cm^3, 4070 cm^2, 7920 cm^2 **9.** 401.92 cm^3, 251.2 cm^2, 452.16 cm^2

10. 38.5 kL **11.** 74.1825 m^3, 80.54 m^2 **12.** 1232 m^3

13. 220 m **16.** 6.4 cm **17.** 1947 m **18.** 693 kg

19. 753.6 cm^3 **20.** 51 min 12 s **21.** (i) 110 (ii) 241.7 m^3

HINTS TO SOME SELECTED QUESTIONS

7. $r = \frac{20}{100}$ m $= \frac{1}{5}$ m and $h = 1$ m.

$l^2 = (r^2 + h^2) = \left(\frac{1}{25} + 1\right) = (0.04 + 1) = 1.04$

$\Rightarrow \quad l = \sqrt{1.04}$ m $= 1.02$ m.

Curved surface area of 50 cones

$= (\pi rl \times 50)\text{ m}^2 = \left(3.14 \times \frac{1}{5} \times 1.02 \times 50\right)\text{ m}^2 = \left(\frac{157 \times 51}{2500}\right)\text{ m}^2.$

Cost of painting $= ₹\left(\frac{157 \times 51}{2500} \times 25\right) = ₹\ 800.70.$

9. $r^2 + h^2 = l^2 \;\Rightarrow\; r^2 + 36 = 100 \;\Rightarrow\; r^2 = 64 \;\Rightarrow\; r = 8$ cm.

Volume $= \frac{1}{3}\pi r^2 h = \left(\frac{1}{3} \times 3.14 \times 64 \times 6\right)\text{ cm}^3 = 401.92\text{ cm}^3.$

Curved surface area $= \pi rl = (3.14 \times 8 \times 10)\text{ cm}^2 = 251.2\text{ cm}^2.$

Total surface area $= (\pi rl + \pi r^2) = \pi r(l + r)$

$= (3.14 \times 8 \times 18)\text{ cm}^2 = 452.16\text{ cm}^2.$

10. $r = \frac{7}{4}$ m and $h = 12$ m.

Capacity = its volume $= \frac{1}{3}\pi r^2 h = \left(\frac{1}{3} \times \frac{22}{7} \times \frac{49}{16} \times 12\right)\text{ m}^3$

$= \frac{77}{2}\text{ m}^3 = 38.5\text{ m}^3 = 38.5\text{ kL} \quad [1\text{ m}^3 = 1\text{ kL}]$

11. $r = \frac{9}{2}$ m and $h = \frac{7}{2}$ m.

$\therefore \quad$ volume $= \frac{1}{3}\pi r^2 h = \left(\frac{1}{3} \times 3.14 \times \frac{81}{4} \times \frac{7}{2}\right)\text{ m}^3 = 74.1825\text{ m}^3.$

$l^2 = (r^2 + h^2) = \left(\frac{81}{4} + \frac{49}{4}\right) = \frac{130}{4} \;\Rightarrow\; l = \frac{\sqrt{130}}{2} = \frac{11.4}{2} = 5.7\text{ m}.$

Area of canvas $= \pi rl = \left(3.14 \times \frac{9}{2} \times \frac{57}{10}\right)\text{ m}^2 = 80.54\text{ m}^2.$

12. Area of canvas in making the tent $= (551 - 1)\text{ m}^2 = 550\text{ m}^2.$

Curved surface area of the tent $= 550\text{ m}^2.$

$\therefore \quad \pi rl = 550 \;\Rightarrow\; \frac{22}{7} \times 7 \times l = 550 \;\Rightarrow\; l = 25$ m.

$h^2 = (l^2 - r^2) = (625 - 49) = 576.$

13. $l = \sqrt{(24)^2 + 7^2}$ m $= \sqrt{625}$ m $= 25$ m.

Area of cloth $= \pi rl = \left(\frac{22}{7} \times 7 \times 25\right)\text{ m}^2 = 550\text{ m}^2.$

Length of cloth $= \frac{\text{area}}{\text{width}} = \left(\frac{550}{2.5}\right)\text{ m} = 220\text{ m}.$

14. Let their heights be h and $3h$ and their radii be $3r$ and r.

Then, $V_1 = \frac{1}{3}\pi(3r)^2 \times h$ and $V_2 = \frac{1}{3}\pi r^2 \times (3h)$

$\Rightarrow \quad \frac{V_1}{V_2} = \frac{3}{1} \Rightarrow V_1 : V_2 = 3 : 1.$

15. Let their curved surfaces be $8x$ and $5x$. Then,

$2\pi rh = 8x$ and $\pi r\sqrt{h^2 + r^2} = 5x$

$\Rightarrow \quad 4\pi^2 r^2 h^2 = 64x^2$ and $\pi^2 r^2 (h^2 + r^2) = 25x^2$

$\Rightarrow \quad \frac{4\pi^2 r^2 h^2}{\pi^2 r^2 (h^2 + r^2)} = \frac{64}{25} \Rightarrow 9h^2 = 16r^2 \Rightarrow \frac{r}{h} = \frac{3}{4}\cdot$

16. $\frac{1}{3}\pi \times 1.6 \times 1.6 \times 3.6 = \frac{1}{3}\pi \times 1.2 \times 1.2 \times h$. Find h.

17. Area of canvas $= (2\pi rH + \pi Rl)$ sq units

$$= \left[\left(2 \times \frac{22}{7} \times \frac{105}{2} \times 3\right) + \left(\frac{22}{7} \times \frac{105}{2} \times 53\right)\right] \text{m}^2.$$

Length of canvas $= \left(\frac{\text{area}}{\text{width}}\right)$ m.

18. Volume of the pillar $= \left(\pi r^2 h + \frac{1}{3}\pi r^2 H\right) \text{cm}^3$

$$= \pi r^2 \left(h + \frac{1}{3}H\right) \text{cm}^3$$

$$= \frac{22}{7} \times 10 \times 10 \left(280 + \frac{1}{3} \times 42\right) \text{cm}^3.$$

Weight of the pillar $= \left(\frac{92400 \times 7.5}{1000}\right)$ kg = 693 kg.

19. Volume of the remaining solid

$$= (\pi \times 6 \times 6 \times 10) \text{ cm}^3 - \left(\frac{1}{3}\pi \times 6 \times 6 \times 10\right) \text{cm}^3$$

$$= \frac{2}{3}\pi \times 6 \times 6 \times 10 \text{ cm}^3 = \left(\frac{2}{3} \times 3.14 \times 360\right) \text{cm}^3 = 753.6 \text{ cm}^3.$$

20. Volume that flows in 1 min $= [x \times (0.25)^2 \times 1000]$ cm^3.

Volume of the conical vessel $= \left[\frac{1}{3}\pi \times (20)^2 \times 24\right] \text{cm}^3.$

Required time $= \left[\frac{\frac{1}{3}\pi \times (20)^2 \times 24}{\pi \times (0.25)^2 \times 1000}\right]$ min = 51 min 12 s.

21. (i) Area of the floor of the tent $= \pi r^2 = \left(\frac{22}{7} \times 5 \times 5\right) \text{m}^2.$

Area occupied by each student $= \frac{5}{7}$ m^2.

Required number of students $= \frac{\left(\frac{22}{7} \times 5 \times 5\right)}{\left(\frac{5}{7}\right)} = 110.$

(ii) Curved surface area of the tent = area of cloth = 165 m^2.

$\pi rl = 165 \Rightarrow \frac{22}{7} \times 5 \times l = 165 \Rightarrow l = \left(\frac{165 \times 7}{22 \times 5}\right) \text{m} = \frac{21}{2}$ m.

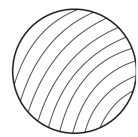
$$h^2 = (l^2 - r^2) = \left\{\left(\frac{21}{2}\right)^2 - 5^2\right\} = \frac{341}{4} = 85.25.$$

$\therefore\quad h = \sqrt{85.25} = 9.23$ m.

Volume of the tent $= \frac{1}{3}\pi r^2 h = \left(\frac{1}{3} \times \frac{22}{7} \times 5 \times 5 \times 9.23\right)$ m^3 = 241.7 m^3.

SPHERE

Objects like *football, volleyball, throwball,* etc., are said to have the shape of a sphere.



Sphere

In geometry, *the solid generated by revolving a circular lamina about any of its diameters is called a sphere.*

The centre and radius of this circle are called respectively the centre and the radius of the sphere.

Formulae:

For a solid sphere of radius r units, we have:

(i) *Volume of the sphere* $= \left(\frac{4}{3}\pi r^3\right)$ *cubic units.*

(ii) *Surface area of the sphere* $= (4\pi r^2)$ *square units.*

SPHERICAL SHELL

SPHERICAL SHELL *The solid enclosed between two concentric spheres is called a spherical shell.*

For a spherical shell with external radius R and internal radius r, we have

volume of the spherical shell $= \frac{4}{3}\pi(R^3 - r^3)$ *cubic units.*

HEMISPHERE

HEMISPHERE *When a plane through the centre of a sphere cuts it into two equal parts, then each part is called a hemisphere.*

For a hemisphere of radius r, we have:

(i) *Volume of the hemisphere* $= \left(\frac{2}{3}\pi r^3\right)$ *cubic units.*

(ii) *Curved surface area of the hemisphere* $= (2\pi r^2)$ *square units.*

(iii) *Total surface area of the hemisphere* $= (3\pi r^2)$ *square units.*

SOLVED EXAMPLES

EXAMPLE 1 *Find the volume and surface area of a sphere of radius* 21 cm.

SOLUTION Radius of the sphere = 21 cm.

Volume of the sphere = $\left(\frac{4}{3}\pi r^3\right)$ cubic units

$$= \left(\frac{4}{3} \times \frac{22}{7} \times 21 \times 21 \times 21\right) \text{cm}^3$$

$$= 38808 \text{ cm}^3.$$

Surface area of the sphere = $(4\pi r^2)$ sq units

$$= \left(4 \times \frac{22}{7} \times 21 \times 21\right) \text{cm}^2 = 5544 \text{ cm}^2.$$

EXAMPLE 2 *Find the surface area of a sphere whose volume is* 4851 cm^3.

SOLUTION Let the radius of the sphere be r cm.

Then, its volume = $\left(\frac{4}{3}\pi r^3\right) \text{cm}^3$.

$$\therefore \quad \frac{4}{3}\pi r^3 = 4851 \Rightarrow \frac{4}{3} \times \frac{22}{7} \times r^3 = 4851$$

$$\Rightarrow r^3 = \left(4851 \times \frac{3}{4} \times \frac{7}{22}\right) = \left(\frac{441 \times 21}{8}\right) = \left(\frac{21}{2}\right)^3$$

$$\Rightarrow r = \frac{21}{2} = 10.5.$$

Thus, the radius of the sphere is 10.5 cm.

Surface area of the sphere = $(4\pi r^2)$ sq units

$$= \left(4 \times \frac{22}{7} \times \frac{21}{2} \times \frac{21}{2}\right) \text{cm}^2$$

$$= 1386 \text{ cm}^2.$$

Hence, the surface area of the given sphere is 1386 cm^2.

EXAMPLE 3 *The surface area of a sphere is* 346.5 cm^2. *Find its radius and hence its volume.*

SOLUTION Let the radius of the sphere be r cm.

Then, its surface area = $(4\pi r^2) \text{ cm}^2$.

$$\therefore \quad 4\pi r^2 = 346.5 \Rightarrow 4 \times \frac{22}{7} \times r^2 = 346.5$$

$$\Rightarrow r^2 = \left(346.5 \times \frac{7}{88}\right) = \left(\frac{3465 \times 7}{10 \times 88}\right) = \frac{441}{16}$$

$$\Rightarrow r = \frac{21}{4} = 5.25.$$

Thus, the radius of the sphere is 5.25 cm.

$$\text{Volume of the sphere} = \left(\frac{4}{3}\pi r^3\right) \text{cm}^2$$
$$= \left(\frac{4}{3} \times \frac{22}{7} \times 5.25 \times 5.25 \times 5.25\right) \text{cm}^3$$
$$= \left(\frac{4}{3} \times \frac{22}{7} \times \frac{525}{100} \times \frac{525}{100} \times \frac{525}{100}\right) \text{cm}^3$$
$$= \frac{4851}{8} \text{cm}^3 = 606.375 \text{ cm}^3.$$

Hence, the volume of the sphere is 606.375 cm^3.

EXAMPLE 4 *How many spherical bullets can be made out of a solid cube of lead whose edge measures* 44 cm, *each bullet being* 4 cm *in diameter?*

SOLUTION Length of each edge of the cube = 44 cm.

Volume of the cube = $(44 \times 44 \times 44)$ cm^3.

Radius of each bullet, $r = 2$ cm.

$$\text{Volume of each bullet} = \left(\frac{4}{3}\pi r^3\right) \text{cm}^3$$
$$= \left(\frac{4}{3} \times \frac{22}{7} \times 2 \times 2 \times 2\right) \text{cm}^3 = \frac{704}{21} \text{cm}^3.$$

$$\text{Number of bullets formed} = \frac{\text{volume of the cube in cm}^3}{\text{volume of each bullet in cm}^3}$$
$$= \left(44 \times 44 \times 44 \times \frac{21}{704}\right) = 2541.$$

Hence, the number of bullets formed is 2541.

EXAMPLE 5 *The diameter of a copper sphere is* 6 cm. *The sphere is melted and drawn into a long wire of uniform circular cross section. If the length of the wire is* 36 cm, *find its radius.*

SOLUTION Radius of the sphere, $R = 3$ cm.

$$\text{Volume of the sphere} = \left(\frac{4}{3}\pi R^3\right) \text{cm}^3$$
$$= \left(\frac{4}{3}\pi \times 3 \times 3 \times 3\right) \text{cm}^3 = (36\pi) \text{ cm}^3.$$

Let the radius of the wire formed be r cm.

Then, volume of the wire = $(\pi r^2 \times h)$ cm^3 = $(\pi r^2 \times 36)$ cm^3.

$\therefore \quad 36\pi r^2 = 36\pi \Rightarrow r^2 = 1 \Rightarrow r = 1.$

Hence, radius of the wire is 1 cm.

EXAMPLE 6 *A metallic sphere of radius* 21 cm *is dropped into a cylindrical vessel, which is partly filled with water. The diameter of the vessel is*

1.68 m. *If the sphere is completely submerged, find by how much the surface of water will rise.*

SOLUTION Radius of the sphere, $r = 21$ cm.

Volume of the sphere $= \left(\frac{4}{3}\pi r^3\right)$ cm^3

$$= \left(\frac{4}{3}\pi \times 21 \times 21 \times 21\right) \text{ cm}^3 = (12348\pi) \text{ cm}^3.$$

Volume of water displaced by the sphere = (12348π) cm^3.

Suppose that the water rises by h cm. Then,

volume of water displaced = volume of cylinder of radius 84 cm and height h cm

$$\therefore \quad \pi \times 84 \times 84 \times h = 12348\pi \Rightarrow h = \left(\frac{12348}{84 \times 84}\right) = \frac{7}{4} \text{ cm}$$

$$= 1.75 \text{ cm}.$$

Hence, the surface of water will rise by 1.75 cm.

EXAMPLE 7 *A hollow sphere of external and internal diameters* 8 cm *and* 4 cm *respectively is melted into a cone of base diameter* 8 cm. *Find the height of the cone.*

SOLUTION External radius of the sphere = 4 cm.

Internal radius of the sphere = 2 cm.

Volume of the material of the sphere $= \frac{4}{3}\pi(4^3 - 2^3)$ cm^3

$$= \left(\frac{224\pi}{3}\right) \text{ cm}^3.$$

Radius of the resulting cone, $r = 4$ cm.

Let the height of this cone formed be h cm.

Volume of the cone $= \left(\frac{1}{3}\pi r^2 h\right)$ cm^3

$$= \left(\frac{1}{3}\pi \times 4 \times 4 \times h\right) \text{ cm}^3 = \left(\frac{16\pi h}{3}\right) \text{ cm}^3.$$

$$\therefore \quad \frac{16\pi h}{3} = \frac{224\pi}{3} \Rightarrow h = \left(\frac{223}{3} \times \frac{3}{16}\right) = 14.$$

Hence, the height of the cone is 14 cm.

EXAMPLE 8 *Find the volume, curved surface area and the total surface area of a hemisphere of diameter* 7 cm.

SOLUTION Radius of the hemisphere, $r = 3.5$ cm.

Volume of the hemisphere $= \left(\frac{2}{3}\pi r^3\right)$ cm^3

$$= \left(\frac{2}{3} \times \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2} \times \frac{7}{2}\right) \text{cm}^3$$

$$= \frac{539}{6} \text{ cm}^3 = 89.83 \text{ cm}^3.$$

Curved surface area of the hemisphere = $(2\pi r^2)$ cm^2

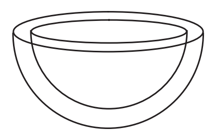
$$= \left(2 \times \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2}\right) \text{cm}^2 = 77 \text{ cm}^2.$$

Total surface area of the hemisphere = $(3\pi r^2)$ cm^2

$$= \left(3 \times \frac{22}{7} \times \frac{7}{2} \times \frac{7}{2}\right) \text{cm}^2 = \frac{231}{2} \text{ cm}^2 = 115.5 \text{ cm}^2.$$

EXAMPLE 9 *The internal and external diameters of a hollow hemispherical vessel are* 20 cm *and* 28 cm *respectively. Find the cost of painting the vessel all over at* 35 *paise per* cm^2.

SOLUTION Outer radius of the vessel, $R = 14$ cm.



Inner radius of the vessel, $r = 10$ cm.

Area of the outer surface = $(2\pi R^2)$ sq units

$$= (2\pi \times 14 \times 14) \text{ cm}^2 = (392\pi) \text{ cm}^2.$$

Area of the inner surface = $(2\pi r^2)$ sq units

$$= (2\pi \times 10 \times 10) \text{ cm}^2 = (200\pi) \text{ cm}^2.$$

Area of the ring at the top = $\pi(R^2 - r^2)$ sq units

$$= \pi[(14)^2 - (10)^2] \text{ cm}^2$$

$$= \pi(14 + 10)(14 - 10) \text{ cm}^2 = (96\pi) \text{ cm}^2.$$

Total area to be painted = $(392\pi + 200\pi + 96\pi)$ cm^2 = (688π) cm^2.

Cost of painting = ₹ $\left(688\pi \times \frac{35}{100}\right)$ = ₹ $\left(688 \times \frac{22}{7} \times \frac{35}{100}\right)$

= ₹ $\frac{3784}{5}$ = ₹ 756.80.

SOME MORE EXAMPLES

EXAMPLE 10 *On increasing the radius of a sphere by* 10%, *find the percentage increase in its volume.*

SOLUTION Let the original radius of the sphere be r units.

Then, its original volume = $\left(\frac{4}{3}\pi r^3\right)$ cubic units.

Radius of the new sphere = (110% of r) units

$$= \left(\frac{110}{100} \times r\right) \text{units} = \left(\frac{11r}{10}\right) \text{units}.$$

Volume of the new sphere $= \left\{\frac{4}{3}\pi \times \left(\frac{11r}{10}\right)^3\right\}$ cubic units

$= \left(\frac{4}{3}\pi r^3 \times \frac{1331}{1000}\right)$ cubic units.

Increase in volume $= \left\{\left(\frac{4}{3}\pi r^3 \times \frac{1331}{1000}\right) - \left(\frac{4}{3}\pi r^3\right)\right\}$ cubic units

$= \left\{\frac{4}{3}\pi r^3 \times \left(\frac{1331}{1000} - 1\right)\right\}$ cubic units

$= \left\{\frac{4}{3}\pi r^3 \times \frac{(1331 - 1000)}{1000}\right\}$ cubic units

$= \left(\frac{4}{3}\pi r^3 \times \frac{331}{1000}\right)$ cubic units.

Percentage increase in volume

$$= \left\{\frac{\frac{4}{3}\pi r^3 \times \frac{331}{1000}}{\frac{4}{3}\pi r^3} \times 100\right\}\% = \left(\frac{331}{1000} \times 100\right)\% = 33.1\%.$$

EXAMPLE 11 *The surface area of a sphere of radius* 5 cm *is five times the area of the curved surface of a cone of radius* 4 cm. *Find the height and the volume of the cone. (Given* $\pi = 3.14$.*)*

SOLUTION Surface area of the given sphere

$$= 4\pi R^2 = (4\pi \times 5 \times 5)\text{ cm}^2 = (100\pi)\text{ cm}^2.$$

Radius of the base of the cone, $r = 4$ cm.

Let the slant height of the cone be l cm. Then,

curved surface area of the given cone

$$= (\pi r l) = (\pi \times 4 \times l)\text{ cm}^2 = (4\pi l)\text{ cm}^2.$$

$\therefore \quad 100\pi = 5 \times (4\pi l) \Rightarrow l = 5$ cm.

Let the height of the cone be h cm. Then,

$$l^2 = h^2 + r^2 \Rightarrow h^2 = (l^2 - r^2) = (5^2 - 4^2) = 9 \Rightarrow h = 3\text{ cm}.$$

Volume of the cone $= \frac{1}{3}\pi r^2 h$

$= \left(\frac{1}{3} \times 3.14 \times 4 \times 4 \times 3\right)\text{ cm}^3 = 50.24\text{ cm}^3.$

Hence, the height of the cone is 3 cm and its volume is 50.24 cm^3.

EXAMPLE 12 *The volumes of two spheres are in the ratio* 64 : 27. *Find the ratio of their surface areas.*

SOLUTION Let the radii of the given spheres be R_1 and R_2 respectively and let their volumes be V_1 and V_2 respectively. Then,

$$\frac{V_1}{V_2} = \frac{64}{27} \Rightarrow \frac{\frac{4}{3}\pi R_1^3}{\frac{4}{3}\pi R_2^3} = \frac{64}{27}$$

$$\Rightarrow \frac{R_1^3}{R_2^3} = \frac{64}{27} \Rightarrow \left(\frac{R_1}{R_2}\right)^3 = \left(\frac{4}{3}\right)^3 \Rightarrow \frac{R_1}{R_2} = \frac{4}{3}. \qquad \ldots \text{(i)}$$

Let the surface areas of the given spheres be S_1 and S_2 respectively.

Then, $\frac{S_1}{S_2} = \frac{4\pi R_1^2}{4\pi R_2^2} = \left(\frac{R_1}{R_2}\right)^2 = \frac{16}{9}$ [using (i)].

Hence, $S_1 : S_2 = 16 : 9$.

EXAMPLE 13 *A sphere and a right circular cylinder of the same radius have equal volumes. By what percentage does the diameter of the cylinder exceed its height?*

SOLUTION Let the given sphere and cylinder have same radius r and let the height of the cylinder be h. Then,

volume of sphere = volume of cylinder

$$\Rightarrow \quad \frac{4}{3}\pi r^3 = \pi r^2 h$$

$$\Rightarrow \quad r = \frac{3h}{4} \Rightarrow 2r = \frac{3h}{2} \Rightarrow d = \frac{3h}{2},$$

where d is the diameter of the cylinder.

Thus, the diameter of the cylinder is $\frac{3h}{2}$.

Excess of diameter of cylinder over its height $= \left(\frac{3h}{2} - h\right) = \frac{h}{2}$.

Excess % $= \left(\frac{h}{2} \times \frac{1}{h} \times 100\right)\% = 50\%$.

EXAMPLE 14 *The water for a factory is stored in a hemispherical tank whose internal diameter is* 14 m. *The tank contains* 50 *kilolitres of water. Water is pumped into the tank to fill it to its capacity. Calculate the volume of water pumped into the tank.*

SOLUTION Radius of the hemispherical tank $= \frac{14}{2}$ m = 7 m.

Capacity of the full tank = volume of the tank

$$= \frac{2}{3}\pi r^3$$

$$= \left(\frac{2}{3} \times \frac{22}{7} \times 7 \times 7 \times 7\right) \text{m}^3 = \frac{2156}{3} \text{ m}^3.$$

Volume of water already contained in the tank = 50 kL = 50 m^3.

Volume of water pumped into the tank $= \left(\frac{2156}{3} - 50\right) \text{m}^3$

$$= \frac{2006}{3} \text{ m}^3 = 668.67 \text{ m}^3.$$

EXAMPLE 15 *A shopkeeper has one spherical laddoo of radius* 5 cm. *With the same amount of material, how many laddoos of radius* 2.5 cm *can be made?*

SOLUTION Volume of one laddoo of radius 5 cm

$$= \left\{\frac{4}{3}\pi \times (5)^3\right\} \text{cm}^3 = \left(\frac{500\pi}{3}\right) \text{cm}^3.$$

From this laddoo, let n laddoos of radius 2.5 cm can be made.

Volume of one laddoo of radius 2.5 cm

$$= \left\{\frac{4}{3}\pi \times \left(\frac{5}{2}\right)^3\right\} \text{cm}^3 = \left(\frac{125\pi}{6}\right) \text{cm}^3.$$

$$\therefore \quad n = \frac{\text{volume of one laddoo of radius 5 cm}}{\text{volume of one laddoo of radius 2.5 cm}}$$

$$= \left(\frac{500\pi}{3} \times \frac{6}{125\pi}\right) = 8.$$

Hence, the required number of laddoos is 8.

EXAMPLE 16 *A shot-put is a metallic sphere of radius* 4.9 cm. *If the density of metal is* 7.8 g *per* cm^3, *find the mass of the shot-put.*

SOLUTION A shot-put is a solid sphere of metal and its mass is equal to the product of its volume and density.

Volume of the shot-put

$$= \frac{4}{3}\pi r^3$$

$$= \left(\frac{4}{3} \times \frac{22}{7} \times \frac{49}{10} \times \frac{49}{10} \times \frac{49}{10}\right) \text{cm}^3$$

$$= \frac{1479016}{3000} \text{ cm}^3 = 493 \text{ cm}^3 \text{ (nearly)}.$$

Mass of 1 cm^3 of the shot-put = 7.8 g.

Mass of the shot-put

$$= (7.8 \times 493) \text{ g}$$

$$= \left(\frac{78}{10} \times \frac{493}{1000}\right) \text{kg} = \frac{38454}{10000} \text{ kg} = 3.85 \text{ kg (nearly)}.$$

Hence, the mass of the given shot-put is 3.85 kg.

EXAMPLE 17 *Two solid spheres made of the same metal have weights* 5920 g *and* 740 g *respectively. Determine the radius of the larger sphere, if the diameter of the smaller one is* 5 cm.

SOLUTION Let d g/cm^3 be the density of the given metal and let the radius of the larger sphere be R cm.

Radius of smaller sphere = 2.5 cm.

Now, $\dfrac{\text{volume of larger sphere} \times d}{\text{volume of smaller sphere} \times d} = \dfrac{5920}{740}$

$$\Rightarrow \quad \frac{\frac{4}{3}\pi R^3 \times d}{\frac{4}{3}\pi\left(\frac{5}{2}\right)^3 \times d} = 8 \Rightarrow R^3 = 8 \times \left(\frac{5}{2}\right)^3 = \frac{8 \times 125}{8} = 125$$

$$\Rightarrow \quad R^3 = 5^3 \Rightarrow R = 5 \text{ cm.}$$

Hence, the radius of the larger sphere is 5 cm.

EXAMPLE 18 *A hemispherical tank is made up of an iron sheet* 1 cm *thick. If the inner radius of the tank is* 1 m *then find the volume of iron used in the tank.*

SOLUTION Inner radius of the tank = 1 m = 100 cm.

Outer radius of the tank = (100 + 1) cm = 101 cm.

Volume of iron used in the hemispherical tank

$$= \frac{2}{3}\pi \times \{(101)^3 - (100)^3\} \text{ cm}^3$$

$$= \frac{2}{3} \times \frac{22}{7} \times (1030301 - 1000000) \text{ cm}^3$$

$$= \left(\frac{2}{3} \times \frac{22}{7} \times 30301\right) \text{ cm}^3 = \frac{1333244}{21} \text{ cm}^3$$

$$= 63487.81 \text{ cm}^3.$$

Hence, the volume of iron used in the tank is 63487.81 cm^3.

EXERCISE 15D

NOTE Take $\pi = \dfrac{22}{7}$, unless stated otherwise.

1. Find the volume and surface area of a sphere whose radius is:

(i) 3.5 cm (ii) 4.2 cm (iii) 5 m

2. The volume of a sphere is 38808 cm^3. Find its radius and hence its surface area.

3. Find the surface area of a sphere whose volume is $606.375\ m^3$.

4. Find the volume of a sphere whose surface area is $154\ cm^2$.

5. The surface area of sphere is $(576\pi)\ cm^2$. Find its volume.

6. How many lead shots, each 3 mm in diameter, can be made from a cuboid with dimensions (12 cm $\times$ 11 cm $\times$ 9 cm)?

7. How many lead balls, each of radius 1 cm, can be made from a sphere of radius 8 cm?

8. A solid sphere of radius 3 cm is melted and then cast into smaller spherical balls, each of diameter 0.6 cm. Find the number of small balls thus obtained.

9. A metallic sphere of radius 10.5 cm is melted and then recast into smaller cones, each of radius 3.5 cm and height 3 cm. How many cones are obtained?

10. How many spheres 12 cm in diameter can be made from a metallic cylinder of diameter 8 cm and height 90 cm?

11. The diameter of a sphere is 6 cm. It is melted and drawn into a wire of diameter 2 mm. Find the length of the wire.

12. The diameter of a copper sphere is 18 cm. It is melted and drawn into a long wire of uniform cross section. If the length of the wire is 108 m, find its diameter.

13. A sphere of diameter 15.6 cm is melted and cast into a right circular cone of height 31.2 cm. Find the diameter of the base of the cone.

14. A spherical cannonball 28 cm in diameter is melted and cast into a right circular cone mould, whose base is 35 cm in diameter. Find the height of the cone.

15. A spherical ball of radius 3 cm is melted and recast into three spherical balls. The radii of two of these balls are 1.5 cm and 2 cm. Find the radius of the third ball.

16. The radii of two spheres are in the ratio 1 : 2. Find the ratio of their surface areas.

17. The surface areas of two spheres are in the ratio 1 : 4. Find the ratio of their volumes.

18. A cylindrical tub of radius 12 cm contains water to a depth of 20 cm. A spherical iron ball is dropped into the tub and thus the level of water is raised by 6.75 cm. What is the radius of the ball?

19. A cylindrical bucket with base radius 15 cm is filled with water up to a height of 20 cm. A heavy iron spherical ball of radius 9 cm is dropped into the bucket to submerge completely in the water. Find the increase in the level of water.

20. The outer diameter of a spherical shell is 12 cm and its inner diameter is 8 cm. Find the volume of metal contained in the shell. Also, find its outer surface area.

21. A hollow spherical shell is made of a metal of density 4.5 g per cm^3. If its internal and external radii are 8 cm and 9 cm respectively, find the weight of the shell.

22. A hemisphere of lead of radius 9 cm is cast into a right circular cone of height 72 cm. Find the radius of the base of the cone.

23. A hemispherical bowl of internal radius 9 cm contains a liquid. This liquid is to be filled into cylindrical shaped small bottles of diameter 3 cm and height 4 cm. How many bottles are required to empty the bowl?

24. A hemispherical bowl is made of steel 0.5 cm thick. The inside radius of the bowl is 4 cm. Find the volume of steel used in making the bowl.

25. A hemispherical bowl is made of steel, 0.25 cm thick. The inner radius of the bowl is 5 cm. Find the outer curved surface area of the bowl.

26. A hemispherical bowl made of brass has inner diameter 10.5 cm. Find the cost of tin-plating it on the inside at the rate of ₹ 32 per 100 cm^2.

27. The diameter of the moon is approximately one fourth of the diameter of the earth. What fraction of the volume of the earth is the volume of the moon?

28. Volume and surface area of a solid hemisphere are numerically equal. What is the diameter of the hemisphere? [CBSE 2017]

ANSWERS (EXERCISE 15D)

1. (i) 179.67 cm^3, 154 cm^2 (ii) 310.464 cm^3, 221.76 cm^2 (iii) 523.81 m^3, 314.28 m^2

2. 21 cm, 5544 cm^2	**3.** 346.5 m^2	**4.** 179.67 cm^3	**5.** (2304π) cm^3
6. 84000	**7.** 512	**8.** 1000	**9.** 126
10. 5	**11.** 36 m	**12.** 0.6 cm	**13.** 15.6 cm
14. 35.84 cm	**15.** 2.5 cm	**16.** 1 : 4	**17.** 1 : 8
18. 9 cm	**19.** 4.32 cm	**20.** 636.95 cm^3, 452.57 cm^2	
21. 4.092 kg	**22.** 4.5 cm	**23.** 54	**24.** 56.83 cm^3
25. 173.25 cm^2	**26.** ₹ 55.44	**27.** $\frac{1}{64}$	**28.** 9 units

HINTS TO SOME SELECTED QUESTIONS

6. Volume of 1 lead shot $=\left(\frac{4}{3}\times\frac{22}{7}\times\frac{3}{20}\times\frac{3}{20}\times\frac{3}{20}\right)cm^3=\frac{99}{7000}cm^3$.

Volume of cuboid = $(12 \times 11 \times 9)\ \text{cm}^3$.

Number of lead shots = $\dfrac{\text{volume of the cuboid}}{\text{volume of 1 lead shot}}$.

8. Volume of solid sphere = $\left(\frac{4}{3}\pi \times 3 \times 3 \times 3\right) \text{cm}^3 = (36\pi)\ \text{cm}^3$.

Volume of 1 small ball = $\left(\frac{4}{3}\pi \times \frac{3}{10} \times \frac{3}{10} \times \frac{3}{10}\right) \text{cm}^3 = \left(\frac{36\pi}{1000}\right) \text{cm}^3$.

Number of small balls = $\dfrac{\text{volume of solid sphere}}{\text{volume of 1 small ball}}$.

9. Volume of metallic sphere = $\left(\frac{4}{3}\pi \times \frac{21}{2} \times \frac{21}{2} \times \frac{21}{2}\right) \text{cm}^3 = \left(\frac{3087\pi}{2}\right) \text{cm}^3$.

Volume of 1 smaller cone = $\left(\frac{1}{3}\pi \times \frac{7}{2} \times \frac{7}{2} \times 3\right) \text{cm}^3 = \left(\frac{49\pi}{4}\right) \text{cm}^3$.

Number of cones = $\dfrac{\text{volume of sphere}}{\text{volume of 1 cone}}$.

11. $\frac{4}{3}\pi \times 3 \times 3 \times 3 = \pi \times \frac{1}{10} \times \frac{1}{10} \times h \Rightarrow h = \frac{36 \times 100}{100}\ \text{m} = 36\ \text{m}$.

13. $\frac{4}{3}\pi \times 7.8 \times 7.8 \times 7.8 = \frac{1}{3} \times \pi \times r^2 \times 31.2 \Rightarrow r^2 = \frac{4 \times 7.8 \times 7.8 \times 7.8}{31.2}$

$\therefore \quad r = 7.8 \Rightarrow d = 2r = 15.6\ \text{cm}$.

15. $\frac{4}{3}\pi \times 3 \times 3 \times 3 = \frac{4}{3}\pi \times \frac{3}{2} \times \frac{3}{2} \times \frac{3}{2} + \frac{4}{3}\pi \times 2 \times 2 \times 2 + \frac{4}{3}\pi r^3$

$\Rightarrow \quad \frac{27}{8} + 8 + r^3 = 27 \Rightarrow r^3 = \left(27 - \frac{91}{8}\right) = \frac{(216 - 91)}{8} = \frac{125}{8} = \left(\frac{5}{2}\right)^3$

$\therefore \quad r = \frac{5}{2}\ \text{cm} = 2.5\ \text{cm}$.

16. Let the radii of the two spheres be x and $2x$ respectively.

Then, $\frac{S_1}{S_2} = \frac{4\pi x^2}{4\pi (2x)^2} = \frac{x^2}{4x^2} = \frac{1}{4} \Rightarrow S_1 : S_2 = 1 : 4$.

17. $\frac{S_1}{S_2} = \frac{1}{4} \Rightarrow \frac{4\pi R_1^2}{4\pi R_2^2} = \frac{1}{4} \Rightarrow \frac{R_1^2}{R_2^2} = \frac{1}{4} \Rightarrow \left(\frac{R_1}{R_2}\right)^2 = \left(\frac{1}{2}\right)^2 \Rightarrow \frac{R_1}{R_2} = \frac{1}{2}$.

$$\frac{V_1}{V_2} = \frac{\frac{4}{3}\pi (R_1^3)}{\frac{4}{3}\pi (R_2^3)} = \frac{R_1^3}{R_2^3} = \left(\frac{R_1}{R_2}\right)^3 = \left(\frac{1}{2}\right)^3 = \frac{1}{8} \Rightarrow V_1 : V_2 = 1 : 8.$$

18. Volume of water displaced = $\{\pi \times (12)^2 \times 6.75\}\ \text{cm}^3$

Volume of the ball = $\frac{4}{3}\pi r^3$.

$\therefore \quad \frac{4}{3}\pi r^3 = \pi \times 144 \times 6.75 \Rightarrow r^3 = \frac{144 \times 6.75 \times 3}{4} = 729 = (9)^3 \Rightarrow r = 9\ \text{cm}$.

$\therefore$ radius of the ball is 9 cm.

19. Let the increase in level be h cm.

Increase in volume of water = volume of the ball.

$\therefore \quad \pi \times 15 \times 15 \times h = \frac{4}{3}\pi \times 9 \times 9 \times 9$. Find h.

20. Volume of iron contained in the shell

$$= \frac{4}{3}\pi \times \{(6)^3 - (4)^3\} \text{ cm}^3 = \left(\frac{4}{3} \times \frac{22}{7} \times 152\right) \text{cm}^3 = 636.95 \text{ cm}^3.$$

Outer surface area $= (4\pi \times 6 \times 6) \text{ cm}^2 = \left(4 \times \frac{22}{7} \times 36\right) \text{cm}^2 = 452.57 \text{ cm}^2$.

21. Weight of the shell $= \left[\frac{4}{3} \times \frac{22}{7} \times \{(9)^3 - (8)^3\} \times \frac{4.5}{1000}\right]$ kg.

22. $\frac{2}{3}\pi \times (9)^3 = \frac{1}{3}\pi r^2 \times 72 \Rightarrow r^2 = \frac{1458}{72} = 20.25 \Rightarrow r = 4.5$ cm.

23. Number of bottles $= \dfrac{\text{volume of bowl}}{\text{volume of each bottle}} = \dfrac{\left\{\frac{2}{3}\pi \times (9)^3\right\}}{\left\{\pi \times \left(\frac{3}{2}\right)^2 \times 4\right\}} = 54.$

24. Inner radius, $r = 4$ cm, and outer radius, $R = 4.5$ cm.

Volume of steel $= \frac{2}{3}\pi \times [(4.5)^3 - (4)^3] \text{ cm}^3 = 56.83 \text{ cm}^3$.

25. Outer radius $= 5.25 \text{ cm} = \frac{525}{100} \text{ cm} = \frac{21}{4}$ cm.

Outer curved surface area $= \left(2 \times \frac{22}{7} \times \frac{21}{4} \times \frac{21}{4}\right) \text{cm}^2 = 173.25 \text{ cm}^2$.

26. Inner radius $= \frac{21}{4}$ cm.

Inner curved surface area $= \left(2 \times \frac{22}{7} \times \frac{21}{4} \times \frac{21}{4}\right) \text{cm}^2 = 173.25 \text{ cm}^2$.

Cost of tin-plating = ₹ $\left(173.25 \times \frac{32}{100}\right)$ = ₹ 55.44.

27. $d = \frac{1}{4}D \Rightarrow 2r = \frac{1}{4} \times 2R \Rightarrow r = \frac{1}{4}R$.

Volume of moon $= \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\left(\frac{R}{4}\right)^3 = \dfrac{\frac{4}{3}\pi R^3}{64}$.

Volume of earth $= \frac{4}{3}\pi R^3$.

$\therefore$ volume of moon $= \frac{1}{64} \times$ volume of earth.

28. $\frac{2}{3}\pi r^3 = 3\pi r^2 \Rightarrow r = \frac{9}{2} \Rightarrow 2r = 9$ units.

MULTIPLE-CHOICE QUESTIONS (MCQ)

Choose the correct answer in each of the following:

1. The length, breadth and height of a cuboid are 15 cm, 12 cm and 4.5 cm respectively. Its volume is

(a) 243 cm^3 (b) 405 cm^3 (c) 810 cm^3 (d) 603 cm^3

2. A cuboid is 12 cm long, 9 cm broad and 8 cm high. Its total surface area is

(a) 864 cm^2 (b) 552 cm^2 (c) 432 cm^2 (d) 276 cm^2

3. The length, breadth and height of a cuboid are 15 m, 6 m and 5 dm respectively. The lateral surface area of the cuboid is

(a) 45 m^2 (b) 21 m^2 (c) 201 m^2 (d) 90 m^2

4. A beam 9 m long, 40 cm wide and 20 cm high is made up of iron which weighs 50 kg per cubic metre. The weight of the beam is

(a) 27 kg (b) 48 kg (c) 36 kg (d) 56 kg

5. The length of the longest rod that can be placed in a room of dimensions (10 m × 10 m × 5 m) is

(a) 15 m (b) 16 m (c) $10\sqrt{5}$ m (d) 12 m

6. What is the maximum length of a pencil that can be placed in a rectangular box of dimensions (8 cm × 6 cm × 5 cm)? (Given $\sqrt{5} = 2.24$.)

(a) 8 cm (b) 9.5 cm (c) 19 cm (d) 11.2 cm

7. The number of planks of dimensions (4 m × 5 m × 2 m) that can be stored in a pit which is 40 m long, 12 m wide and 16 m deep, is

(a) 190 (b) 192 (c) 184 (d) 180

8. How many planks of dimensions (5 m × 25 cm × 10 cm) can be stored in a pit which is 20 m long, 6 m wide and 50 cm deep?

(a) 480 (b) 450 (c) 320 (d) 360

9. How many bricks will be required to construct a wall 8 m long, 6 m high and 22.5 cm thick if each brick measures (25 cm × 11.25 cm × 6 cm)?

(a) 4800 (b) 5600 (c) 6400 (d) 5200

10. How many persons can be accommodated in a dining hall of dimensions (20 m × 15 m × 4.5 m), assuming that each person requires 5 m^3 of air?

(a) 250 (b) 270 (c) 320 (d) 300

11. A river 1.5 m deep and 30 m wide is flowing at the rate of 3 km per hour. The volume of water that runs into the sea per minute is

(a) 2000 m^3 (b) 2250 m^3 (c) 2500 m^3 (d) 2750 m^3

12. The lateral surface area of a cube is 256 m^2. The volume of the cube is

(a) 64 m^3 (b) 216 m^3 (c) 256 m^3 (d) 512 m^3

13. The total surface area of a cube is 96 cm^2. The volume of the cube is

(a) 8 cm^3 (b) 27 cm^3 (c) 64 cm^3 (d) 512 cm^3

14. The volume of a cube is 512 cm^3. Its total surface area is

(a) 256 cm^2 (b) 384 cm^2 (c) 512 cm^2 (d) 64 cm^2

15. The length of the longest rod that can fit in a cubical vessel of side 10 cm, is

(a) 10 cm (b) 20 cm (c) $10\sqrt{2}$ cm (d) $10\sqrt{3}$ cm

16. If the length of diagonal of a cube is $8\sqrt{3}$ cm then its surface area is

(a) 192 cm^2 (b) 384 cm^2 (c) 512 cm^2 (d) 768 cm^2

17. If each edge of a cube is increased by 50% then the percentage increase in its surface area is

(a) 50% (b) 75% (c) 100% (d) 125%

18. Three cubes of metal with edges 3 cm, 4 cm and 5 cm respectively are melted to form a single cube. The lateral surface area of the new cube formed is

(a) 72 cm^2 (b) 144 cm^2 (c) 128 cm^2 (d) 256 cm^2

19. In a shower, 5 cm of rain falls. What is the volume of water that falls on 2 hectares of ground?

(a) 500 m^3 (b) 750 m^3 (c) 800 m^3 (d) 1000 m^3

20. Two cubes have their volumes in the ratio 1 : 27. The ratio of their surface areas is

(a) 1 : 3 (b) 1 : 8 (c) 1 : 9 (d) 1 : 18

21. If each side of a cube is doubled then its volume

(a) is doubled (b) becomes 4 times

(c) becomes 6 times (d) becomes 8 times

22. The diameter of the base of a cylinder is 6 cm and its height is 14 cm. The volume of the cylinder is

(a) 198 cm^3 (b) 396 cm^3 (c) 495 cm^3 (d) 297 cm^3

23. If the diameter of a cylinder is 28 cm and its height is 20 cm then its curved surface area is

(a) 880 cm^2 (b) 1760 cm^2 (c) 3520 cm^2 (d) 2640 cm^2

24. If the curved surface area of a cylinder is 1760 cm^2 and its base radius is 14 cm then its height is

(a) 10 cm (b) 15 cm (c) 20 cm (d) 40 cm

25. The height of a cylinder is 14 cm and its curved surface area is 264 cm^2. The volume of the cylinder is

(a) 308 cm^3 (b) 396 cm^3 (c) 1232 cm^3 (d) 1848 cm^3

26. The curved surface area of a cylindrical pillar is 264 m^2 and its volume is 924 m^3. The height of the pillar is

(a) 4 m (b) 5 m (c) 6 m (d) 7 m

27. The radii of two cylinders are in the ratio 2 : 3 and their heights are in the ratio 5 : 3. The ratio of their curved surface areas is

(a) 2 : 5 (b) 8 : 7 (c) 10 : 9 (d) 16 : 9

28. The radii of two cylinders are in the ratio 2 : 3 and their heights are in the ratio 5 : 3. The ratio of their volumes is

(a) 27 : 20 (b) 20 : 27 (c) 4 : 9 (d) 9 : 4

29. The ratio between the radius of the base and the height of a cylinder is 2 : 3. If its volume is 1617 cm^3 then its total surface area is

(a) 308 cm^2 (b) 462 cm^2 (c) 540 cm^2 (d) 770 cm^2

30. Two circular cylinders of equal volume have their heights in the ratio 1 : 2. The ratio of their radii is

(a) $1 : \sqrt{2}$ (b) $\sqrt{2} : 1$ (c) 1 : 2 (d) 1 : 4

31. The ratio between the curved surface area and the total surface area of a right circular cylinder is 1 : 2. If the total surface area is 616 cm^2 then the volume of the cylinder is

(a) 1078 cm^3 (b) 1232 cm^3 (c) 1848 cm^3 (d) 924 cm^3

32. In a cylinder, if the radius is halved and the height is doubled then the volume will be

(a) the same (b) doubled (c) halved (d) four times

33. The number of coins 1.5 cm in diameter and 0.2 cm thick to be melted to form a right circular cylinder of height 10 cm and diameter 4.5 cm is

(a) 540 (b) 450 (c) 380 (d) 472

34. The radius of a wire is decreased to one third. If volume remains the same, the length will become

(a) 2 times (b) 3 times (c) 6 times (d) 9 times

35. The diameter of a roller, 1 m long, is 84 cm. If it takes 500 complete revolutions to level a playground, the area of the playground is

(a) 1440 m^2 (b) 1320 m^2 (c) 1260 m^2 (d) 1550 m^2

36. 2.2 dm^3 of lead is to be drawn into a cylindrical wire 0.50 cm in diameter. The length of the wire is

(a) 110 m (b) 112 m (c) 98 m (d) 124 m

37. The lateral surface area of a cylinder is

(a) $\pi r^2 h$ (b) πrh (c) $2\pi rh$ (d) $2\pi r^2$

38. The height of a cone is 24 cm and the diameter of its base is 14 cm. The curved surface area of the cone is

(a) 528 cm^2 (b) 550 cm^2 (c) 616 cm^2 (d) 704 cm^2

39. The volume of a right circular cone of height 12 cm and base radius 6 cm, is

(a) (12π) cm^3 (b) (36π) cm^3 (c) (72π) cm^3 (d) (144π) cm^3

40. How much cloth 2.5 m wide will be required to make a conical tent having base radius 7 m and height 24 m?

(a) 120 m (b) 180 m (c) 220 m (d) 550 m

41. The volume of a cone is 1570 cm^3 and its height is 15 cm. What is the radius of the cone? (Use $\pi = 3.14$.)

(a) 10 cm (b) 9 cm (c) 12 cm (d) 8.5 cm

42. The height of a cone is 21 cm and its slant height is 28 cm. The volume of the cone is

(a) 7356 cm^3 (b) 7546 cm^3 (c) 7506 cm^3 (d) 7564 cm^3

43. The volume of a right circular cone of height 24 cm is 1232 cm^3. Its curved surface area is

(a) 1254 cm^2 (b) 704 cm^2 (c) 550 cm^2 (d) 462 cm^2

44. If the volumes of two cones be in the ratio 1 : 4 and the radii of their bases be in the ratio 4 : 5 then the ratio of their heights is

(a) 1 : 5 (b) 5 : 4 (c) 25 : 16 (d) 25 : 64

45. If the height of a cone is doubled then its volume is increased by

(a) 100% (b) 200% (c) 300% (d) 400%

46. The curved surface area of one cone is twice that of the other while the slant height of the latter is twice that of the former. The ratio of their radii is

(a) 2 : 1 (b) 4 : 1 (c) 8 : 1 (d) 1 : 1

47. The ratio of the volumes of a right circular cylinder and a right circular cone of the same base and the same height will be

(a) 1 : 3 (b) 3 : 1 (c) 4 : 3 (d) 3 : 4

48. A right circular cylinder and a right circular cone have the same radius and the same volume. The ratio of the height of the cylinder to that of the cone is

(a) 3 : 5 (b) 2 : 5 (c) 3 : 1 (d) 1 : 3

49. The radii of the bases of a cylinder and a cone are in the ratio 3 : 4 and their heights are in the ratio 2 : 3. Then, their volumes are in the ratio

(a) 9 : 8 (b) 8 : 9 (c) 3 : 4 (d) 4 : 3

50. If the height and the radius of a cone are doubled, the volume of the cone becomes

(a) 3 times (b) 4 times (c) 6 times (d) 8 times

51. A solid metallic cylinder of base radius 3 cm and height 5 cm is melted to make n solid cones of height 1 cm and base radius 1 mm. The value of n is

(a) 450 (b) 1350 (c) 4500 (d) 13500

52. A conical tent is to accommodate 11 persons such that each person occupies 4 m^2 of space on the ground. They have 220 m^3 of air to breathe. The height of the cone is

(a) 14 m (b) 15 m (c) 16 m (d) 20 m

53. The volume of a sphere of radius $2r$ is

(a) $\frac{32\pi r^3}{3}$ (b) $\frac{16\pi r^3}{3}$ (c) $\frac{8\pi r^3}{3}$ (d) $\frac{64\pi r^3}{3}$

54. The volume of a sphere of radius 10.5 cm is

(a) 9702 cm^3 (b) 4851 cm^3 (c) 19404 cm^3 (d) 14553 cm^3

55. The surface area of a sphere of radius 21 cm is

(a) 2772 cm^2 (b) 1386 cm^2 (c) 4158 cm^2 (d) 5544 cm^2

56. The surface area of a sphere is 1386 cm^2. Its volume is

(a) 1617 cm^3 (b) 3234 cm^3 (c) 4851 cm^3 (d) 9702 cm^3

57. If the surface area of a sphere is (144π) m^2 then its volume is

(a) (288π) m^3 (b) (188π) m^3 (c) (300π) m^3 (d) (316π) m^3

58. The volume of a sphere is 38808 cm^3. Its curved surface area is

(a) 5544 cm^2 (b) 8316 cm^2 (c) 4158 cm^2 (d) 1386 cm^2

59. If the ratio of the volumes of two spheres is 1 : 8 then the ratio of their surface areas is

(a) 1 : 2 (b) 1 : 4 (c) 1 : 8 (d) 1 : 16

60. A solid metal ball of radius 8 cm is melted and cast into smaller balls, each of radius 2 cm. The number of such balls is

(a) 8 (b) 16 (c) 32 (d) 64

61. A cone is 8.4 cm high and the radius of its base is 2.1 cm. It is melted and recast into a sphere. The radius of the sphere is

(a) 4.2 cm (b) 2.1 cm (c) 2.4 cm (d) 1.6 cm

62. A solid lead ball of radius 6 cm is melted and then drawn into a wire of diameter 0.2 cm. The length of wire is

(a) 272 m (b) 288 m (c) 292 m (d) 296 m

63. A metallic sphere of radius 10.5 cm is melted and then recast into small cones, each of radius 3.5 cm and height 3 cm. The number of such cones will be

(a) 21 (b) 63 (c) 126 (d) 130

64. How many lead shots, each 0.3 cm in diameter, can be made from a cuboid of dimensions 9 cm $\times$ 11 cm $\times$ 12 cm?

(a) 7200 (b) 8400 (c) 72000 (d) 84000

65. The diameter of a sphere is 6 cm. It is melted and drawn into a wire of diameter 2 mm. The length of the wire is

(a) 12 m (b) 18 m (c) 36 m (d) 66 m

66. A sphere of diameter 12.6 cm is melted and cast into a right circular cone of height 25.2 cm. The radius of the base of the cone is

(a) 6.3 cm (b) 2.1 cm (c) 6 cm (d) 4 cm

67. A spherical ball of radius 3 cm is melted and recast into three spherical balls. The radii of two of these balls are 1.5 cm and 2 cm. The radius of the third ball is

(a) 1 cm (b) 1.5 cm (c) 2.5 cm (d) 0.5 cm

68. The radius of a hemispherical balloon increases from 6 cm to 12 cm as air is being pumped into it. The ratio of the surface areas of the balloons in two cases is

(a) 1 : 4 (b) 1 : 3 (c) 2 : 3 (d) 1 : 2

69. The volumes of two spheres are in the ratio 64 : 27 and the sum of their radii is 7 cm. The difference in their total surface areas is

(a) 38 cm^2 (b) 58 cm^2 (c) 78 cm^2 (d) 88 cm^2

70. A hemispherical bowl of radius 9 cm contains a liquid. This liquid is to be filled into cylindrical small bottles of diameter 3 cm and height 4 cm. How many bottles will be needed to empty the bowl?

(a) 27 (b) 35 (c) 54 (d) 63

71. A cone and a hemisphere have equal bases and equal volumes. The ratio of their heights is

(a) 1 : 2 (b) 2 : 1 (c) 4 : 1 (d) $\sqrt{2} : 1$

72. A cone, a hemisphere and a cylinder stand on equal bases and have the same height. The ratio of their volumes is

(a) 1 : 2 : 3 (b) 2 : 1 : 3 (c) 2 : 3 : 1 (d) 3 : 2 : 1

73. If the volume and the surface area of a sphere are numerically the same then its radius is

(a) 1 unit (b) 2 units (c) 3 units (d) 4 units

ANSWERS (MCQ)

1. (c)	**2.** (b)	**3.** (b)	**4.** (c)	**5.** (a)	**6.** (d)	**7.** (b)	**8.** (a)
9. (c)	**10.** (b)	**11.** (b)	**12.** (d)	**13.** (c)	**14.** (b)	**15.** (d)	**16.** (b)
17. (d)	**18.** (b)	**19.** (d)	**20.** (c)	**21.** (d)	**22.** (b)	**23.** (b)	**24.** (c)
25. (b)	**26.** (c)	**27.** (c)	**28.** (b)	**29.** (d)	**30.** (b)	**31.** (a)	**32.** (c)
33. (b)	**34.** (d)	**35.** (b)	**36.** (b)	**37.** (c)	**38.** (b)	**39.** (d)	**40.** (c)

41. (a)	**42.** (b)	**43.** (c)	**44.** (d)	**45.** (a)	**46.** (b)	**47.** (b)	**48.** (d)
49. (a)	**50.** (d)	**51.** (d)	**52.** (b)	**53.** (a)	**54.** (b)	**55.** (d)	**56.** (c)
57. (a)	**58.** (a)	**59.** (b)	**60.** (d)	**61.** (b)	**62.** (b)	**63.** (c)	**64.** (d)
65. (c)	**66.** (a)	**67.** (c)	**68.** (a)	**69.** (d)	**70.** (c)	**71.** (b)	**72.** (a)
73. (c)							

HINTS TO SOME SELECTED QUESTIONS

2. Total surface area $= 2(lb + bh + lb)$.

3. Lateral surface area of the cuboid $= [2(l + b) \times h]$

$$= \left\{2(15+6) \times \frac{5}{10}\right\} \text{m}^2 = 21 \text{ m}^2.$$

5. Length of the longest rod = length of the diagonal

$$= \sqrt{l^2 + b^2 + h^2} = \sqrt{100 + 100 + 25} = \sqrt{225} = 15 \text{ m}.$$

6. Required length $= \sqrt{l^2 + b^2 + h^2}$

$$= \sqrt{64 + 36 + 25} = \sqrt{125} = 5\sqrt{5} = (5 \times 2.24) = 11.2 \text{ cm}.$$

11. Volume of water running into the sea per hour

$$= \left(\frac{3}{2} \times 30 \times 3000\right) \text{m}^3 = (45 \times 3000) \text{ m}^3.$$

Volume of water running into the sea per minute

$$= \frac{(45 \times 3000)}{60} \text{ m}^3 = 2250 \text{ m}^3.$$

12. $4a^2 = 256 \Rightarrow a^2 = 64 = 8^2 \Rightarrow a = 8.$

$\therefore$ volume $= a^3 = (8 \times 8 \times 8) \text{ m}^3 = 512 \text{ m}^3.$

13. $6a^2 = 96 \Rightarrow a^2 = 16 = 4^2 \Rightarrow a = 4.$

$\therefore$ volume $= a^3 = (4 \times 4 \times 4) \text{ cm}^3 = 64 \text{ cm}^3.$

14. $a^3 = 512 = 8^3 \Rightarrow a = 8$ cm.

Total surface area $= 6a^2 = (6 \times 8 \times 8) \text{ cm}^2 = 384 \text{ cm}^2.$

15. Required length = length of its diagonal

$$= \sqrt{3}a = (\sqrt{3} \times 10) \text{ cm} = 10\sqrt{3} \text{ cm}.$$

16. $\sqrt{3}a = 8\sqrt{3} \Rightarrow a = 8$ cm.

$\therefore$ its surface area $= 6a^2 = (6 \times 8 \times 8) \text{ cm}^2 = 384 \text{ cm}^2.$

18. Volume of new cube formed $= (3^3 + 4^3 + 5^3) \text{ cm}^3$

$$= (27 + 64 + 125) \text{ cm}^3 = 216 \text{ cm}^3.$$

Let $a^3 = 216 = 6^3$. Then, $a = 6$ cm.

Lateral surface area of the new cube $= 4a^2 = (4 \times 6 \times 6) \text{ cm}^2 = 144 \text{ cm}^2.$

19. Required volume $= \left(2 \times 10000 \times \frac{5}{100}\right) \text{m}^3 = 1000 \text{ m}^3.$

20. $\frac{a^3}{b^3} = \frac{1}{27} \Rightarrow \left(\frac{a}{b}\right)^3 = \left(\frac{1}{3}\right)^3 \Rightarrow \frac{a}{b} = \frac{1}{3}$

$$\Rightarrow \frac{a^2}{b^2} = \left(\frac{1}{3}\right)^2 = \frac{1}{9} \Rightarrow \frac{6a^2}{6b^2} = \frac{6 \times 1}{6 \times 9} = \frac{1}{9}.$$

$\therefore$ required ratio = 1 : 9.

21. Let original side be a cm. Then, original volume = a^3 cm^3.

New side = $2a$ cm. So, new volume = $(2a)^3$ cm^3 = $(8a^3)$ cm^3.

So, the volume becomes 8 times.

22. Here $r = 3$ cm and $h = 14$ cm.

$\therefore$ volume $= \pi r^2 h = \left(\frac{22}{7} \times 3 \times 3 \times 14\right)$ cm^3 = 396 cm^3.

23. Here $r = 14$ cm and $h = 20$ cm.

$\therefore$ curved surface area $= 2\pi rh = \left(2 \times \frac{22}{7} \times 14 \times 20\right)$ cm^2 = 1760 cm^2.

24. Here $r = 14$ cm and $2\pi rh = 1760$ cm^2.

$\therefore$ $2 \times \frac{22}{7} \times 14 \times h = 1760 \Rightarrow h = \frac{1760}{88} = 20$ cm.

$\therefore$ height = 20 cm.

25. Here $h = 14$ cm and $2\pi rh = 264$ cm^2.

$\therefore$ $2 \times \frac{22}{7} \times r \times 14 = 264 \Rightarrow r = \frac{264}{88} = 3$ cm.

$\therefore$ volume $= \pi r^2 h = \left(\frac{22}{7} \times 3 \times 3 \times 14\right)$ cm^3 = 396 cm^3.

26. Given: $2\pi rh = 264$ m^2 and $\pi r^2 h = 924$ m^3.

$\therefore$ $\frac{\pi r^2 h}{2\pi rh} = \frac{924}{264} = \frac{7}{2} \Rightarrow r = \left(\frac{7}{2} \times 2\right) = 7$ m.

$\therefore$ $2 \times \frac{22}{7} \times 7 \times h = 264 \Rightarrow h = \frac{264}{44} = 6$ m.

27. Let $\frac{R_1}{R_2} = \frac{2}{3}$ and $\frac{H_1}{H_2} = \frac{5}{3}$.

Required ratio $= \frac{2\pi R_1 H_1}{2\pi R_2 H_2} = \left(\frac{R_1}{R_2}\right) \times \left(\frac{H_1}{H_2}\right) = \left(\frac{2}{3} \times \frac{5}{3}\right) = \frac{10}{9} = 10:9.$

28. Let $\frac{R_1}{R_2} = \frac{2}{3}$ and $\frac{H_1}{H_2} = \frac{5}{3}$.

Ratio of their volumes $= \frac{\pi R_1^2 H_1}{\pi R_2^2 H_2} = \left(\frac{R_1}{R_2}\right)^2 \times \left(\frac{H_1}{H_2}\right) = \left(\frac{2}{3}\right)^2 \times \frac{5}{3} = \left(\frac{4}{9} \times \frac{5}{3}\right) = \frac{20}{27} = 20:27.$

29. Let the radius be $2x$ cm and the height be $3x$ cm.

Then, volume $= \pi \times (2x)^2 \times 3x = \frac{22}{7} \times 12 \times x^3$.

$\therefore$ $\frac{22}{7} \times 12 \times x^3 = 1617 \Rightarrow x^3 = \left(1617 \times \frac{1}{12} \times \frac{7}{22}\right) = \frac{49 \times 7}{8} = \left(\frac{7}{2}\right)^3$

$$\Rightarrow x = \frac{7}{2} \Rightarrow r = 7 \text{ cm and } h = \frac{21}{2} \text{ cm.}$$

$\therefore$ total surface area $= 2\pi rh + 2\pi r^2 = 2\pi r(h + r)$

$$= \left\{2 \times \frac{22}{7} \times 7 \times \left(\frac{21}{2} + 7\right)\right\} \text{ cm}^2 = 770 \text{ cm}^2.$$

30. Let their heights be x cm and $2x$ cm respectively and let their radii be R_1 and R_2 respectively. Then,

$$\pi \times R_1^2 \times x = \pi \times R_2^2 \times 2x$$

$$\Rightarrow \quad \left(\frac{R_1}{R_2}\right)^2 = 2 \Rightarrow \frac{R_1}{R_2} = \sqrt{2} = \frac{\sqrt{2}}{1} = \sqrt{2} : 1.$$

31. $\dfrac{2\pi RH}{2\pi RH + 2\pi R^2} = \dfrac{1}{2} \Rightarrow \dfrac{2\pi R(H)}{2\pi R(H+R)} = \dfrac{1}{2} \Rightarrow \dfrac{H}{H+R} = \dfrac{1}{2}$

$$\Rightarrow H + R = 2H \Rightarrow H = R.$$

Now, $2\pi R(H+R) = 616 \Rightarrow 4\pi R^2 = 616 \quad [\because H = R]$.

$$\therefore \quad 4 \times \frac{22}{7} \times R^2 = 616 \Rightarrow R^2 = \left(616 \times \frac{7}{88}\right) = 49 = 7^2.$$

34. Let the radii be R and $\left(\frac{R}{3}\right)$ and the heights be h and H respectively. Then,

$$\pi R^2 h = \pi \times \left(\frac{R}{3}\right)^2 \times H \Rightarrow H = 9h.$$

35. Radius of the roller, $R = 42$ cm and its length, $h = 100$ cm.

Area covered by the roller in 1 revolution

$$= 2\pi Rh = \left(2 \times \frac{22}{7} \times 42 \times 100\right) \text{cm}^2 = 26400 \text{ cm}^2.$$

Area covered by the roller in 500 revolutions

$$= \left(\frac{26400 \times 500}{100 \times 100}\right) \text{m}^2 = 1320 \text{ m}^2.$$

36. Volume of the wire $= (2.2 \times 10 \times 10 \times 10)\ \text{cm}^3 = 2200\ \text{cm}^3$.

Radius of the wire $= 0.25$ cm $= \dfrac{25}{100}$ cm $= \dfrac{1}{4}$ cm.

Let the length of wire be h cm. Then,

$$\pi \times \frac{1}{4} \times \frac{1}{4} \times h = 2200 \Rightarrow \frac{22}{7} \times \frac{1}{16} \times h = 2200.$$

$$\therefore \quad h = \left(2200 \times \frac{7}{22} \times 16\right) \text{cm} = \left(\frac{11200}{100}\right) \text{m} = 112 \text{ m}.$$

38. $l^2 = (r^2 + h^2) = 7^2 + (24)^2 = (49 + 576) = 625$

$$\therefore \quad l = \sqrt{625} = 25 \text{ cm}.$$

Curved surface area $= \pi rl = \left(\dfrac{22}{7} \times 7 \times 25\right) \text{cm}^2 = 550\ \text{cm}^2$.

40. Given: $r = 7$ m and $h = 24$ m.

$$\therefore \quad l^2 = (r^2 + h^2) = [7^2 + (24)^2] = (49 + 576) = 625$$

$$\Rightarrow \quad l = \sqrt{625} = 25 \text{ m}.$$

Area of the cloth needed $= \pi rl = \left(\dfrac{22}{7} \times 7 \times 25\right) \text{m}^2 = 550\ \text{m}^2$.

Length of the cloth $= \dfrac{\text{area}}{\text{width}} = \left(550 \times \dfrac{2}{5}\right) \text{m} = 220$ m.

41. $\frac{1}{3}\pi R^2 h = 1570 \Rightarrow \frac{1}{3} \times 3.14 \times R^2 \times 15 = 1570$

$$\Rightarrow 15.7R^2 = 1570 \Rightarrow R^2 = \left(1570 \times \frac{10}{157}\right) = 100 = (10)^2.$$

44. Let the radii of the cones be $4x$ cm and $5x$ cm respectively and their volumes be v cm^3 and $4v$ cm^3 respectively. Let their heights be h cm and H cm respectively. Then,

$$\frac{\frac{1}{3}\pi \times (4x)^2 \times h}{\frac{1}{3}\pi \times (5x)^2 \times H} = \frac{v}{4v} \Rightarrow \frac{16h}{25H} = \frac{1}{4} \Rightarrow \frac{h}{H} = \frac{25}{64}.$$

46. Let their slant heights be l and $2l$ and their radii be R_1 and R_2. Then,

$$\frac{\pi R_1 l}{\pi R_2 (2l)} = \frac{2}{1} \Rightarrow \frac{R_1}{R_2} = \frac{4}{1} = 4:1.$$

47. Required ratio $= \dfrac{\pi r^2 h}{\frac{1}{3}\pi r^2 h} = \dfrac{3}{1} = 3:1.$

48. $\pi r^2 h = \frac{1}{3}\pi r^2 H \Rightarrow \frac{h}{H} = \frac{1}{3} = 1:3.$

49. Let the radii of the bases of a cylinder and a cone be $3x$ cm and $4x$ cm respectively and let their heights be $2y$ cm and $3y$ cm respectively. Then,

ratio of their volumes $= \dfrac{\pi \times (3x)^2 \times 2y}{\frac{1}{3}\pi \times (4x)^2 \times 3y} = \dfrac{54}{48} = \dfrac{9}{8} = 9:8.$

50. Volume of a cone of height h and radius $r = \frac{1}{3}\pi r^2 h = V$.

New height $= 2h$ and new radius $= 2r$.

$\therefore$ new volume $= \frac{1}{3}\pi (2r)^2 \times 2h = 8 \times \left(\frac{1}{3}\pi r^2 h\right) = 8V.$

51. n = number of cones $= \dfrac{\text{volume of the cylinder}}{\text{volume of 1 cone}} = \dfrac{\pi \times 3 \times 3 \times 5}{\frac{1}{3}\pi \times \frac{1}{10} \times \frac{1}{10} \times 1} = 13500.$

69. Let their radii be x cm and $(7-x)$ cm. Then,

$$\frac{\frac{4}{3}\pi \times x^3}{\frac{4}{3}\pi \times (7-x)^3} = \frac{64}{27} = \left(\frac{4}{3}\right)^3 \Rightarrow \frac{x^3}{(7-x)^3} = \left(\frac{4}{3}\right)^3 \Rightarrow \left(\frac{x}{7-x}\right)^3 = \left(\frac{4}{3}\right)^3.$$

$\therefore$ $\frac{x}{7-x} = \frac{4}{3} \Rightarrow 3x = 28 - 4x \Rightarrow 7x = 28 \Rightarrow x = 4.$

So, their radii are 4 cm and 3 cm.

Required difference $= 4\pi \times (4)^2 - 4\pi \times (3)^2$.

72. Let R be the radius of each.

Height of hemisphere = its radius = R cm.

$\therefore$ height of each = R cm.

Ratio of their volumes $= \left(\frac{1}{3}\pi R^2 \times R\right) : \left(\frac{2}{3}\pi R^3\right) : (\pi R^2 \times R)$.

REVIEW OF FACTS AND FORMULAE

1. CUBOID Let length = l, breadth = b and height = h. Then, we have:

(i) Volume of cuboid = $(l \times b \times h)$ cubic units.

(ii) Total surface area of the cuboid = $2(lb + bh + lh)$ sq units.

(iii) Lateral surface area of the cuboid = $\{2(l + b) \times h\}$ sq units.

(iv) Diagonal of the cuboid = $\{\sqrt{l^2 + b^2 + h^2}\}$ units.

2. CUBE Let each edge of the cube be a. Then, we have:

(i) Volume of the cube = a^3 cubic units.

(ii) Total surface area of the cube = $(6a^2)$ sq units.

(iii) Lateral surface area of the cube = $(4a^2)$ sq units.

(iv) Diagonal of the cube = $(a\sqrt{3})$ units.

3. CYLINDER Let the radius of the base be r and height be h. Then, we have:

(i) Volume of the cylinder = $(\pi r^2 h)$ cubic units.

(ii) Total surface area of the cylinder = $\{2\pi r(r + h)\}$ sq units.

(iii) Curved surface area of the cylinder = $(2\pi rh)$ sq units.

4. CONE Let the radius of its base be r, height be h and slant height be l. Then, we have:

(i) Volume of the cone = $\left(\frac{1}{3}\pi r^2 h\right)$ cubic units.

(ii) Curved surface area of the cone = (πrl) sq units.

(iii) Total surface area of the cone = $(\pi rl + \pi r^2)$ sq units.

(iv) Slant height of the cone $l = \sqrt{h^2 + r^2}$.

5. SPHERE Let the radius of the sphere be r. Then, we have:

(i) Volume of the sphere = $\left\{\frac{4}{3}\pi r^3\right\}$ cubic units.

(ii) Surface area of the sphere = $(4\pi r^2)$ sq units.

6. HEMISPHERE Let the radius of the hemisphere be r. Then, we have:

(i) Volume of the hemisphere = $\left(\frac{2}{3}\pi r^3\right)$ cubic units.

(ii) Curved surface area of the hemisphere = $(2\pi r^2)$ sq units.

(iii) Total surface area of the hemisphere = $(3\pi r^2)$ sq units.

✿

16 Presentation of Data in Tabular Form

FREQUENCY DISTRIBUTION

INTRODUCTION

The present-day society is essentially information-oriented. In various fields, we need information in the form of numerical figures, called *data*.

These data may relate to the profits of a company during last few years, the monthly wages earned by workers in a factory, the expenditure in various sectors of a five-year plan, the marks obtained by the students of a class in a certain examination, etc.

DATA *The word data means information or set of given facts in numerical figures.*

STATISTICS *It is the science which deals with the collection, presentation, analysis and interpretation of numerical data.*

In plural form, *statistics means data*.

In singular form, *statistics is taken as a subject*.

FUNDAMENTAL CHARACTERISTICS OF DATA

(i) Numerical facts alone constitute data.

(ii) Qualitative characteristics like intelligence, poverty, etc., which cannot be measured numerically, do not form data.

(iii) Data are aggregate of facts. A single observation does not form data.

(iv) Data collected for a definite purpose may not be suited for another purpose.

(v) Data in different experiments are comparable.

LIMITATIONS OF STATISTICS

(i) Statistics deals with groups and does not study individuals.

(ii) Statistics is not suited to the study of qualitative phenomenon, like honesty, poverty, etc.

(iii) Statistical laws are not exact. They are true on averages only.

TYPES OF DATA

(i) PRIMARY DATA The data collected by the investigator himself with a definite plan in mind are known as *primary data*. These data are, therefore, highly reliable and relevant.

(ii) SECONDARY DATA The data collected by someone, other than the investigator, are known as *secondary data*.

Secondary data should be carefully used, since they are collected with a purpose different from that of the investigator and may not be fully relevant to the investigation.

RAW OR UNGROUPED DATA *The data obtained in original form are called raw data or ungrouped data.*

GROUPED DATA *We may condense data into classes or groups. Such a presentation is known as a grouped data.*

ARRAY *An arrangement of raw numerical data in ascending or descending order of magnitude, is called an array.*

PRESENTATION OF DATA *Putting the data, in the form of tables, in condensed form, is known as the presentation of data.*

FREQUENCY OF AN OBSERVATION *The number of times an observation occurs is called its frequency.*

FREQUENCY DISTRIBUTION OF AN UNGROUPED DATA

The tabular arrangement of data, showing the frequency of each observation, is called a frequency distribution, as shown in the example given below.

EXAMPLE 1 *The following data gives the number of children in* 20 *families:*

4, 5, 2, 4, 2, 2, 1, 3, 3, 2, 5, 3, 2, 1, 1, 4, 3, 2, 1, 1.

Make an array of the above data and construct a frequency table.

SOLUTION Arranging the numerical data in ascending order of magnitude, we get the array as under:

1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 3, 3, 3, 3, 4, 4, 4, 5, 5.

For counting, we use tally marks $\vert$, $\vert\vert$, $\vert\vert\vert$, $\vert\vert\vert\vert$, and the fifth tally mark is entered as $\cancel{\vert\vert\vert\vert}$ by crossing diagonally the four tally marks already entered.

Frequency table may now be presented as under.

Frequency Table

No. of children	Tally marks	No. of families (Frequency)
1	$\cancel{\vert\vert\vert\vert}$	5
2	$\cancel{\vert\vert\vert\vert}\ \vert$	6
3	$\vert\vert\vert\vert$	4
4	$\vert\vert\vert$	3
5	$\vert\vert$	2

TYPES OF FREQUENCY DISTRIBUTION

(I) EXCLUSIVE FORM (OR CONTINUOUS FORM) *A frequency distribution in which each upper limit of each class is excluded and lower limit is included, is called an exclusive form.*

Example Suppose the marks obtained by some students are given.

We may consider the classes 0–10, 10–20, etc. In class 0–10, we include 0 and exclude 10. In class 10–20, we include 10 and exclude 20.

(ii) INCLUSIVE FORM (OR DISCONTINUOUS FORM) *A frequency distribution in which each upper limit as well as lower limit is included, is called an inclusive form.* Thus, we have classes of the form 0–10, 11–20, 21–30, etc. In 0–10, both 0 and 10 are included.

EXAMPLE 2 *The following data gives marks out of* 50 *obtained by* 30 *students of a class in a test:*

30, 27, 17, 37, 46, 12, 40, 6, 23, 2, 19, 5, 33, 25, 39, 21, 19, 12, 17, 19, 17, 41, 8, 10, 12, 1, 9, 13, 21, 48.

Arrange them in ascending order and present it as a grouped data (i) in exclusive form, (ii) in inclusive form.

SOLUTION Arranging the marks in ascending order, we get

1, 2, 5, 6, 8, 9, 10, 12, 12, 12, 13, 17, 17, 17, 19, 19, 19, 21, 21, 23, 25, 27, 30, 33, 37, 39, 40, 41, 46, 48.

We may now classify them into groups as shown below.

(i) In exclusive form:

Marks (Class interval)	Number of students (Frequency)
0–10	6
10–20	11
20–30	5
30–40	4
40–50	4

In 0–10, we include 0 but exclude 10.

In 10–20, we include 10 but exclude 20, and so on.

(ii) In inclusive form:

Marks (Class interval)	Number of students (Frequency)
0–10	7

11–20	10
21–30	6
31–40	4
41–50	3

Here, in 0–10, we include both 0 and 10.

In 11–20, we include both 11 and 20.

SOME DEFINITIONS

(I) VARIATE OR VARIABLE *Any character which is capable of taking several different values is called a variable.*

Example Suppose we make a survey of the workers in a factory and record the age, sex and income of each worker. Then, the three variables are: age, sex and income. Here, age and income are represented by numbers. So, age and income are quantitative variables. But the variable sex can take the value 'male' or 'female'. So, sex is a qualitative variable.

(II) CLASS INTERVAL *Each group into which the raw data is condensed, is called a class interval.*

Each class is bounded by two figures, which are called *class limits*. The figure on the left side of a class is called its *lower limit* and that on the right is called its *upper limit*.

(III) CLASS BOUNDARIES OR TRUE UPPER AND TRUE LOWER LIMITS

RULE I. In the exclusive form, the upper and lower limits of a class are respectively known as the true upper limit and true lower limit.

RULE II. In an inclusive form of frequency distribution, the true lower limit of a class is obtained by subtracting 0.5 from the lower limit of the class.

And, the true upper limit of the class is obtained by adding 0.5 to the upper limit.

Example In class 11–20 above, we have true lower limit = 10.5 and true upper limit = 20.5.

(IV) CLASS SIZE *The difference between the true upper limit and the true lower limit of a class is called its class size.*

(V) CLASS MARK OF A CLASS $= \left(\dfrac{\text{Upper limit + Lower limit}}{2}\right)$.

REMARK The difference between any two successive class marks gives the class size.

(VI) RANGE *The difference between the maximum value and the minimum value of the variate is called the range.*

METHOD OF FORMING CLASSES OF A DATA

(i) Determine the maximum and minimum values of the variate occurring in the data.

(ii) Decide upon the number of classes to be formed.

(iii) Find the range, i.e., the difference between the maximum value and the minimum value.

Divide the range by the number of classes to be formed.

(iv) Be sure that there must be classes having minimum and maximum values occurring in the data.

(v) By counting, we obtain the frequency of each class.

EXAMPLE 3 *The marks obtained by* 40 *students of class IX in an examination are given below:*

23, 5, 12, 16, 8, 6, 12, 8, 18, 2, 16, 10, 2, 23, 7, 9, 12, 20, 0, 3, 5, 16, 18, 3, 17, 7, 23, 18, 13, 10, 21, 7, 1, 24, 20, 15, 13, 21, 13, 5.

Present the data in the form of a frequency distribution using the same class size, starting with class 0–5 *(where* 5 *is not included).*

SOLUTION Minimum observation is 0 and maximum observation is 24. The classes of equal size covering the given data are:

0–5, 5–10, 10–15, 15–20 and 20–25.

Thus, the frequency distribution may be given as under:

Marks	Tally marks	Frequency
0–5	~~||||~~ |	6
5–10	~~||||~~ ~~||||~~	10
10–15	~~||||~~ |||	8
15–20	~~||||~~ |||	8
20–25	~~||||~~ |||	8
Total		40

EXAMPLE 4 *The water-tax bills (in ₹) of* 30 *houses in a locality are given below:*

144, 184, 130, 195, 132, 134, 196, 114, 212, 174, 188, 210, 202, 145, 175, 154, 174, 178, 166, 146, 135, 115, 120, 114, 140, 188, 176, 166, 210, 208.

Construct a frequency distribution table with class size 10.

SOLUTION The minimum observation is 114 and the maximum observation is 212.

$\therefore$ range = (212 − 114) = 98.

Class size = 10.

Since 98 ÷ 10 = 9.8, we should have 10 classes, each of size 10.

The classes of equal size, covering the above data are:

114–124, 124–134, 134–144,144–154, 154–164, 164–174, 174–184, 184–194, 194–204 and 204–214.

The frequency distribution table may be presented as shown below:

Bill (in ₹)	Tally marks	Frequency
114–124	||||	4
124–134	||	2
134–144	|||	3
144–154	|||	3
154–164	|	1
164–174	||	2
174–184	$\cancel{\|\|\|\|}$	5
184–194	|||	3
194–204	|||	3
204–214	||||	4
Total		30

EXAMPLE 5 *The weights in grams of* 50 *apples picked at random from a consignment are as follows:*

131, 113, 82, 75, 204, 81, 84, 118, 104, 110, 80, 107, 111, 141, 136, 123, 90, 78, 90, 115, 110, 98, 106, 99, 107, 84, 76, 186, 82, 100, 109, 128, 115, 107, 115, 119, 93, 187, 139, 129, 130, 68, 195, 123, 125, 111, 92, 86, 70, 126.

Form the grouped frequency table by dividing the variable range into intervals of equal width of 20 g *such that the mid-value of the first class interval is* 70 g.

SOLUTION Size of each class = 20.

Let the lower limit of the first class interval be a.

Then, its upper limit $= (a + 20)$.

Mid-value of the first class interval = 70.

$$\therefore \quad \frac{a+(a+20)}{2} = 70 \Rightarrow 2a = 120 \Rightarrow a = 60.$$

$\therefore$ the first class interval is 60–80.

So, we may give the frequency distribution as under.

Weight (in grams)	Tally marks	Frequency
60–80	$\cancel{\|\|\|\|}$	5
80–100	$\cancel{\|\|\|\|}$ $\cancel{\|\|\|\|}$ $\|\|\|$	13
100–120	$\cancel{\|\|\|\|}$ $\cancel{\|\|\|\|}$ $\cancel{\|\|\|\|}$ $\|\|$	17
120–140	$\cancel{\|\|\|\|}$ $\cancel{\|\|\|\|}$	10
140–160	$\|$	1
160–180		0
180–200	$\|\|\|$	3
200–220	$\|$	1
Total		50

EXAMPLE 6 *The relative humidity (in %) of a certain city for a month of 30 days was as follows:*

89.2	92.3	97.1	93.5	92.7	95.1	97.2	93.3	95.2	97.3
92.1	89	97.3	96.1	95.7	92.1	84.9	98.3	96.2	90.2
95.1	94.2	96.3	92.9	95.3	86.5	90.3	98.1	98.6	99.2

(i) Construct a grouped frequency distribution table with classes 84–86, 86–88, *etc.*

(ii) Which season do you think this data is about?

(iii) What is the range of this data?

SOLUTION (i) The required frequency table is given below:

Relative humidity (in %)	Tally marks	Frequency
84–86	$\|$	1
86–88	$\|$	1
88–90	$\|\|$	2
90–92	$\|\|$	2
92–94	$\cancel{\|\|\|\|}$ $\|\|$	7
94–96	$\cancel{\|\|\|\|}$ $\|$	6
96–98	$\cancel{\|\|\|\|}$ $\|\|$	7
98–100	$\|\|\|\|$	4
Total		30

(ii) Since the humidity in the air is more in percentage, the data is therefore related to the rainy season.

(iii) Range of the data = (99.2 − 84.9) = 14.3.

CUMULATIVE FREQUENCY The cumulative frequency corresponding to a class is the sum of all frequencies up to and including that class.

A table which shows the cumulative frequencies over various classes is called a cumulative frequency distribution table.

EXAMPLE 7 *The marks obtained by* 35 *students in an examination are given below:*

380, 405, 378, 410, 170, 290, 370, 175, 318, 241, 275, 315, 305, 375, 288, 325, 402, 355, 261, 380, 253, 178, 428, 240, 154, 175, 210, 405, 425, 440, 306, 328, 370, 460, 380.

Form a cumulative frequency table with class intervals of length 50.

SOLUTION The minimum marks are 154 and the maximum marks are 460.

So, we take the class intervals of length 50 as 150–200, 200–250, 250–300, …, 450–500. Thus, we may form the cumulative frequency table as under:

Class intervals	Tally marks	Freq.	Cumulative freq.
150–200	卌	5	5
200–250	|||	3	8
250–300	卌	5	13
300–350	卌 |	6	19
350–400	卌 |||	8	27
400–450	卌 ||	7	34
450–500	|	1	35

EXERCISE 16

1. Define statistics as a subject.
2. Define some fundamental characteristics of statistics.
3. What are primary data and secondary data? Which of the two is more reliable and why?
4. Explain the meaning of each of the following terms:

(i) Variate (ii) Class interval (iii) Class size
(iv) Class mark (v) Class limit (vi) True class limits
(vii) Frequency of a class (viii) Cumulative frequency of a class

5. The blood groups of 30 students of a class are recorded as under:

A, B, O, O, AB, O, A, O, A, B, O, B, A, O, O, A, AB, O, A, A, O, O, AB, B, A, O, B, A, B, O.

(i) Represent this data in the form of a frequency distribution table.

(ii) Find out which is the most common and which is the rarest blood group among these students.

6. Three coins are tossed 30 times. Each time the number of heads occurring was noted down as follows:

0, 1, 2, 2, 1, 2, 3, 1, 3, 0, 1, 3, 1, 1, 2, 2, 0, 1, 2, 1, 0, 3, 0, 2, 1, 1, 3, 2, 0, 2.

Prepare a frequency distribution table.

7. Following data gives the number of children in 40 families:

1, 2, 6, 5, 1, 5, 1, 3, 2, 6, 2, 3, 4, 2, 0, 4, 4, 3, 2, 2, 0, 0, 1, 2, 2, 4, 3, 2, 1, 0, 5, 1, 2, 4, 3, 4, 1, 6, 2, 2.

Represent it in the form of a frequency distribution, taking classes 0–2, 2–4, etc.

8. Thirty children were asked about the number of hours they watched TV programmes in the previous week. The results were found as under:

8, 4, 8, 5, 1, 6, 2, 5, 3, 12, 3, 10, 4, 12, 2, 8, 15, 1, 6, 17, 5, 8, 2, 3, 9, 6, 7, 8, 14, 12.

(i) Make a grouped frequency distribution table for this data, taking class width 5 and one of the class interval as 5–10.

(ii) How many children watched television for 15 or more hours a week?

9. The marks obtained by 40 students of a class in an examination are given below.

3, 20, 13, 1, 21, 13, 3, 23, 16, 13, 18, 12, 5, 12, 5, 24, 9, 2, 7, 18, 20, 3, 10, 12, 7, 18, 2, 5, 7, 10, 16, 8, 16, 17, 8, 23, 24, 6, 23, 15.

Present the data in the form of a frequency distribution using equal class size, one such class being 10–15 (15 not included).

10. Construct a frequency table for the following ages (in years) of 30 students using equal class intervals, one of them being 9–12, where 12 is not included.

18, 12, 7, 6, 11, 15, 21, 9, 8, 13, 15, 17, 22, 19, 14, 21, 23, 8, 12, 17, 15, 6, 18, 23, 22, 16, 9, 21, 11, 16.

11. Construct a frequency table with equal class intervals from the following data on the daily wages (in ₹) of 28 labourers working in a factory, taking one of the class intervals as 210–230 (230 not included).

220, 268, 258, 242, 210, 268, 272, 242, 311, 290, 300, 320, 319, 304, 302, 318, 306, 292, 254, 278, 210, 240, 280, 316, 306, 215, 256, 236.

12. The weights (in grams) of 40 oranges picked at random from a basket are as follows:

40, 50, 60, 65, 45, 55, 30, 90, 75, 85, 70, 85, 75, 80, 100, 110, 70, 55, 30, 35, 45, 70, 80, 85, 95, 70, 60, 70, 75, 40, 100, 65, 60, 40, 100, 75, 110, 30, 45, 84.

Construct a frequency table as well as a cumulative frequency table.

13. The heights (in cm) of 30 students of a class are given below:

161, 155, 159, 153, 150, 158, 154, 158, 160, 148, 149, 162, 163, 159, 148, 153, 157, 151, 154, 157, 153, 156, 152, 156, 160, 152, 147, 155, 155, 157.

Prepare a frequency table as well as a cumulative frequency table with 160–165 (165 not included) as one of the class intervals.

14. Following are the ages (in years) of 360 patients, getting medical treatment in a hospital:

Age (in years)	10–20	20–30	30–40	40–50	50–60	60–70
Number of patients	90	50	60	80	50	30

Construct the cumulative frequency table for the above data.

15. Present the following as an ordinary grouped frequency table:

Marks (below)	10	20	30	40	50	60
Number of students	5	12	32	40	45	48

16. Given below is a cumulative frequency table:

Marks	Number of students
Below 10	17
Below 20	22
Below 30	29
Below 40	37
Below 50	50
Below 60	60

Extract a frequency table from the above.

17. Make a frequency table from the following:

Marks obtained	Number of students
More than 60	0
More than 50	16
More than 40	40
More than 30	75
More than 20	87
More than 10	92
More than 0	100

18. The marks obtained by 17 students in a mathematics test (out of 100) are given below:

90, 79, 76, 82, 65, 96, 100, 91, 82, 100, 49, 46, 64, 48, 72, 66, 68.

Find the range of the above data.

19. (i) Find the class mark of the class 90–120.

(ii) In a frequency distribution, the mid-value of the class is 10 and width of the class is 6. Find the lower limit of the class.

(iii) The width of each of five continuous classes in a frequency distribution is 5 and lower class limit of the lowest class is 10. What is the upper class limit of the highest class?

(iv) The class marks of a frequency distribution are 15, 20, 25, … . Find the class corresponding to the class mark 20.

(v) In the class intervals 10–20, 20–30, find the class in which 20 is included.

20. Find the values of a, b, c, d, e, f, g from the following frequency distribution of the heights of 50 students in a class:

Height (in cm)	Frequency	Cumulative frequency
160–165	15	a
165–170	b	35
170–175	12	c
175–180	d	50
180–185	e	55
185–190	5	f
	g	

ANSWERS (EXERCISE 16)

5. (i) Frequency distribution table:

Blood group	A	B	O	AB
Number of students	9	6	12	3

(ii) Clearly, the blood group O is most common and AB is the rarest.

6. Frequency distribution table:

Number of heads	0	1	2	3
Frequency	6	10	9	5

7. Frequency distribution table:

Class	0–2	2–4	4–6	6–8
Frequency	11	17	9	3

8. (i) Frequency distribution table:

Class interval	0–5	5–10	10–15	15–20
Frequency	10	13	5	2

(ii) 2 children watched TV for 15 or more hours.

9.

Class	0–5	5–10	10–15	15–20	20–25
Frequency	6	10	8	8	8

10.

Class	6–9	9–12	12–15	15–18	18–21	21–24
Frequency	5	4	4	7	3	7

11.

Class	210–230	230–250	250–270	270–290	290–310	310–330
Frequency	4	4	5	3	7	5

12.

Class	30–40	40–50	50–60	60–70	70–80	80–90	90–100	100–110	110–120
Frequency	4	6	3	5	9	6	2	3	2
Cumulative frequency	4	10	13	18	27	33	35	38	40

13.

Class	145–150	150–155	155–160	160–165
Frequency	4	9	12	5
Cumulative frequency	4	13	25	30

14.

Age (in years)	Below 20	Below 30	Below 40	Below 50	Below 60	Below 70
No. of patients	90	140	200	280	330	360

15.

Marks	0–10	10–20	20–30	30–40	40–50	50–60
Number of students	5	7	20	8	5	3

16.

Marks	0–10	10–20	20–30	30–40	40–50	50–60
Number of students	17	5	7	8	13	10

17.

Class	0–10	10–20	20–30	30–40	40–50	50–60
Frequency	8	5	12	35	24	16

18. 54

19. (i) 105 (ii) 7 (iii) 35 (iv) 17.5–22.5 (v) 20–30

20. $a = 15, b = 20, c = 47, d = 3, e = 5, f = 60, g = 60$.

HINTS TO SOME SELECTED QUESTIONS

18. Range = (maximum value) – (minimum value) = (100 – 46) = 54.

19. (i) Class mark $= \dfrac{(\text{upper limit} + \text{lower limit})}{2} = \dfrac{(120 + 90)}{2} = \dfrac{210}{2} = 105$.

(ii) Let the lower limit of the required class be a.

Then, its upper limit is $(a + 6)$.

$\therefore \quad \dfrac{a + (a + 6)}{2} = 10 \Rightarrow 2a + 6 = 20 \Rightarrow 2a = 14 \Rightarrow a = 7$.

Hence, the lower limit of the class is 7.

(iii) The classes are 10–15, 15–20, 20–25, 25–30 and 30–35.

Upper class limit of the highest class is 35.

(iv) Class size = (20 – 15) = 5.

Let $= \dfrac{a + (a + 5)}{2} = 20$. Then, $2a = 35 \Rightarrow a = 17.5$.

Required class is 17.5–22.5.

(v) Clearly, 20 is included in 20–30.

MULTIPLE-CHOICE QUESTIONS (MCQ)

Choose the correct answer in each of the following questions:

1. The range of the data

12, 25, 15, 18, 17, 20, 22, 6, 16, 11, 8, 19, 10, 30, 20, 32 is

(a) 10 (b) 15 (c) 18 (d) 26

2. The class mark of the class 100–120 is

(a) 100 (b) 110 (c) 115 (d) 120

3. In the class intervals 10–20, 20–30, the number 20 is included in

(a) 10–20 (b) 20–30
(c) in each of 10–20 and 20–30 (d) in none of 10–20 and 20–30

4. The class marks of a frequency distribution are 15, 20, 25, 30, ...
The class corresponding to the class mark 20 is

(a) 12.5–17.5 (b) 17.5–22.5 (c) 18.5–21.5 (d) 19.5–20.5

5. In a frequency distribution, the mid-value of a class is 10 and width of each class is 6. The lower limit of the class is

(a) 6 (b) 7 (c) 8 (d) 12

6. The mid-value of a class interval is 42 and the class size is 10. The lower and upper limits are

(a) 37–47 (b) 37.5–47.5 (c) 36.5–47.5 (d) 36.5–46.5

7. Let m be the midpoint and u be the upper class limit of a class in a continuous frequency distribution. The lower class limit of the class is

(a) $2m-u$ (b) $2m+u$ (c) $m-u$ (d) $m+u$

8. The width of each of the five continuous classes in a frequency distribution is 5 and the lower class limit of the lowest class is 10. The upper class limit of the highest class is

(a) 45 (b) 25 (c) 35 (d) 40

9. Let L be the lower class boundary of a class in a frequency distribution and m be the midpoint of the class. Which one of the following is the upper class boundary of the class?

(a) $m+\frac{(m+L)}{2}$ (b) $L+\frac{m+L}{2}$ (c) $2m-L$ (d) $m-2L$

ANSWERS (MCQ)

1. (d) **2.** (b) **3.** (b) **4.** (b) **5.** (b) **6.** (a) **7.** (a) **8.** (c)
9. (c)

HINTS TO SOME SELECTED QUESTIONS

1. Range = (maximum value) – (minimum value) = (32 – 6) = 26.
2. Class mark $=\frac{1}{2}(100+120)=\frac{220}{2}=110.$
3. Clearly, 20 is included in 20–30.
4. Class size = (20 – 15) = 5.

   Class mark = 20.

   Lower limit $=\left(20-\frac{5}{2}\right)=\frac{35}{2}=17.5$

Upper limit $= \left(20 + \frac{5}{2}\right) = \frac{45}{2} = 22.5$

Required class is 17.5–22.5.

5. Let the lower limit be a. Then, upper limit $= a + 6$.

Mid-value $= \frac{a + a + 6}{2} = \frac{2a + 6}{2} = a + 3.$

$\therefore \quad a + 3 = 10 \Rightarrow a = 7.$

$\therefore$ lower limit of the class is 7.

6. Let the lower limit be x. Then, upper limit $= (x + 10)$.

$\therefore \quad \frac{x + (x + 10)}{2} = 42 \Rightarrow 2x + 10 = 84 \Rightarrow 2x = 84 \Rightarrow x = 37.$

$\therefore$ lower limit = 37 and upper limit = 47.

7. Let l be the lower limit. Then,

$$m = \frac{l + u}{2} \Rightarrow l = 2m - u.$$

8. Upper class limit of the highest class $= 10 + 5 \times 5 = 35$.

9. $m = \frac{L + U}{2} \Rightarrow U = (2m - L)$.

REVIEW OF FACTS AND FORMULAE

1. (I) TABULATION *The arrangement of the raw data under various heads in the form of a table is called tabulation.*

(II) RANGE OF A DATA *The difference between the largest and the smallest observations is called the range.*

(III) FREQUENCY *The number of observations in a particular class is called its frequency.*

(IV) CUMULATIVE FREQUENCY *The cumulative frequency of a particular class is the sum of all frequencies up to this class.*

(V) FREQUENCY DISTRIBUTION *The distribution of frequency in various classes is known as frequency distribution.*

2. TYPES OF FREQUENCY DISTRIBUTION

(I) EXCLUSIVE OR CONTINUOUS FORM Here, the classes are of the form 0–10, 10–20, 20–30, etc.

Here, 10–20 means 10 and more but less than 20.

Thus, in 10–20 we include 10 and exclude 20.

Here in 10–20, we have

lower limit = 10, upper limit = 20.

True lower limit = 10, true upper limit = 20.

Class size = (20 – 10) = 10.

(II) INCLUSIVE OR DISCONTINUOUS FORM Here, the classes are of the form 0–10, 11–20, 21–30, etc.

Here, 11–20 means 11 and more but less than 20.

Thus, in 11–20 both 11 and 20 are included.

In 11–20, we have

lower limit = 11, upper limit = 20.

True lower limit = 10.5, true upper limit = 20.5.

Class size = (20.5 – 10.5) = 10.

(III) Class mark of a class $= \dfrac{(\text{upper limit} + \text{lower limit})}{2}$.

(IV) Range = (maximum value) – (minimum value).

✿

17 Bar Graph, Histogram and Frequency Polygon

BAR GRAPH

GRAPHICAL REPRESENTATION OF DATA

The tabular representation of data is an ideal way of presenting them in a systematic manner. When these numerical figures are represented pictorially or graphically, they become more noticeable and easily intelligible, leaving a more lasting effect on the mind of the observer. With the help of these pictures or graphs, data can be compared easily.

In this chapter, we shall deal with three types of graphs, namely

(i) *bar graph,* (ii) *histogram* and (iii) *frequency polygon.*

BAR GRAPH, OR COLUMN GRAPH

As compared to written statements, the graphical representation of statistical data has a more soothing effect on mind. We understand them more clearly.

In a bar graph, bars of uniform width are drawn with various heights. The height of a bar represents the frequency of the corresponding observation.

Same space is left between consecutive bars.

The following examples will give a clear idea about a bar graph.

SOLVED EXAMPLES

EXAMPLE 1 70 *students from a locality use different modes of transport to go to school as given below:*

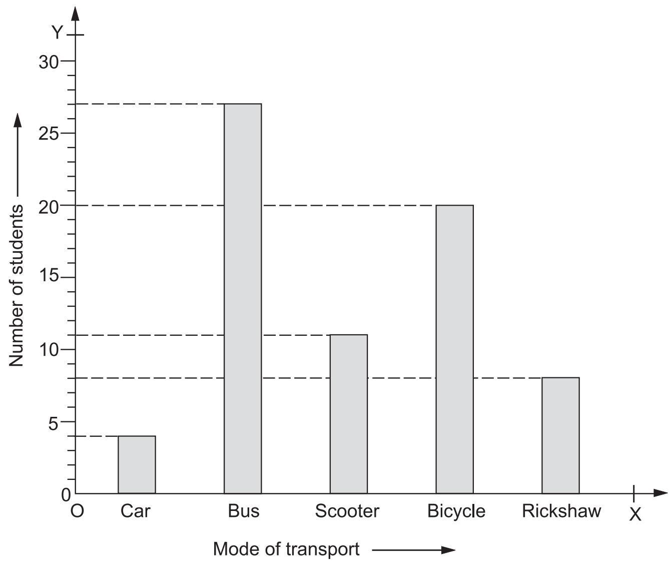
Mode of transport	*Car*	*Bus*	*Scooter*	*Bicycle*	*Rickshaw*
Number of students	4	27	11	20	8

Draw the bar graph representing the above data.

SOLUTION Take the mode of transport along the x-axis and the number of students along the y-axis.

All the bars should be of the same width and same space should be left between the consecutive bars.

These bars may be drawn as shown below.



EXAMPLE 2 *There are 840 creatures in a zoo as per list given below:*

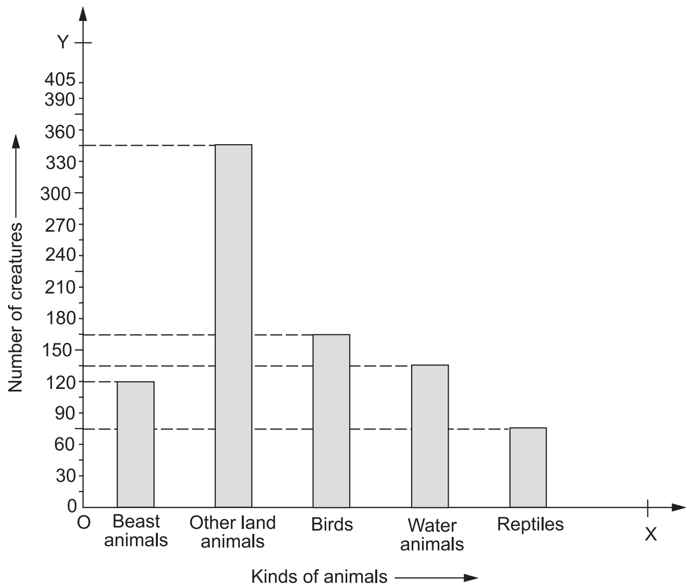
Beast animals	*Other land animals*	*Birds*	*Water animals*	*Reptiles*
120	345	165	135	75

Represent the above data by a bar chart.

SOLUTION Take the kinds of animals along the x-axis and the corresponding number of these animals along the y-axis.

Draw the bars of the same width and the same space should be left between the consecutive bars.

The heights of these bars should be proportional to their number, as shown below.



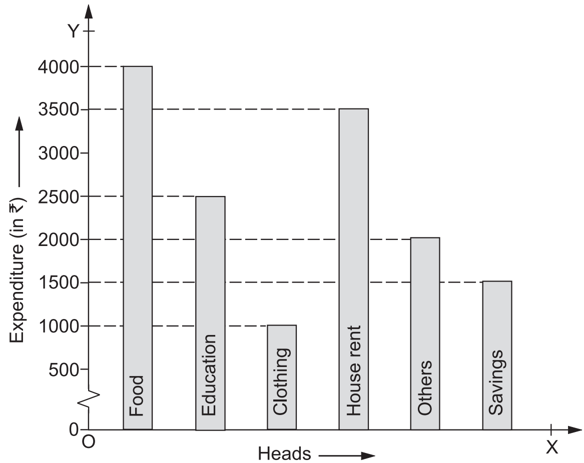
EXAMPLE 3 *The expenditure of a family on different heads in a month is given below:*

Head	*Food*	*Education*	*Clothing*	*House rent*	*Others*	*Savings*
Expenditure (in ₹)	4000	2500	1000	3500	2000	1500

Draw a bar graph to represent the above data.

SOLUTION Take the difference heads of expenditure along the x-axis and the corresponding expenditure (in ₹) along the y-axis.

Draw the bars of the same width having their heights proportional to the expenditure in corresponding heads, keeping the same space between two consecutive bars, as shown below.



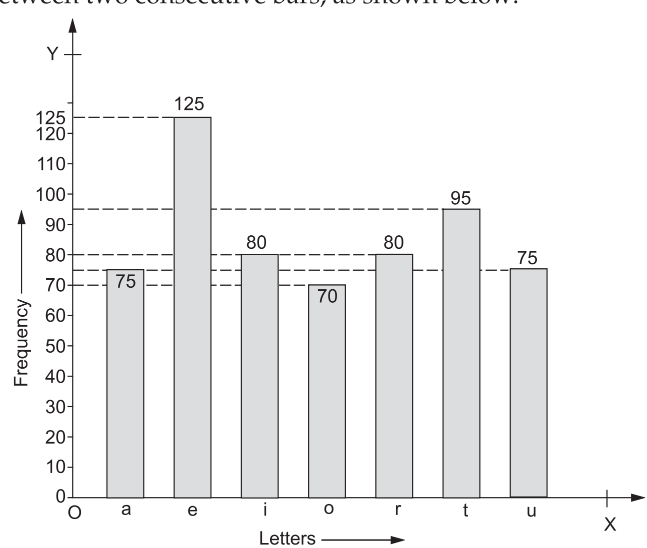
EXAMPLE 4 *The following table gives the frequencies of most commonly used letters a, e, i, o, r, t, u from a page of a book:*

Letters	*a*	*e*	*i*	*o*	*r*	*t*	*u*
Frequency	75	125	80	70	80	95	75

Represent the above information by a bar graph.

SOLUTION Take the letters along the x-axis and their corresponding frequencies along the y-axis.

Draw the bars of the same width having their heights proportional to their frequencies, keeping the same space between two consecutive bars, as shown below.



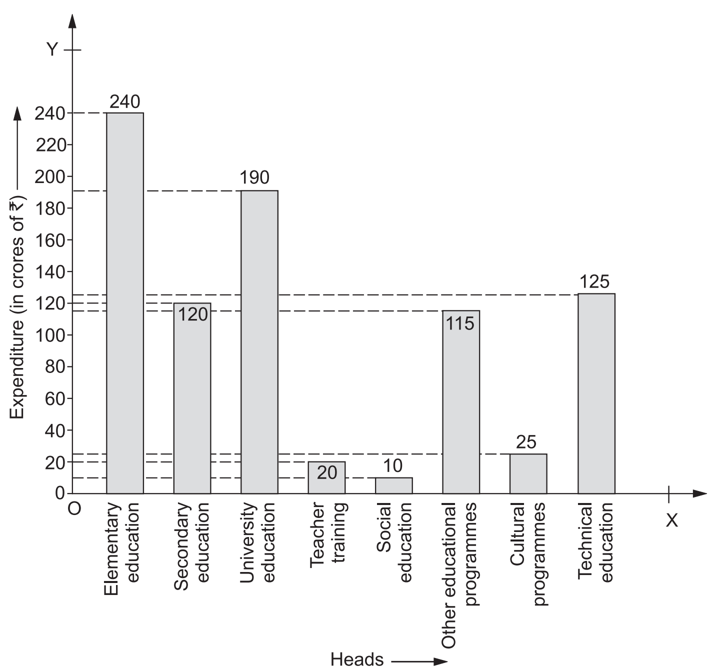
EXAMPLE 5 *Expenditure on education of a country during a five-year period* (2013–17), *in crores of rupees, is given below:*

Head	*Expenditure (in crores of rupees)*
Elementary education	240
Secondary education	120
University education	190
Teacher training	20
Social education	10
Other educational programmes	115
Cultural programmes	25
Technical education	125

Represent the above information by a bar graph.

SOLUTION Take the heads along the x-axis and their corresponding expenditures (in crores of ₹) along the y-axis.

Draw the bars of the same width having their heights proportional to their expenditures, keeping the same space between two consecutive bars, as shown below.



EXAMPLE 6 *A survey conducted by an organisation for the cause of illness and death among the women between the ages* 15–44 *(in years) worldwide, found the following figure (in %):*

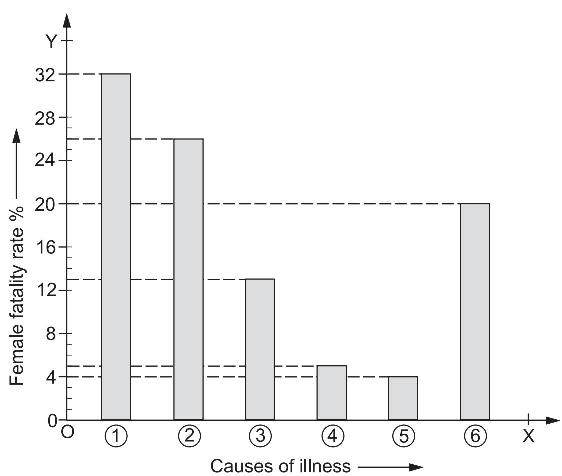
S. No.	*Cause of illness and death*	*Female fatality rate %*
1.	*Reproductive health conditions*	32
2.	*Neuropsychiatric conditions*	26
3.	*Injuries*	13
4.	*Cardiovascular conditions*	5
5.	*Respiratory conditions*	4
6.	*Other causes*	20

(i) Represent the information given above graphically.

(ii) Which condition is the major cause of women's ill health and death worldwide?

SOLUTION Take the causes of illness and death among the women along the x-axis and the female fatality rate % along the y-axis.

Draw the bars of the same width having their heights proportional to female fatality rate %, keeping the same space between two consecutive bars, as shown below.



(i) The required information has been given above graphically.

(ii) Clearly, the reproductive health conditions are the major cause of women's ill health and death worldwide.

EXAMPLE 7 *The following data on the number of girls per thousand boys in different sections of the society is given below:*

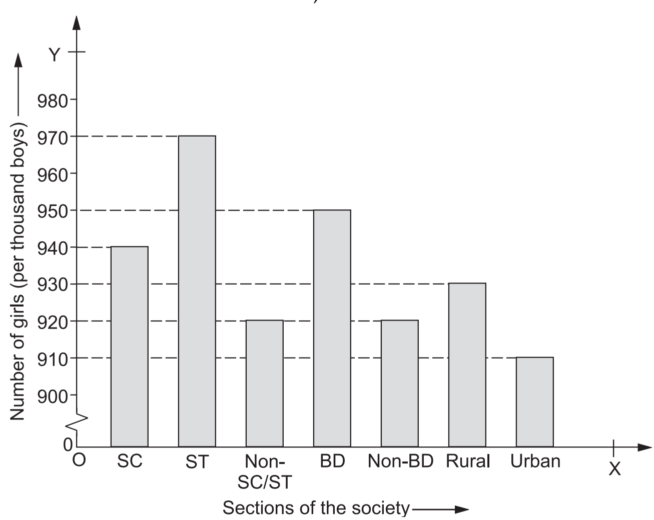
Section of the society	*Number of girls per thousand boys*
Scheduled caste (SC)	940
Scheduled tribe (ST)	970
Non-SC/ST	920
Backward districts (BD)	950
Non-BD	920
Rural	930
Urban	910

(i) Represent the above information by a bar graph.

(ii) Write two conclusions derived from the graph with justification.

SOLUTION We denote the different sections of the society along the x-axis and the corresponding number of girls (per thousand boys) along the y-axis.

Draw the bars of the same width having their heights proportional to the number of girls, keeping the same space between two consecutive bars, as shown below.

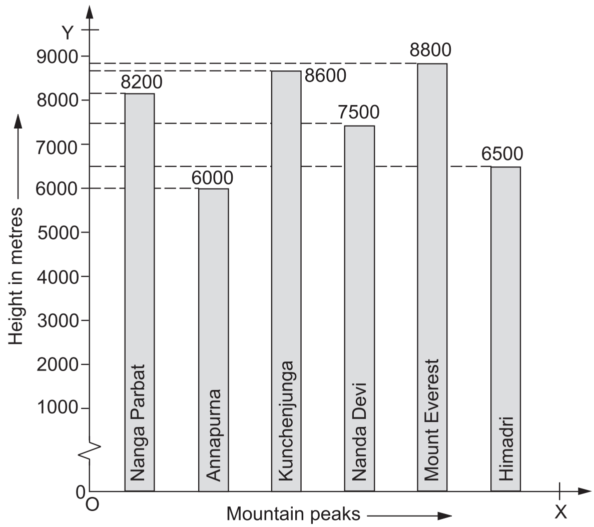


(i) The given information has been represented above graphically.

(ii) From the graph, it is clear that the number of girls per thousand boys, is maximum in schedule tribes and it is minimum in urban.

READING A BAR DIAGRAM

EXAMPLE 8 *Given below is a bar graph showing the heights of six mountain peaks.*



Read the above diagram and answer the following questions:

(i) Which is the highest peak and what is its height?

(ii) Write down the ratio of the heights of the highest peak and the lowest peak.

(iii) Write the heights of the given peaks in ascending order.

(iv) Which peak is the second highest and what is its height?

SOLUTION (i) Clearly, Mount Everest is the highest peak and its height is 8800 m.

(ii) Height of the highest peak = 8800 m.

Height of the lowest peak = 6000 m.

$\therefore$ required ratio = 8800 : 6000 = 22 : 15.

(iii) The heights of the given peaks in ascending order are

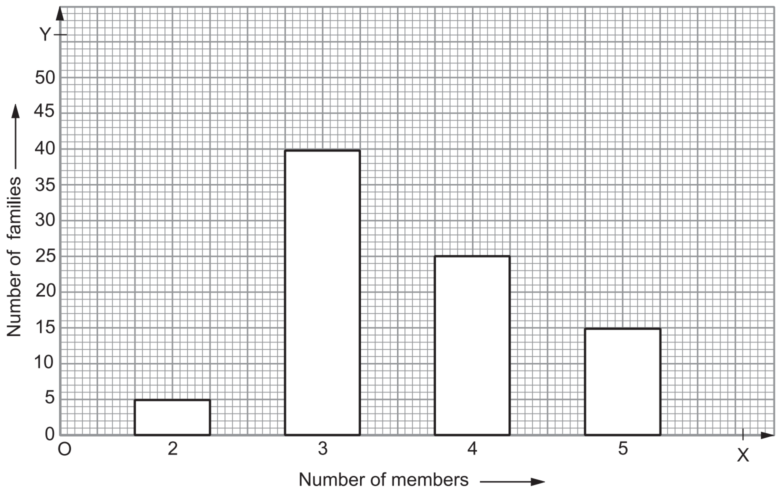
6000 m < 6500 m < 7500 m < 8200 m < 8600 m < 8800 m.

(iv) The height of Kunchenjunga is next to that of Mount Everest.

So, the second highest peak is Kunchenjunga.

Its height is 8600 m.

EXAMPLE 9 *In a survey of 85 families of a colony, the number of members in each family was recorded and the data has been represented by the following bar graph:*



Read the bar graph carefully and answer the following questions:

(i) What information does the bar graph give?

(ii) How many families have 3 members?

(iii) How many people live alone?

(iv) Which type of family is the most common? How many members are there in each family of this kind?

SOLUTION (i) The bar graph shows the number of members in each of the 85 families of a colony.

(ii) Clearly, 40 families have 3 members each.

(iii) Clearly, 5 couples, i.e., 10 people live alone.

(iv) Most common is the family having 3 members.

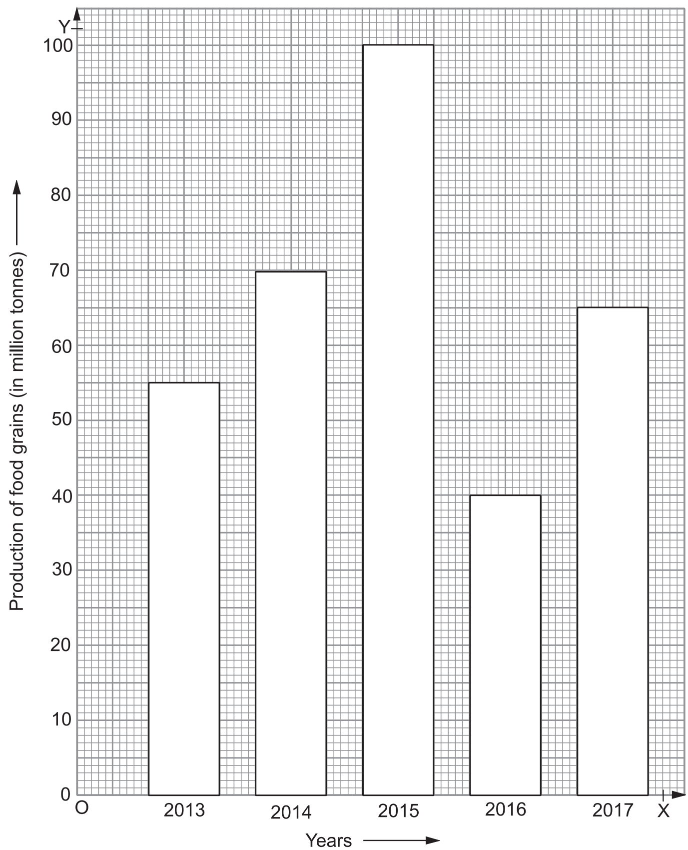
EXAMPLE 10 *Read the given bar graph and answer the questions given below:*

(i) What information is given by the bar graph?

(ii) In which year was the production maximum?

(iii) After which year was there a sudden fall in the production?

(iv) Find the ratio between the maximum production and the minimum production during the given period.



Production of food grains in an Indian state during 5 consecutive years

SOLUTION

(i) The given bar graph shows the annual production (in million tonnes) of food grains in an Indian state during the period from 2013 to 2017.

(ii) It is clear that the bar of maximum height corresponds to the year 2015. So, the production was maximum in that year.

(iii) From the bar graph, we find that there was a sudden fall in the production after the year 2015.

(iv) The maximum production in a year during the given period = 100 million tonnes.

The minimum production in a year during the given period = 40 million tonnes.

$\therefore$ maximum production : minimum production

$= 100 : 40 = 5 : 2.$

EXERCISE 17A

1. The following table shows the number of students participating in various games in a school.

Game	Cricket	Football	Basketball	Tennis
Number of students	27	36	18	12

Draw a bar graph to represent the above data.

2. On a certain day, the temperature in a city was recorded as under:

Time	5 a.m.	8 a.m.	11 a.m.	3 p.m.	6 p.m.
Temperature (in °C)	20	24	26	22	18

Illustrate the data by a bar graph.

3. The approximate velocities of some vehicles are given below:

Name of vehicle	Bicycle	Scooter	Car	Bus	Train
Velocity (in km/hr)	27	45	90	72	63

Draw a bar graph to represent the above data.

4. The following table shows the favourite sports of 250 students of a school. Represent the data by a bar graph.

Sports	Cricket	Football	Tennis	Badminton	Swimming
No. of students	75	35	50	25	65

5. Given below is a table which shows the yearwise strength of a school. Represent this data by a bar graph.

Year	2012–13	2013–14	2014–15	2015–16	2016–17
No. of students	800	975	1100	1400	1625

6. The following table shows the number of scooters sold by a dealer during six consecutive years. Draw a bar graph to represent this data.

Year	2011	2012	2013	2014	2015	2016
Number of scooters sold (in thousands)	16	20	32	36	40	48

7. The air distances of four cities from Delhi (in km) are given below:

City	Kolkata	Mumbai	Chennai	Hyderabad
Distance from Delhi (in km)	1340	1100	1700	1220

Draw a bar graph to represent the above data.

8. The birth rate per thousand in five countries over a period of time is shown below:

Country	China	India	Germany	UK	Sweden
Birth rate per thousand	42	35	14	28	21

Represent the above data by a bar graph.

9. The following table shows the life expectancy (average age to which people live) in various countries in a particular year. Represent the data by a bar graph.

Country	Japan	India	Britain	Ethiopia	Cambodia	UK
Life expectancy (in years)	84	68	80	64	62	73

10. Given below are the seats won by different political parties in the polling outcome of a state assembly elections:

Political party	A	B	C	D	E	F
Seats won	65	52	34	28	10	31

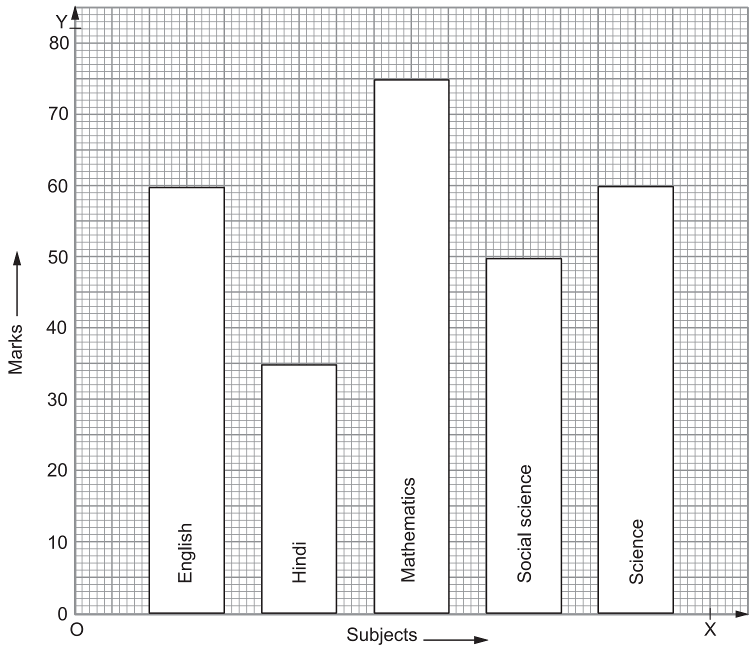
Draw a bar graph to represent the polling results.

11. Various modes of transport used by 1850 students of a school are given below:

School bus	Private bus	Bicycle	Rickshaw	By foot
640	360	490	210	150

Draw a bar graph to represent the above data.

12. Look at the bar graph given below.



Bar graph showing the marks obtained by a student in an examination

Read it carefully and answer the following questions.

(i) What information does the bar graph give?

(ii) In which subject is the student very good?

(iii) In which subject is he poor?

(iv) What is the average of his marks?

ANSWERS (EXERCISE 17A)

12. (i) The bar graph shows the marks obtained by a student in various subjects in an examination.

(ii) The student is very good in mathematics.

(iii) He is poor in Hindi.

(iv) Average marks $= \dfrac{(60 + 35 + 75 + 50 + 60)}{5} = \dfrac{280}{5} = 56.$

HISTOGRAM AND FREQUENCY POLYGON

HISTOGRAM *A histogram is the graphical representation of a frequency distribution (in exclusive form) in the form of rectangles with class intervals as bases and the corresponding frequencies as heights, there being no gap between any two successive rectangles.*

METHOD OF DRAWING A HISTOGRAM

Steps:

(i) Convert the frequency distribution in an exclusive form, if it is in inclusive form.

(ii) Taking suitable scale, mark the class intervals on the x-axis.

(iii) Taking suitable scale, mark the corresponding frequencies on the y-axis.

(iv) Construct rectangles with class intervals as bases and the corresponding frequencies as heights.

CASE 1. HISTOGRAM WHEN FREQUENCY DISTRIBUTION IS IN EXCLUSIVE FORM

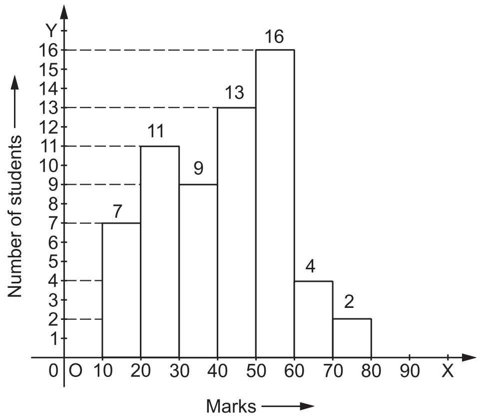
EXAMPLE 1 *Represent the following frequency distribution by means of a histogram:*

Marks	10–20	20–30	30–40	40–50	50–60	60–70	70–80
Number of students (Frequency)	7	11	9	13	16	4	2

SOLUTION Clearly, the given frequency distribution is in exclusive form. We represent the class intervals along the x-axis on a suitable scale and the corresponding frequencies along the y-axis on another suitable scale.

We construct rectangles with class intervals as bases and the corresponding frequencies as heights.

Thus, we obtain a histogram as shown below.



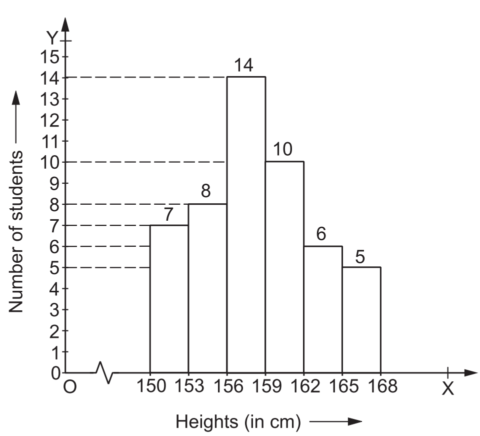
EXAMPLE 2 *Draw a histogram of the following distribution:*

Height (in cm)	150–153	153–156	156–159	159–162	162–165	165–168
Number of students	7	8	14	10	6	5

SOLUTION Clearly, the given frequency distribution is in exclusive form. We represent the class intervals (showing heights in cm) along the x-axis on a suitable scale and the corresponding frequencies (showing the number of students) along the y-axis on another suitable scale.

We construct rectangles with class intervals as bases and the corresponding frequencies as heights.

Thus, we obtain a histogram, as shown below.



EXAMPLE 3 *Depict the following frequency distribution by a histogram:*

Daily wages (in ₹)	500–525	525–550	550–575	575–600	600–625
Number of workers (Frequency)	20	15	25	30	10

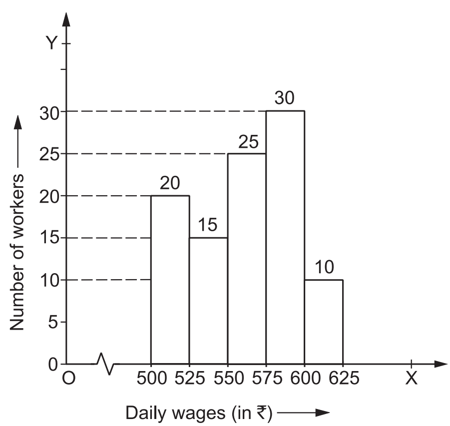
SOLUTION Clearly, the given frequency distribution is in exclusive form. Here, the class intervals represent the daily wages in rupees and the corresponding frequencies represent the number of workers.

We represent the class intervals along the x-axis on a suitable scale and the corresponding frequencies along the y-axis on another suitable scale.

Since the scale along the x-axis starts at ₹ 500, so a kink, i.e. a break ( ), is indicated at the origin.

We construct rectangles with class intervals as bases and the corresponding frequencies as heights.

Thus, we obtain a histogram, as shown below.



CASE 2. HISTOGRAM WHEN FREQUENCY DISTRIBUTION IS IN INCLUSIVE FORM

METHOD When the frequency distribution is in inclusive form, then first we convert it into an exclusive form and then draw the histogram, as shown above.

EXAMPLE 4 *The following table shows the number of literate females in the age group (10–57 years) in a village:*

Age group (in years)	10–17	18–25	26–33	34–41	42–49	50–57	Total
No. of females (Frequency)	300	980	740	580	260	140	3000

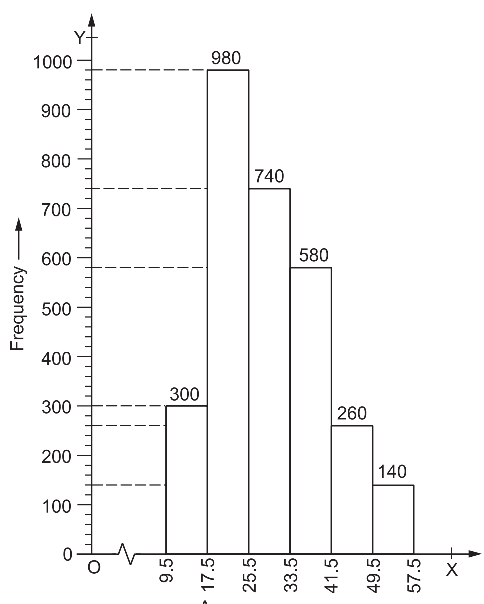
Draw a histogram to represent the above data.

SOLUTION The given frequency distribution is in inclusive form.

So, first we convert it into exclusive form, as shown below.

Age group (in years)	Frequency
9.5–17.5	300
17.5–25.5	980
25.5–33.5	740
33.5–41.5	580
41.5–49.5	260
49.5–57.5	140
Total	3000

We represent the class intervals along the x-axis on a suitable scale and mark frequencies along the y-axis on another suitable scale.



Since reading along the x-axis do not start with 0, we use a kink ( ) at the origin.

We construct rectangles with class intervals as bases and the corresponding frequencies as heights.

Thus, we obtain the histogram as shown just above.

EXAMPLE 5 *The lengths of 62 leaves of a plant are measured in millimetres and the data is represented in the following table:*

Length (in mm*)*	118–126	127–135	136–144	145–153	154–162	163–171	172–180
Number of leaves	8	10	12	17	7	5	3

Draw a histogram to represent the above given data.

SOLUTION The given frequency distribution is in inclusive form.

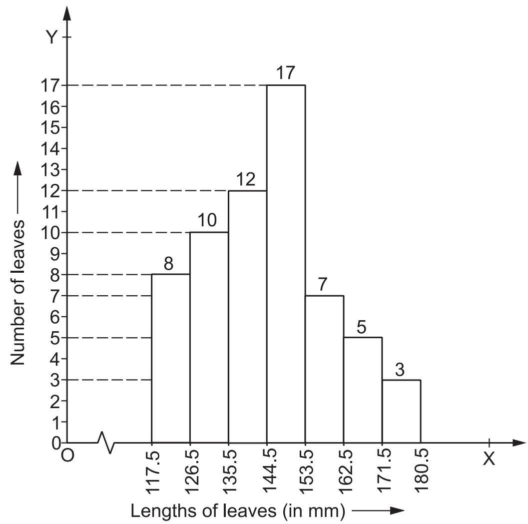
So, we convert it into an exclusive form, as shown below.

Length (in mm)	Number of leaves (Frequency)
117.5–126.5	8
126.5–135.5	10
135.5–144.5	12
144.5–153.5	17
153.5–162.5	7
162.5–171.5	5
171.5–180.5	3

We represent the class intervals along the x-axis, using a suitable scale and mark frequencies along the y-axis, using another suitable scale.

Since we do not start the length from 0, so we use a kink ( ) in the beginning of the x-axis, as shown below.

Now, we obtain the required histogram, as shown below.



CASE 3. HISTOGRAM WHEN CLASS INTERVALS ARE OF UNEQUAL SIZE

In this case, for each class interval, we calculate the adjusted frequency by using the formula:

$$\text{Adjusted frequency} = \frac{\text{minimum class size}}{\text{class size of this class}} \times \text{its frequency.}$$

EXAMPLE 6 *Following is the frequency distribution of the total marks obtained by the students of all sections of a class in an examination:*

Marks	100–150	150–200	200–300	300–500	500–800
Number of students	60	100	100	80	180

Draw a histogram for the above given distribution.

SOLUTION In the given frequency distribution, the class intervals are not of equal width. So, we would make modifications in the heights of rectangles in the histogram in such a way that the ***areas of rectangles are proportional to the frequencies.***

Here, minimum class size = 50.

Height of the rectangle for a given class interval

$$= \left\{\frac{\text{its frequency}}{\text{its width}} \times 50\right\}.$$

For example, the height for class interval 300–500 is

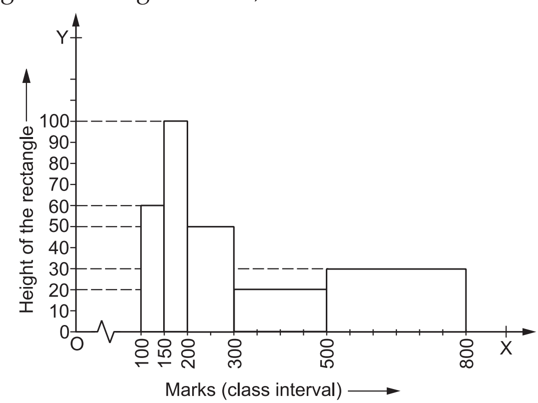
$$\left(\frac{80}{200} \times 50\right) = 20.$$

Thus, we have

Marks (Class interval)	Frequency	Width of the class	Height of the rectangle
100–150	60	50	$\frac{60}{50} \times 50 = 60$
150–200	100	50	$\frac{100}{50} \times 50 = 100$
200–300	100	100	$\frac{100}{100} \times 50 = 50$
300–500	80	200	$\frac{80}{200} \times 50 = 20$
500–800	180	300	$\frac{180}{300} \times 50 = 30$

Since along the x-axis we do not start with 0, so we use a kink ( ) in the beginning.

Now, we draw rectangles with given class intervals along the x-axis on a suitable scale and their corresponding heights along the y-axis on another suitable scale. Thus, we obtain the histogram of the given data, as shown below.



EXAMPLE 7 *A random survey of the number of children of various age groups playing in a park was found as follows:*

Age (in years)	1–2	2–3	3–5	5–7	7–10	10–15	15–17
Number of children	5	3	6	12	9	10	4

Draw a histogram to represent the above given data.

SOLUTION In the given frequency distribution, the class intervals are not of equal width. So, we would make modifications in the heights of the rectangles in the histogram in such a way that the areas of the rectangles are proportional to the frequencies.

Here, minimum class size = 1.

$\therefore$ height of the rectangle for a given class interval

$$= \left(\frac{\text{its frequency}}{\text{its width}} \times 1\right).$$

For example, the height of the rectangle for 10–15 is

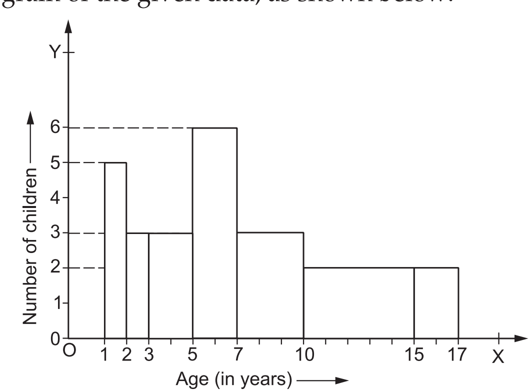
$$\left(\frac{10}{5} \times 1\right) = 2.$$

The height of the rectangle for 7–10 is $\left(\frac{9}{3} \times 1\right) = 3$, etc.

Thus, we have

Age (in years) (Class interval)	Frequency	Width of the class	Height of the rectangle
1–2	5	1	$\left(\frac{5}{1} \times 1\right) = 5$
2–3	3	1	$\left(\frac{3}{1} \times 1\right) = 3$
3–5	6	2	$\left(\frac{6}{2} \times 1\right) = 3$
5–7	12	2	$\left(\frac{12}{2} \times 1\right) = 6$
7–10	9	3	$\left(\frac{9}{3} \times 1\right) = 3$
10–15	10	5	$\left(\frac{10}{5} \times 1\right) = 2$
15–17	4	2	$\left(\frac{4}{2} \times 1\right) = 2$

Now, we draw rectangles with given class intervals along the x-axis on a suitable scale and their corresponding heights along the y-axis on another suitable scale. Thus, we obtain the histogram of the given data, as shown below.



FREQUENCY POLYGON

Let $x_1, x_2, x_3, \ldots, x_n$ be the class marks (i.e., midpoints) of the given frequency distribution and let $f_1, f_2, f_3, \ldots, f_n$ be the corresponding frequencies. We plot the points $(x_1, f_1), (x_2, f_2), (x_3, f_3), \ldots, (x_n, f_n)$ on a graph paper and join these points by line segments. We complete the diagram in the form of a polygon by taking two more classes (called imagined classes), one at the beginning and the other at the end.

This polygon is known as the frequency polygon of the given frequency distribution.

METHOD OF DRAWING A FREQUENCY POLYGON

Steps:

(i) Calculate the class marks $x_1, x_2, x_3, \ldots, x_n$ of the given class intervals.

(ii) Mark $x_1, x_2, x_3, \ldots, x_n$ along the x-axis on some suitable scale.

(iii) Mark the frequencies $f_1, f_2, f_3, \ldots, f_n$ along the y-axis on some suitable scale.

(iv) Join the points $(x_1, f_1), (x_2, f_2), (x_3, f_3), \ldots, (x_n, f_n)$ by line segments.

(v) Take two class intervals each of frequency zero, one at the beginning and the other at the end of the frequency table. Find their class marks. (These classes are imagined classes.)

(vi) Join the midpoint of the first class interval to the midpoint of the imagined class at the beginning. Also, join the midpoint of the last class interval to the midpoint of the imagined class at the end.

The method will become clear from the examples given below.

EXAMPLE 8 *Following table shows a frequency distribution for the speeds of cars passing through a particular spot on a highway:*

Class interval (km/hr)	30–40	40–50	50–60	60–70	70–80	80–90	90–100
Frequency (Number of cars)	3	6	25	65	50	28	14

Draw a histogram and frequency polygon representing the above data.

SOLUTION Adding two more class intervals, namely 20–30 and 100–110, each with frequency 0, we have the frequency distribution table as under:

Class interval	Class mark	Frequency
20–30	25	0
30–40	35	3
40–50	45	6
50–60	55	25
60–70	65	65
70–80	75	50
80–90	85	28
90–100	95	14
100–110	105	0

We represent the class intervals along the x-axis on a suitable scale and the corresponding frequencies along the y-axis on another suitable scale.

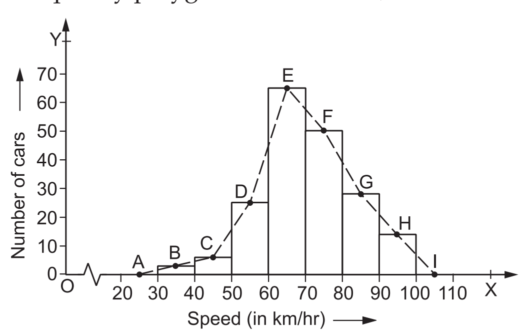
We construct rectangles with class intervals as the bases and the respective frequencies as the heights.

Thus, we obtain the histogram, shown below.

We take the imagined classes 20–30 and 100–110, each with frequency 0.

Now, we plot the points $A(25, 0)$, $B(35, 3)$, $C(45, 6)$, $D(55, 25)$, $E(65, 65)$, $F(75, 50)$, $G(85, 28)$, $H(95, 14)$ and $I(105, 0)$.

Joining these points in pairs successively, we obtain the required frequency polygon $ABCDEFGHI$, as shown below.



EXAMPLE 9 *The daily pocket expenses of* 150 *students in a school are given below:*

Pocket expenses (in ₹)	50–60	60–70	70–80	80–90	90–100	100–110	110–120
Number of students (Frequency)	12	16	25	20	32	27	18

Construct a frequency polygon representing the above data.

SOLUTION We take two imagined classes, namely 40–50 and 120–130, each with frequency 0.

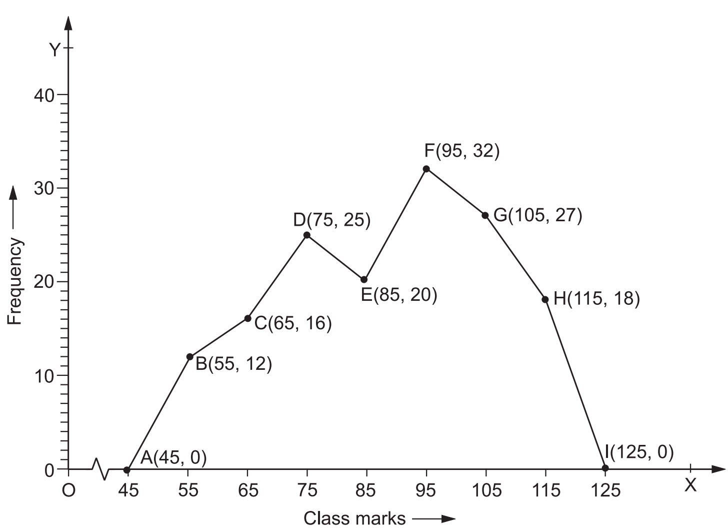
Now, we have the following frequency table:

Class interval (in ₹)	Class mark	Frequency
40–50	45	0
50–60	55	12
60–70	65	16
70–80	75	25
80–90	85	20
90–100	95	32
100–110	105	27
110–120	115	18
120–130	125	0

Taking suitable scale along the x-axis and another along the y-axis, we plot the following points:

$A(45, 0)$, $B(55, 12)$, $C(65, 16)$, $D(75, 25)$, $E(85, 20)$, $F(95, 32)$, $G(105, 27)$, $H(115, 18)$ and $I(125, 0)$.

Join these points successively in pairs to get the required frequency polygon $ABCDEFGHI$, as shown below.



EXAMPLE 10 *Draw the frequency polygon representing the following frequency distribution:*

Class interval	30–34	35–39	40–44	45–49	50–54	55–59
Frequency	12	16	20	8	10	4

SOLUTION Though the given frequency table is in inclusive form, yet we find that class marks in case of inclusive and exclusive forms are the same.

We take the imagined classes 25–29 at the beginning and 60–64 at the end, each with frequency zero.

Thus, we have

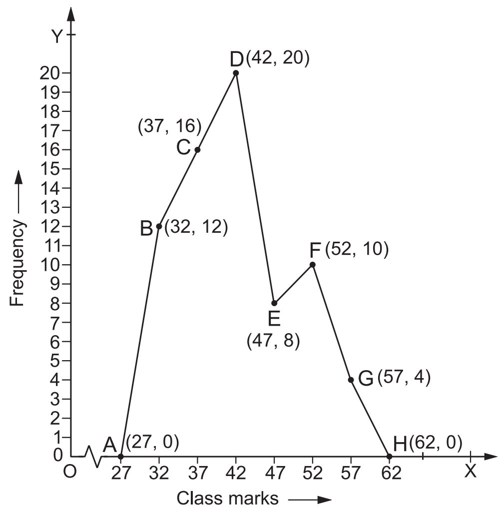
Class interval	Class mark	Frequency
25–29	27	0
30–34	32	12
35–39	37	16
40–44	42	20
45–49	47	8
50–54	52	10
55–59	57	4
60–64	62	0

Along the x-axis, we mark 27, 32, 37, 42, 47, 52, 57 and 62 on a suitable scale.

Along the y-axis, we mark 0, 12, 16, 20, 8, 10, 4 and 0 on another suitable scale.

Plot the points $A(27, 0)$, $B(32, 12)$, $C(37, 16)$, $D(42, 20)$, $E(47, 8)$, $F(52, 10)$, $G(57, 4)$ and $H(62, 0)$.

We draw line segments AB, BC, CD, DE, EF, FG and GH to obtain the required frequency polygon, shown below.



EXAMPLE 11 *The following two tables give the distribution of students of two sections according to the marks obtained by them:*

Section A

Marks	*Frequency*
0–10	3
10–20	9
20–30	17
30–40	12
40–50	9

Section B

Marks	*Frequency*
0–10	5
10–20	19
20–30	15
30–40	10
40–50	1

Represent the marks of both the sections on the same graph by two frequency polygons.

SOLUTION First we obtain the class marks as shown below.

Marks	Class marks	Frequency	
		Section A	Section B
0–10	5	3	5
10–20	15	9	19
20–30	25	17	15
30–40	35	12	10
40–50	45	9	1

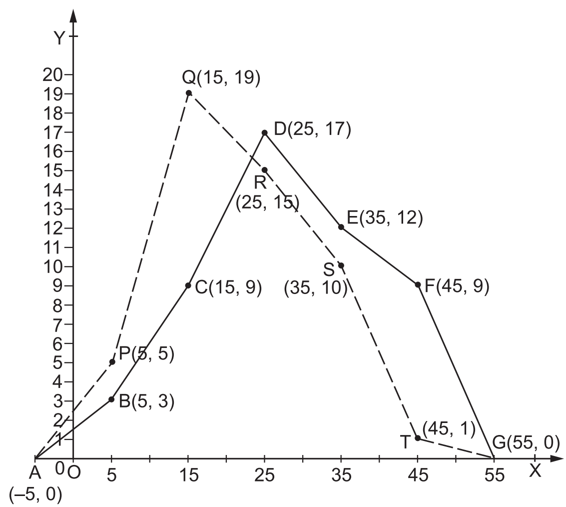
We represent the class marks along the x-axis on a suitable scale and frequencies along the y-axis on another suitable scale.

To obtain the frequency polygon of section A, we plot the points $A(-5, 0)$, $B(5, 3)$, $C(15, 9)$, $D(25, 17)$, $E(35, 12)$, $F(45, 9)$ and $G(55, 0)$.

Now, join these points in pairs successively to obtain the frequency polygon $ABCDEFG$, as shown below.

To obtain the frequency polygon of section B, we plot the points $A(-5, 0)$, $P(5, 5)$, $Q(15, 19)$, $R(25, 15)$, $S(35, 10)$, $T(45, 1)$ and $G(55, 0)$.

Now, join these points successively to obtain the frequency polygon $APQRSTG$, as shown below.



EXERCISE 17B

1. The daily wages of 50 workers in a factory are given below:

Daily wages (in ₹)	340–380	380–420	420–460	460–500	500–540	540–580
Number of workers	16	9	12	2	7	4

Construct a histogram to represent the above frequency distribution.

2. The following table shows the average daily earnings of 40 general stores in a market, during a certain week:

Daily earning (in rupees)	700–750	750–800	800–850	850–900	900–950	950–1000
Number of stores	6	9	2	7	11	5

Draw a histogram to represent the above data.

3. The heights of 75 students in a school are given below:

Height (in cm)	130–136	136–142	142–148	148–154	154–160	160–166
Number of students	9	12	18	23	10	3

Draw a histogram to represent the above data.

4. The following table gives the lifetimes of 400 neon lamps:

Lifetime (in hr)	300–400	400–500	500–600	600–700	700–800	800–900	900–1000
Number of lamps	14	56	60	86	74	62	48

(i) Represent the given information with the help of a histogram.

(ii) How many lamps have a lifetime of more than 700 hours?

5. Draw a histogram for the frequency distribution of the following data:

Class interval	8–13	13–18	18–23	23–28	28–33	33–38	38–43
Frequency	320	780	160	540	260	100	80

6. Construct a histogram for the following frequency distribution:

Class interval	5–12	13–20	21–28	29–36	37–44	45–52
Frequency	6	15	24	18	4	9

7. The following table shows the number of illiterate persons in the age group (10–58 years) in a town:

Age group (in years)	10–16	17–23	24–30	31–37	38–44	45–51	52–58
Number of illiterate persons	175	325	100	150	250	400	525

Draw a histogram to represent the above data.

8. Draw a histogram to represent the following data:

Class interval	10–14	14–20	20–32	32–52	52–80
Frequency	5	6	9	25	21

9. 100 surnames were randomly picked up from a local telephone directory and frequency distribution of the number of letters in the English alphabet in the surnames was found as follows:

Number of letters	1–4	4–6	6–8	8–12	12–20
Number of surnames	6	30	44	16	4

(i) Draw a histogram to depict the given information.

(ii) Write the class interval in which the maximum number of surnames lie.

10. Draw a histogram to represent the following information:

Class interval	5–10	10–15	15–25	25–45	45–75
Frequency	6	12	10	8	18

11. Draw a histogram to represent the following information:

Marks	0–10	10–30	30–45	45–50	50–60
Number of students	8	32	18	10	6

12. In a study of diabetic patients in a village, the following observations were noted.

Age in years	10–20	20–30	30–40	40–50	50–60	60–70
Number of patients	2	5	12	19	9	4

Represent the above data by a frequency polygon.

13. Draw a frequency polygon for the following frequency distribution:

Class interval	1–10	11–20	21–30	31–40	41–50	51–60
Frequency	8	3	6	12	2	7

14. The ages (in years) of 360 patients treated in a hospital on a particular day are given below:

Age in years	10–20	20–30	30–40	40–50	50–60	60–70
Number of patients	90	40	60	20	120	30

Draw a histogram and a frequency polygon on the same graph to represent the above data.

15. Draw a histogram and the frequency polygon from the following data:

Class interval	20–25	25–30	30–35	35–40	40–45	45–50
Frequency	30	24	52	28	46	10

16. Draw a histogram for the following data:

Class interval	600–640	640–680	680–720	720–760	760–800	800–840
Frequency	18	45	153	288	171	63

Using this histogram, draw the frequency polygon on the same graph.

HINTS TO SOME SELECTED QUESTIONS

4. (ii) Number of lamps having a lifetime of more than 700 hours = $(74 + 62 + 48) = 184$.

6. Convert the given inclusive form into an exclusive form by taking intervals (4.5–12.5), (12.5–20.5), (20.5–28.5), (28.5–36.5), (36.5–44.5), and (44.5–52.5).

8. The class sizes are 4, 6, 12, 20 and 28.
Minimum class size = 4.

$$\text{Adjusted frequency} = \frac{\text{minimum class size}}{\text{class size of this class}} \times \text{its frequency}.$$

So, the adjusted frequencies of the given class intervals are

$$\left(\frac{4}{4}\times 5\right) = 5, \left(\frac{4}{6}\times 6\right) = 4, \left(\frac{4}{12}\times 9\right) = 3, \left(\frac{4}{20}\times 25\right) = 5, \left(\frac{4}{28}\times 21\right) = 3.$$

9. The class sizes of the given frequency distribution are

$$(4 - 1) = 3, (6 - 4) = 2, (8 - 6) = 2, (12 - 8) = 4 \text{ and } (20 - 12) = 8.$$

Minimum class size = 2.
Adjusted frequencies of the given class intervals are

$$\left(\frac{2}{3}\times 6\right) = 4, \left(\frac{2}{2}\times 30\right) = 30, \left(\frac{2}{2}\times 44\right) = 44, \left(\frac{2}{4}\times 16\right) = 8, \left(\frac{2}{8}\times 4\right) = 1.$$

(ii) Maximum surnames lie in class interval 6–8.

10. The class sizes of the given class intervals are

$$(10 - 5) = 5, (15 - 10) = 5, (25 - 15) = 10, (45 - 25) = 20 \text{ and } (75 - 45) = 30.$$

Minimum class size = 5.
Adjusted frequencies of the given class intervals are

$$\left(\frac{5}{5}\times 6\right) = 6, \left(\frac{5}{5}\times 12\right) = 12, \left(\frac{5}{10}\times 10\right) = 5, \left(\frac{5}{20}\times 8\right) = 2, \left(\frac{5}{15}\times 15\right) = 5, \left(\frac{5}{30}\times 18\right) = 3.$$

✿

18 Mean, Median and Mode of Ungrouped Data

ARITHMETIC MEAN

ARITHMETIC MEAN The average of a given set of numbers is called the *arithmetic mean*, or simply the *mean*, of the given numbers.

$$\text{Mean} = \frac{\text{sum of observations}}{\text{number of observations}}.$$

Thus, the mean of n observations $x_1, x_2, \ldots, x_n$ is given by

$$\bar{x} = \frac{(x_1 + x_2 + \ldots + x_n)}{n} = \frac{\sum_{i=1}^{n} x_i}{n},$$

where the symbol Σ, called *sigma*, stands for summation, and we write,

$$(x_1 + x_2 + \ldots + x_n) = \sum_{i=1}^{n} x_i.$$

SOLVED EXAMPLES

EXAMPLE 1 *The heights of five players are* 153 cm, 140 cm, 148 cm, 150 cm *and* 154 cm *respectively. Find the mean height of the five players.*

SOLUTION
$$\text{Mean height} = \frac{\text{sum of the heights of the players}}{\text{number of players}}$$

$$= \frac{(153 + 140 + 148 + 150 + 154)}{5} \text{ cm}$$

$$= \frac{745}{5} \text{ cm} = 149 \text{ cm}.$$

Hence, the mean height of the five players is 149 cm.

EXAMPLE 2 *Find the mean of the first six multiples of* 3.

SOLUTION The first six multiples of 3 are 3, 6, 9, 12, 15 and 18.

$$\therefore \quad \text{their mean} = \frac{(3 + 6 + 9 + 12 + 15 + 18)}{6} = \frac{63}{6} = 10\frac{1}{2}.$$

EXAMPLE 3 *Find the mean of first eight prime numbers.*

SOLUTION The first eight prime numbers are 2, 3, 5, 7, 11, 13, 17, 19.

$$\therefore \quad \text{their mean} = \frac{(2+3+5+7+11+13+17+19)}{8}$$

$$= \frac{77}{8} = 9\frac{5}{8}.$$

EXAMPLE 4 *If the mean of* 6, 8, 9, x, 13 *is* 10, *find the value of* x.

SOLUTION Mean of the given numbers $= \frac{(6+8+9+x+13)}{5} = \frac{(36+x)}{5}$.

But, mean = 10 (given).

$$\therefore \quad \frac{36+x}{5} = 10 \Rightarrow 36+x = 50$$

$$\Rightarrow \quad x = (50-36) = 14.$$

Hence, the value of x is 14.

EXAMPLE 5 *If the mean of the observations* x, $x+3$, $x+5$, $x+7$, $x+10$ *is* 9, *find the mean of the last three observations.*

SOLUTION Mean of the given five observations

$$= \frac{x+(x+3)+(x+5)+(x+7)+(x+10)}{5}$$

$$= \frac{(5x+25)}{5} = \frac{5(x+5)}{5} = (x+5).$$

But, mean of these observations is 9 (given).

$$\therefore \quad x+5 = 9 \Rightarrow x = 9-5 = 4.$$

So, the last three observations are $(4+5), (4+7), (4+10)$, i.e., 9, 11, 14.

Mean of these three observations $= \frac{(9+11+14)}{3} = \frac{34}{3} = 11\frac{1}{3}$.

EXAMPLE 6 *The mean of five numbers is* 30. *If one number is excluded, their mean becomes* 28. *Find the excluded number.*

SOLUTION Mean of the given five numbers = 30.

Sum of these five numbers = $(30 \times 5) = 150$.

Mean of the remaining four numbers = 28.

Sum of these four numbers = $(28 \times 4) = 112$.

Excluded number = $(150 - 112) = 38$.

EXAMPLE 7 *The mean of the heights of six girls is* 148 cm. *If the individual heights of five of them are* 150 cm, 154 cm, 146 cm, 142 cm *and* 145 cm, *find the height of the sixth girl.*

SOLUTION Mean of the heights of six girls = 148 cm.

Sum of the heights of six girls = (148×6) cm = 888 cm.

Sum of the heights of the five girls

$$= (150 + 154 + 146 + 142 + 145) \text{ cm} = 737 \text{ cm}.$$

Height of the sixth girl = $(888 - 737)$ cm = 151 cm.

Hence, the height of the sixth girl is 151 cm.

EXAMPLE 8 *The mean of* 40 *numbers was found to be* 38. *Later on, it was detected that a number* 56 *was misread as* 36. *Find the correct mean of the given numbers.*

SOLUTION Calculated mean of 40 numbers = 38.

$\therefore$ calculated sum of these numbers = $(38 \times 40) = 1520$.

Correct sum of these numbers

$$= [1520 - (\text{wrong item}) + (\text{correct item})]$$

$$= (1520 - 36 + 56) = 1540.$$

$\therefore$ the correct mean $= \dfrac{1540}{40} = 38.5$.

EXAMPLE 9 *The mean of* 25 *observations is* 36. *Out of these observations, if the mean of first* 13 *observations is* 32 *and that of the last* 13 *observations is* 40, *find the* 13*th observation.*

SOLUTION Mean of 25 observations = 36.

Sum of 25 observations = $(36 \times 25) = 900$.

Mean of first 13 observations = 32.

Sum of first 13 observations = $(32 \times 13) = 416$.

Mean of last 13 observations = 40.

Sum of last 13 observations = $(40 \times 13) = 520$.

13th observation

$$= (\text{sum of first 13 observations}) + (\text{sum of last 13 observations}) - (\text{sum of 25 observations})$$

$$= (416 + 520) - (900) = 936 - 900 = 36.$$

Hence, the 13th observation is 36.

EXAMPLE 10 *The mean of* 100 *items was found to be* 64. *Later on it was discovered that two items were misread as* 26 *and* 9 *instead of* 36 *and* 90 *respectively. Find the correct mean.*

SOLUTION Mean of 100 items, as calculated = 64.

Sum of 100 items, as calculated $= (64 \times 100) = 6400$.

Correct sum of 100 items

$$= \{6400 - (\text{sum of wrong items}) + (\text{sum of correct items})\}$$
$$= \{6400 - (26 + 9) + (36 + 90)\}$$
$$= (6400 - 35 + 126) = (6526 - 35) = 6491.$$

Correct mean $= \left(\dfrac{6491}{100}\right) = 64.91$.

EXAMPLE 11 *The mean weight of a class of* 35 *students is* 45 kg. *If the weight of the teacher be included, the mean weight increases by* 500 g. *Find the weight of the teacher.*

SOLUTION Mean weight of 35 students = 45 kg.

Total weight of 35 students $= (45 \times 35)$ kg = 1575 kg.

Mean weight of 35 students and the teacher

$$= (45 + 0.5) \text{ kg} = 45.5 \text{ kg}.$$

Total weight of 35 students and the teacher

$$= (45.5 \times 36) \text{ kg} = 1638 \text{ kg}.$$

Weight of the teacher $= (1638 - 1575)$ kg = 63 kg.

Hence, the weight of the teacher is 63 kg.

EXAMPLE 12 *The average temperature of Monday, Tuesday and Wednesday was* 40 °C. *The average temperature of Tuesday, Wednesday and Thursday was* 41 °C. *If the temperature on Thursday was* 42 °C, *what was the temperature on Monday?*

SOLUTION Sum of temperatures on Monday, Tuesday and Wednesday

$$= (40 \times 3)\,°\text{C} = 120\,°\text{C}. \qquad \ldots \text{(i)}$$

Sum of temperatures on Tuesday, Wednesday and Thursday

$$= (41 \times 3)\,°\text{C} = 123\,°\text{C}. \qquad \ldots \text{(ii)}$$

Temperature on Thursday = 42 °C. … (iii)

$\therefore$ the temperature on Monday $= (120 + 42 - 123)\,°\text{C} = 39\,°\text{C}$

[by {(i) + (iii) – (ii)}].

Hence, the temperature on Monday was 39 °C.

EXAMPLE 13 *A cricketer has a mean score of* 58 *runs in nine innings. Find out how many runs are to be scored by him in the tenth innings to raise the mean score to* 61.

SOLUTION Mean score of 9 innings = 58 runs.

Total score of 9 innings = (58×9) runs = 522 runs.

Required mean score of 10 innings = 61 runs.

Required total score of 10 innings = (61×10) runs = 610 runs.

Number of runs to be scored in the 10th innings

= (total score of 10 innings) – (total score of 9 innings)

$= 610 - 522 = 88.$

Hence, the number of runs to be scored in the 10th innings = 88.

EXAMPLE 14 *A batsman in his* 12*th innings makes a score of* 63 *runs and thereby increases his average score by* 2. *What is his average after the* 12*th innings?*

SOLUTION Let the average score of 12 innings be x.

Then, the average score of 11 innings = $(x-2)$.

Total score of 12 innings = $12x$.

Total score of 11 innings = $11(x-2) = (11x-22)$.

Score of the 12th innings

= (total score of 12 innings) – (total score of 11 innings)

$= [12x - (11x - 22)] = (x + 22)$.

$\therefore \quad x + 22 = 63 \quad \Rightarrow \quad x = 41.$

Hence, the average score after the 12th innings is 41.

EXAMPLE 15 *A train travels between two stations A and B. While going from A to B, its average speed is* 80 km/hr, *and when coming back from B to A, its average speed is* 120 km/hr. *Find the average speed of the train during the whole journey.*

SOLUTION Let the distance between A and B be x km.

Average speed while going from A to B = 80 km/hr.

Time taken to cover x km from A to B = $\left(\frac{x}{80}\right)$ hours.

Average speed while coming from B to A = 120 km/hr.

Time taken to cover x km from B to A = $\left(\frac{x}{120}\right)$ hours.

Total time taken to cover $2x$ km = $\left(\frac{x}{80} + \frac{x}{120}\right)$ hours = $\left(\frac{x}{48}\right)$ hours.

Average speed during the whole journey

$$= \frac{\text{distance}}{\text{time}} = \frac{2x}{(x/48)} \text{ km/hr} = 96 \text{ km/hr}.$$

Hence, the average speed of the train during the whole journey is 96 km/hr.

PROPERTIES OF ARITHMETIC MEAN

THEOREM 1 *If $\bar{x}$ is the arithmetic mean of n observations $x_1, x_2, x_3, \ldots, x_n$ then* $(x_1 - \bar{x}) + (x_2 - \bar{x}) + (x_3 - \bar{x}) + \ldots + (x_n - \bar{x}) = 0$.

PROOF We know that

$$\bar{x} = \frac{(x_1 + x_2 + x_3 + \ldots + x_n)}{n} \Rightarrow (x_1 + x_2 + x_3 + \ldots + x_n) = n\bar{x}. \quad \ldots \text{(i)}$$

$$\therefore \quad (x_1 - \bar{x}) + (x_2 - \bar{x}) + (x_3 - \bar{x}) + \ldots + (x_n - \bar{x})$$
$$= (x_1 + x_2 + x_3 + \ldots + x_n) - n\bar{x}$$
$$= (n\bar{x} - n\bar{x}) = 0 \quad \text{[using (i)]}.$$

Hence, $(x_1 - \bar{x}) + (x_2 - \bar{x}) + (x_3 - \bar{x}) + \ldots + (x_n - \bar{x}) = 0$.

THEOREM 2 *The mean of n observations $x_1, x_2, \ldots, x_n$ is $\bar{x}$. If each observation is increased by p, the mean of the new observations is $(\bar{x} + p)$.*

PROOF $$\bar{x} = \frac{(x_1 + x_2 + \ldots + x_n)}{n} \Rightarrow (x_1 + x_2 + \ldots + x_n) = n\bar{x}. \quad \ldots \text{(i)}$$

Mean of $(x_1 + p), (x_2 + p), \ldots, (x_n + p)$

$$= \frac{(x_1 + p) + (x_2 + p) + \ldots + (x_n + p)}{n} = \frac{(x_1 + x_2 + \ldots + x_n) + np}{n}$$
$$= \frac{n\bar{x} + np}{n} \quad \text{[using (i)]}$$
$$= \frac{n(\bar{x} + p)}{n} = (\bar{x} + p).$$

Hence, the mean of the new observations is $(\bar{x} + p)$.

THEOREM 3 *The mean of n observations $x_1, x_2, \ldots, x_n$ is $\bar{x}$. If each observation is decreased by p, the mean of the new observations is $(\bar{x} - p)$.*

PROOF Left as an exercise.

THEOREM 4 *The mean of n observations $x_1, x_2, \ldots, x_n$ is $\bar{x}$. If each observation is multiplied by a nonzero number p, the mean of the new observations is $p\bar{x}$.*

PROOF $$\bar{x} = \frac{(x_1 + x_2 + \ldots + x_n)}{n} \Rightarrow (x_1 + x_2 + \ldots + x_n) = n\bar{x}. \quad \ldots \text{(i)}$$

Mean of $px_1, px_2, \ldots, px_n$

$$= \frac{px_1 + px_2 + \ldots + px_n}{n} = \frac{p(x_1 + x_2 + \ldots + x_n)}{n} = \frac{p(n\overline{x})}{n} \quad \text{[using (i)].}$$

$= p\overline{x}.$

Hence, the mean of the new observations is $p\overline{x}$.

THEOREM 5 *The mean of n observations $x_1, x_2, \ldots, x_n$ is $\overline{x}$. If each observation is divided by a nonzero number p, the mean of the new observations is $\left(\frac{\overline{x}}{p}\right)$.*

PROOF $\overline{x} = \dfrac{(x_1 + x_2 + \ldots + x_n)}{n} \Rightarrow (x_1 + x_2 + \ldots + x_n) = n\overline{x}.$ … (i)

Mean of $\left(\frac{x_1}{p}\right), \left(\frac{x_2}{p}\right), \ldots, \left(\frac{x_n}{p}\right)$

$$= \frac{1}{n} \cdot \left(\frac{x_1}{p} + \frac{x_2}{p} + \ldots + \frac{x_n}{p}\right) = \frac{(x_1 + x_2 + \ldots + x_n)}{np} = \frac{n\overline{x}}{np} \quad \text{[using (i)].}$$

$= \left(\frac{\overline{x}}{p}\right).$

SOME MORE EXAMPLES

EXAMPLE 16 *The mean of 8 numbers is 25. If 5 is subtracted from each number, what will be the new mean?*

SOLUTION Let the given numbers be $x_1, x_2, \ldots, x_8$.

Then, the mean of these numbers $= \dfrac{(x_1 + x_2 + \ldots + x_8)}{8}$.

$\therefore \quad \dfrac{x_1 + x_2 + \ldots + x_8}{8} = 25 \Rightarrow (x_1 + x_2 + \ldots + x_8) = 200.$ … (i)

The new numbers are $(x_1 - 5), (x_2 - 5), \ldots, (x_8 - 5)$.

$$\text{Mean of the new numbers} = \frac{(x_1 - 5) + (x_2 - 5) + \ldots + (x_8 - 5)}{8}$$

$$= \frac{(x_1 + x_2 + \ldots + x_8) - 40}{8} = \frac{200 - 40}{8}$$

[using (i)].

$$= \frac{160}{8} = 20.$$

Hence, the new mean is 20.

EXAMPLE 17 *The mean of 16 numbers is 8. If 2 is added to every number, what will be the new mean?*

SOLUTION Let the given numbers be $x_1, x_2, x_3, \ldots, x_{16}$.

Then, the mean of these numbers $= \dfrac{x_1 + x_2 + x_3 + \ldots + x_{16}}{16}$.

$$\therefore \quad \frac{x_1 + x_2 + x_3 + \ldots + x_{16}}{16} = 8$$

$$\Rightarrow \quad (x_1 + x_2 + x_3 + \ldots + x_{16}) = 128. \qquad \ldots \text{(i)}$$

The new numbers are $(x_1 + 2), (x_2 + 2), (x_3 + 2), \ldots, (x_{16} + 2)$.

Mean of the new numbers

$$= \frac{(x_1 + 2) + (x_2 + 2) + (x_3 + 2) + \ldots + (x_{16} + 2)}{16}$$

$$= \frac{(x_1 + x_2 + x_3 + \ldots + x_{16}) + 32}{16} = \frac{(128 + 32)}{16} \quad \text{[using (i)]}.$$

$$= \frac{160}{16} = 10.$$

Hence, the new mean is 10.

EXAMPLE 18 *There are 50 numbers. Each number is subtracted from 53 and the mean of the numbers so obtained is found to be –3.5. Find the mean of the given numbers.*

SOLUTION Let the given numbers be $x_1, x_2, x_3, \ldots, x_{50}$. Then,

$$\frac{(53 - x_1) + (53 - x_2) + \ldots + (53 - x_{50})}{50} = -3.5$$

$$\Rightarrow \quad (53 + 53 + \ldots \text{ to 50 times}) - (x_1 + x_2 + \ldots + x_{50}) = (-3.5 \times 50)$$

$$\Rightarrow \quad (x_1 + x_2 + \ldots + x_{50}) = (53 \times 50) + (3.5 \times 50) = (56.5 \times 50)$$

$$\Rightarrow \quad \frac{(x_1 + x_2 + \ldots + x_{50})}{50} = 56.5$$

$\Rightarrow$ mean of the given numbers = 56.5.

Hence, the mean of the given numbers is 56.5.

EXERCISE 18A

1. Find the mean of:

(i) the first eight natural numbers

(ii) the first ten odd numbers

(iii) the first seven multiples of 5

(iv) all the factors of 20

(v) all prime numbers between 50 and 80.

2. The number of children in 10 families of a locality are

2, 4, 3, 4, 2, 0, 3, 5, 1, 6.

Find the mean number of children per family.

3. The following are the numbers of books issued in a school library during a week:

   105, 216, 322, 167, 273, 405 and 346.

   Find the average number of books issued per day.

4. The daily minimum temperature recorded (in °F) at a place during six days of a week was as under:

Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
35.5	30.8	27.3	32.1	23.8	29.9

   Find the mean temperature.

5. If the mean of five observations x, $x+2$, $x+4$, $x+6$, $x+8$ is 13, find the value of x and hence find the mean of the last three observations.

6. The mean weight of 6 boys in a group is 48 kg. The individual weights of five of them are 51 kg, 45 kg, 49 kg, 46 kg and 44 kg. Find the weight of the sixth boy.

7. The mean of the marks scored by 50 students was found to be 39. Later on it was discovered that a score of 43 was misread as 23. Find the correct mean.

8. The mean of 24 numbers is 35. If 3 is added to each number, what will be the new mean?

9. The mean of 20 numbers is 43. If 6 is subtracted from each of the numbers, what will be the new mean?

10. The mean of 15 numbers is 27. If each number is multiplied by 4, what will be the mean of the new numbers?

11. The mean of 12 numbers is 40. If each number is divided by 8, what will be the mean of the new numbers?

12. The mean of 20 numbers is 18. If 3 is added to each of the first ten numbers, find the mean of the new set of 20 numbers.

13. The mean of six numbers is 23. If one of the numbers is excluded, the mean of the remaining numbers is 20. Find the excluded number.

14. The average height of 30 boys was calculated to be 150 cm. It was detected later that one value of 165 cm was wrongly copied as 135 cm for the computation of the mean. Find the correct mean.

15. The mean weight of a class of 34 students is 46.5 kg. If the weight of the teacher is included, the mean rises by 500 g. Find the weight of the teacher.

16. The mean weight of a class of 36 students is 41 kg. If one of the students leaves the class then the mean is decreased by 200 g. Find the weight of the student who left.

17. The average weight of a class of 39 students is 40 kg. When a new student is admitted to the class, the average decreases by 200 g. Find the weight of the new student.

18. The average weight of 10 oarsmen in a boat is increased by 1.5 kg when one of the crew who weighs 58 kg is replaced by a new man. Find the weight of the new man.

19. The mean of 8 numbers is 35. If a number is excluded then the mean is reduced by 3. Find the excluded number.

20. The mean of 150 items was found to be 60. Later on, it was discovered that the values of two items were misread as 52 and 8 instead of 152 and 88 respectively. Find the correct mean.

21. The mean of 31 results is 60. If the mean of the first 16 results is 58 and that of the last 16 results is 62, find the 16th result.

22. The mean of 11 numbers is 42. If the mean of the first 6 numbers is 37 and that of the last 6 numbers is 46, find the 6th number.

23. The mean weight of 25 students of a class is 52 kg. If the mean weight of the first 13 students of the class is 48 kg and that of the last 13 students is 55 kg, find the weight of the 13th student.

24. The mean score of 25 observations is 80 and the mean score of another 55 observations is 60. Determine the mean score of the whole set of observations.

25. Arun scored 36 marks in English, 44 marks in Hindi, 75 marks in mathematics and x marks in science. If he has secured an average of 50 marks, find the value of x.

26. A ship sails out to an island at the rate of 15 km/hr and sails back to the starting point at 10 km/hr. Find the average sailing speed for the whole journey.

27. There are 50 students in a class, of which 40 are boys. The average weight of the class is 44 kg and that of the girls is 40 kg. Find the average weight of the boys.

28. The aggregate monthly expenditure of a family was ₹ 18720 during the first 3 months, ₹ 20340 during the next 4 months and ₹ 21708 during the last 5 months of a year. If the total savings during the year be ₹ 35340 find the average monthly income of the family.

29. The average weekly payment to 75 workers in a factory is ₹ 5680. The mean weekly payment to 25 of them is ₹ 5400 and that of 30 others is ₹ 5700. Find the mean weekly payment of the remaining workers.

30. The mean marks (out of 100) of boys and girls in an examination are 70 and 73 respectively. If the mean marks of all the students in that examination is 71, find the ratio of the number of boys to the number of girls.

31. The average monthly salary of 20 workers in an office is ₹ 45900. If the manager's salary is added, the average salary becomes ₹ 49200 per month. What's manager's monthly salary?

ANSWERS (EXERCISE 18A)

1. (i) 4.5 (ii) 10 (iii) 20 (iv) 7 (v) $66\frac{1}{7}$ **2.** 3 **3.** 262

4. 29.9 °F	**5.** $x = 9$, mean = 15		**6.** 53 kg	**7.** 39.4
8. 38	**9.** 37	**10.** 108	**11.** 5	**12.** 19.5
13. 38	**14.** 151 cm	**15.** 64 kg	**16.** 48 kg	**17.** 32 kg
18. 73 kg	**19.** 56	**20.** 61.2	**21.** 60	**22.** 36
23. 39 kg	**24.** 66.25	**25.** 45	**26.** 12 km/hr	**27.** 45 kg
28. ₹ 23450	**29.** ₹ 6000	**30.** 2 : 1	**31.** ₹ 115200	

HINTS TO SOME SELECTED QUESTIONS

1. (v) Required mean $= \dfrac{(53+59+61+67+71+73+79)}{7} = \dfrac{463}{7} = 66\dfrac{1}{7}$.

12. New mean $= \dfrac{(20\times 18)+(3\times 10)}{20} = \dfrac{390}{20} = \dfrac{39}{2} = 19.5$.

15. Total weight of 34 students = (46.5×34) kg = 1581 kg.

Total weight of (34 students + 1 teacher) = (47×35) kg = 1645 kg.

16. Total weight of 36 students = (41×36) kg = 1476 kg.

New mean = $(41-0.2)$ kg = 40.8 kg.

Total weight of 35 students = (40.8×35) kg = 1428 kg.

17. Total weight of 39 students = (40×39) kg = 1560 kg.

New mean = $(40-0.2)$ kg = 39.8 kg.

Total weight of 40 students = (39.8×40) kg = 1592 kg.

18. Total weight increased = (1.5×10) kg = 15 kg.

Weight of the new man = $(58+15)$ kg = 73 kg.

26. Let the distance of one side journey be x km. Then,

total distance covered = $2x$ km.

Total time taken $= \left(\dfrac{x}{15}+\dfrac{x}{10}\right)$ hr $= \left(\dfrac{5x}{30}\right)$ hr $= \left(\dfrac{x}{6}\right)$ hr.

Average speed for the whole journey $= \left\{\dfrac{2x}{(x/6)}\right\}$ km/hr = 12 km/hr.

28. Total yearly income

$= ₹\,(18720 \times 3 + 20340 \times 4 + 21708 \times 5 + 35340)$

$= ₹\,(56160 + 81360 + 108540 + 35340) = ₹\,281400.$

29. Total weekly payment to remaining 20 workers

$= ₹\,[(5680 \times 75) - (5400 \times 25 + 5700 \times 30)].$

30. Let the required ratio be $x : 1$. Then,

$70x + 73 \times 1 = 71 \times (x + 1)$

$\Rightarrow \quad 70x + 73 = 71x + 71 \Rightarrow x = 2.$

Hence, the ratio of boys and girls is 2 : 1.

31. Manager's monthly salary

$= ₹\,(49200 \times 21 - 45900 \times 20) = ₹\,(1033200 - 918000) = ₹\,115200.$

MEAN FOR AN UNGROUPED FREQUENCY DISTRIBUTION (DIRECT METHOD)

Let n observations consist of values $x_1, x_2, \ldots, x_n$ of a variable x, occurring with frequencies $f_1, f_2, \ldots, f_n$ respectively. Then, the mean, $\overline{x}$, of these observations is given by

$$\overline{x} = \frac{(f_1x_1 + f_2x_2 + \ldots + f_nx_n)}{(f_1 + f_2 + \ldots + f_n)} = \frac{\Sigma(f_i \times x_i)}{\Sigma f_i}.$$

SOLVED EXAMPLES

EXAMPLE 1 *The following table shows the weights of 12 persons of a health club:*

Weight (in kg)	67	70	72	73	75
Number of persons	4	3	2	2	1

Find the mean weight per person.

SOLUTION For calculating the mean, we prepare the table given below:

Weight (in kg) x_i	Number of persons f_i	$(f_i \times x_i)$
67	4	268
70	3	210
72	2	144
73	2	146
75	1	75
	$\Sigma f_i = 12$	$\Sigma(f_i \times x_i) = 843$

$\therefore \quad \text{mean, } \bar{x} = \dfrac{\Sigma(f_i \times x_i)}{\Sigma f_i} = \dfrac{843}{12} = 70.25.$

Hence, the mean weight is 70.250 kg.

EXAMPLE 2 *The heights of* 100 *plants in a garden are given below:*

Height (in cm*)* x_i	60	64	68	72	76
Number of plants f_i	5	16	40	30	9

Find the mean height per plant.

SOLUTION For calculating the mean, we prepare the table given below:

Height (in cm) x_i	Number of plants f_i	$(f_i \times x_i)$
60	5	300
64	16	1024
68	40	2720
72	30	2160
76	9	684
	$\Sigma f_i = 100$	$\Sigma(f_i \times x_i) = 6888$

$\therefore \quad \text{mean, } \bar{x} = \dfrac{\Sigma(f_i \times x_i)}{\Sigma f_i} = \dfrac{6888}{100} = 68.88.$

Hence, the mean height per plant is 68.88 cm.

EXAMPLE 3 *If the mean of the following data is* 18.75, *find the value of p.*

x_i	10	15	p	25	30
f_i	5	10	7	8	2

SOLUTION We may prepare the frequency table as given below:

x_i	f_i	$f_i \times x_i$
10	5	50
15	10	150
p	7	$7p$
25	8	200
30	2	60
	$\Sigma f_i = 32$	$\Sigma(f_i \times x_i) = 460 + 7p$

$\therefore \quad \text{mean} = \dfrac{\Sigma(f_i \times x_i)}{\Sigma f_i} = \dfrac{460 + 7p}{32}$.

But, mean = 18.75 (given).

$\therefore \quad \dfrac{460+7p}{32} = 18.75 \Rightarrow 460 + 7p = 18.75 \times 32$

$\Rightarrow 460 + 7p = 600$

$\Rightarrow 7p = 140 \Rightarrow p = 20.$

Hence, $p = 20$.

EXAMPLE 4 *The mean of the following data is* 21.6. *Find the value of p.*

x_i	6	12	18	24	30	36
f_i	5	4	p	6	4	6

SOLUTION For calculating the mean, we prepare the table given below:

x_i	f_i	$f_i \times x_i$
6	5	30
12	4	48
18	p	$18p$
24	6	144
30	4	120
36	6	216
	$\Sigma f_i = 25 + p$	$\Sigma(f_i \times x_i) = 558 + 18p$

$\therefore \quad \text{mean} = \dfrac{\Sigma(f_i \times x_i)}{\Sigma f_i} = \dfrac{558 + 18p}{25 + p}$.

But, mean = 21.6 (given).

$\therefore \quad \dfrac{558+18p}{25+p} = 21.6 \Rightarrow 558 + 18p = 540 + 21.6p$

$\Rightarrow 3.6p = 18$

$\Rightarrow p = \left(\dfrac{18}{3.6}\right) = \left(\dfrac{180}{36}\right) = 5.$

Hence, $p = 5$.

EXAMPLE 5 *The mean of the following frequency distribution is* 1.46.

No. of accidents (x)	0	1	2	3	4	5	Total
No. of drivers (f)	46	?	?	25	10	5	200

Find the missing frequencies.

SOLUTION Let the missing frequencies be p and q respectively. For calculating the mean, we prepare the table given below:

x_i	f_i	$f_i \times x_i$
0	46	0
1	p	p
2	q	$2q$
3	25	75
4	10	40
5	5	25
	$\Sigma f_i = 86 + p + q$	$\Sigma(f_i \times x_i) = 140 + p + 2q$

Here, $\Sigma f_i = 86 + p + q$.

But, $\Sigma f_i = 200$ (given).

$\therefore \quad 86 + p + q = 200 \Rightarrow p + q = 114.$ … (i)

Also, mean $= \dfrac{\Sigma(f_i \times x_i)}{\Sigma f_i} = \dfrac{140 + p + 2q}{86 + p + q} = \dfrac{140 + 114 + q}{86 + 114}$ [using (i)]

$= \dfrac{254 + q}{200}.$

But, mean = 1.46.

$\therefore \quad \dfrac{254 + q}{200} = 1.46 \Rightarrow 254 + q = 292$

$\Rightarrow q = 292 - 254 = 38.$

Putting $q = 38$ in (i), we get $p = 76$.

Hence, the missing frequencies are 76 and 38 respectively.

EXERCISE 18B

1. Obtain the mean of the following distribution:

Variable (x_i)	4	6	8	10	12
Frequency (f_i)	4	8	14	11	3

2. The following table shows the weights of 12 workers in a factory:

Weight (in kg)	60	63	66	69	72
No. of workers	4	3	2	2	1

Find the mean weight of the workers.

3. The measurements (in mm) of the diameters of the heads of 50 screws are given below:

Diameter (in mm) (x_i)	34	37	40	43	46
Number of screws (f_i)	5	10	17	12	6

Calculate the mean diameter of the heads of the screws.

4. The following data give the number of boys of a particular age in a class of 40 students.

Age (in years)	15	16	17	18	19	20
Frequency (f_i)	3	8	9	11	6	3

Calculate the mean age of the students.

5. Find the mean of the following frequency distribution:

Variables (x_i)	10	30	50	70	89
Frequency (f_i)	7	8	10	15	10

6. Find the mean of daily wages of 40 workers in a factory as per data given below:

Daily wages (in ₹) (x_i)	250	300	350	400	450
Number of workers (f_i)	8	11	6	10	5

7. If the mean of the following data is 20.2, find the value of p.

Variable (x_i)	10	15	20	25	30
Frequency (f_i)	6	8	p	10	6

8. If the mean of the following data is 8, find the value of p.

x	3	5	7	9	11	13
f	6	8	15	p	8	4

9. Find the missing frequency p for the following frequency distribution whose mean is 28.25.

x	15	20	25	30	35	40
f	8	7	p	14	15	6

10. Find the value of p for the following frequency distribution whose mean is 16.6.

x	8	12	15	p	20	25	30
f	12	16	20	24	16	8	4

11. Find the missing frequencies in the following frequency distribution whose mean is 34.

x	10	20	30	40	50	60	Total
f	4	f_1	8	f_2	3	4	35

12. Find the missing frequencies in the following frequency distribution whose mean is 50.

x	10	30	50	70	90	Total
f	17	f_1	32	f_2	19	120

13. Find the value of p, when the mean of the following distribution is 20.

x	15	17	19	$20 + p$	23
f	2	3	4	$5p$	6

14. The mean of the following distribution is 50.

x	10	30	50	70	90
f	17	$5a + 3$	32	$7a - 11$	19

Find the value of a and hence the frequencies of 30 and 70.

ANSWERS (EXERCISE 18B)

1. 8.05 **2.** 64.250 kg **3.** 40.24 mm **4.** 17.45 years **5.** 55
6. ₹ 341.25 **7.** $p = 20$ **8.** $p = 25$ **9.** $p = 10$ **10.** $p = 18$
11. $f_1 = 6, f_2 = 10$ **12.** $f_1 = 28, f_2 = 24$ **13.** $p = 1$
14. $a = 5, f_{30} = 28, f_{70} = 24$

HINTS TO SOME SELECTED QUESTIONS

14. $$\frac{170 + 30(5a + 3) + 1600 + 70(7a - 11) + 1710}{60 + 12a} = 50$$

$$\Rightarrow \quad 640a + 2800 = 3000 + 600a \Rightarrow 40a = 200 \Rightarrow a = 5.$$

MEDIAN OF UNGROUPED DATA

MEDIAN *After arranging the given data in an ascending or a descending order of magnitude, the value of the middle-most observation(s) is called the median of the data.*

MEDIAN OF AN UNGROUPED DATA

METHOD Arrange the data in an increasing or a decreasing order of magnitude. Let the total number of observations be n.

(i) If n is odd, then median = value of $\left(\frac{n+1}{2}\right)$th observation.

(ii) If n is even, then median = mean of $\left(\frac{n}{2}\right)$th and $\left(\frac{n}{2}+1\right)$th observations.

SOLVED EXAMPLES

EXAMPLE 1 *The runs scored by* 11 *members of a cricket team are*

15, 29, 43, 13, 31, 50, 20, 0, 27, 56, 34.

Find the median score.

SOLUTION Arranging the runs in an ascending order, we have

0, 13, 15, 20, 27, 29, 31, 34, 43, 50, 56.

Here, $n = 11$, which is odd.

$$\therefore \quad \text{median score} = \text{value of } \left(\frac{11+1}{2}\right)\text{th term}$$

$$= \text{value of 6th term} = 29.$$

Hence, the median score is 29.

EXAMPLE 2 *The weights of* 10 *students (in kilograms) are*

36, 45, 31, 52, 35, 40, 55, 60, 38, 44.

Find the median weight.

SOLUTION Arranging the weights in an ascending order, we have

31, 35, 36, 38, 40, 44, 45, 52, 55, 60.

Here, $n = 10$, which is even.

$$\therefore \quad \text{median weight} = \text{average of } \left(\frac{10}{2}\right)\text{th and } \left(\frac{10}{2}+1\right)\text{th terms}$$

$$= \text{average of 5th and 6th terms}$$

$$= \frac{1}{2}(40+44)\text{ kg} = \frac{84}{2}\text{ kg} = 42\text{ kg}.$$

Hence, the median weight is 42 kg.

EXAMPLE 3 *Ten observations* 6, 14, 15, 17, $x+1$, $2x-13$, 30, 32, 34, 43 *are written in an ascending order. The median of the data is* 24. *Find the value of* x.

SOLUTION Here $n = 10$, which is even.

$\therefore$ median $= \text{mean of } \left(\frac{10}{2}\right)\text{th and } \left(\frac{10}{2}+1\right)\text{th terms}$

$= \text{mean of 5th and 6th terms}$

$= \frac{1}{2}[(x+1)+(2x-13)] = \frac{1}{2}(3x-12)$.

But, median = 24 (given).

$\therefore \quad \frac{1}{2}(3x-12) = 24 \Rightarrow 3x-12 = 48$

$\Rightarrow \quad 3x = 60 \Rightarrow x = 20$.

Hence, $x = 20$.

EXAMPLE 4 *Find the median of the data* 61, 92, 41, 57, 43, 71, 58, 99, 108. *If* 58 *is replaced by* 85, *what will be the new median?*

SOLUTION Arranging the given data in an ascending order, we get

41, 43, 57, 58, 61, 71, 92, 99, 108.

The given number of observations, $n = 9$ (odd).

$\therefore$ median $= \text{value of } \left(\frac{n+1}{2}\right)\text{th observation}$

$= \text{value of } \left(\frac{9+1}{2}\right)\text{th observation}$

$= \text{value of 5th observation} = 61$.

On replacing 58 by 85 and arranging the new observations in an ascending order, we get

41, 43, 57, 61, 71, 85, 92, 99, 108.

$\therefore$ new median = value of 5th observation = 71.

EXERCISE 18C

1. Find the median of:

(i) 2, 10, 9, 9, 5, 2, 3, 7, 11

(ii) 15, 6, 16, 8, 22, 21, 9, 18, 25

(iii) 20, 13, 18, 25, 6, 15, 21, 9, 16, 8, 22

(iv) 7, 4, 2, 5, 1, 4, 0, 10, 3, 8, 5, 9, 2

2. Find the median of:

(i) 17, 19, 32, 10, 22, 21, 9, 35

(ii) 72, 63, 29, 51, 35, 60, 55, 91, 85, 82

(iii) 10, 75, 3, 15, 9, 47, 12, 48, 4, 81, 17, 27

3. The marks of 15 students in an examination are

25, 19, 17, 24, 23, 29, 31, 40, 19, 20, 22, 26, 17, 35, 21.

Find the median score.

4. The heights (in cm) of 9 students of a class are

148, 144, 152, 155, 160, 147, 150, 149, 145.

Find the median height.

5. The weights (in kg) of 8 children are

13.4, 10.6, 12.7, 17.2, 14.3, 15, 16.5, 9.8.

Find the median weight.

6. The ages (in years) of 10 teachers in a school are

32, 44, 53, 47, 37, 54, 34, 36, 40, 50.

Find the median age.

7. If 10, 13, 15, 18, $x+1$, $x+3$, 30, 32, 35, 41 are ten observations in an ascending order with median 24, find the value of x.

8. The following observations are arranged in ascending order:

26, 29, 42, 53, x, $x+2$, 70, 75, 82, 93.

If the median is 65, find the value of x.

9. The numbers 50, 42, 35, $(2x+10)$, $(2x-8)$, 12, 11, 8 have been written in a descending order. If their median is 25, find the value of x.

10. Find the median of the data

46, 41, 77, 58, 35, 64, 87, 92, 33, 55, 90.

In the above data, if 41 and 55 are replaced by 61 and 75 respectively, what will be the new median?

ANSWERS (EXERCISE 18C)

1. (i) 7 (ii) 16 (iii) 16 (iv) 4 **2.** (i) 20 (ii) 61.5 (iii) 16
3. 23 **4.** 149 cm **5.** 13.85 kg **6.** 42 years
7. $x = 22$ **8.** $x = 64$ **9.** $x = 12$ **10.** 58, 64

MODE OF UNGROUPED DATA

Mode is the most frequently occurring observation.

SOLVED EXAMPLES

EXAMPLE 1 *Find the mode of the following data:*

14, 25, 14, 28, 18, 17, 18, 14, 23, 22, 14, 18.

SOLUTION Arranging the given data in an ascending order, we get

14, 14, 14, 14, 17, 18, 18, 18, 22, 23, 25, 28.

Here, 14 occurs most often.

Hence, mode of the given data = 14.

EXAMPLE 2 *The points scored by a basketball team in a series of matches are as follows:*

17, 2, 7, 27, 25, 5, 14, 18, 10, 24, 48, 10, 8, 7, 10, 28.

Find the median and mode of the above data.

SOLUTION Arranging the scores in an ascending order, we get

2, 5, 7, 7, 8, 10, 10, 10, 14, 17, 18, 24, 25, 27, 28, 48.

Here, $n = 16$ (even).

$$\text{Median score} = \text{mean of } \left(\frac{n}{2}\right)\text{th and } \left(\frac{n}{2}+1\right)\text{th scores}$$

$$= \text{mean of } \left(\frac{16}{2}\right)\text{th and } \left(\frac{16}{2}+1\right)\text{th scores}$$

$$= \text{mean of 8th and 9th scores}$$

$$= \left(\frac{10+14}{2}\right) = \frac{24}{2} = 12.$$

Mode = most occurring score = 10.

EXAMPLE 3 *The following number of goals were scored by a team in a series of* 10 *matches:*

2, 3, 4, 5, 0, 1, 3, 3, 4, 3.

Find the mean, median and mode of these scores.

SOLUTION Arranging the number of goals scored in an ascending order, we get

0, 1, 2, 3, 3, 3, 3, 4, 4, 5.

$$\text{Mean} = \left(\frac{0+1+2+4\times 3+2\times 4+5}{10}\right) = \frac{28}{10} = 2.8.$$

Mode = most occurring score = 3.

Total number of matches = 10 (even).

$$\text{Median score} = \text{mean of } \left(\frac{10}{2}\right)\text{th and } \left(\frac{10}{2}+1\right)\text{th scores}$$

$$= \text{mean of 5th and 6th scores}$$

$$= \left(\frac{3+3}{2}\right) = \frac{6}{2} = 3.$$

$\therefore$ mean = 2.8, mode = 3 and median = 3.

EXAMPLE 4 *In a mathematics test given to* 15 *students, the following marks (out of* 100*) are recorded:*

52, 60, 42, 40, 98, 52, 48, 39, 41, 62, 46, 52, 54, 40, 96.

Find the mean, median and mode of the given data.

SOLUTION $\text{Mean} = \dfrac{\text{sum of the given observations}}{\text{total number of observations}}$

$= \dfrac{822}{15} = \dfrac{274}{5} = 54.8.$

Now, we arrange the given data in an ascending order, as under:

39, 40, 40, 41, 42, 46, 48, 52, 52, 52, 54, 60, 62, 96, 98.

In this data, the most occurring item is 52.

$\therefore$ mode = 52.

Total number of terms = 15 (odd).

$\therefore$ median = marks obtained by $\left(\dfrac{15+1}{2}\right)$th student

= marks obtained by 8th student = 52.

$\therefore$ mean = 54.8, median = 52 and mode = 52.

EXERCISE 18D

1. Find the mode of the following items.

0, 6, 5, 1, 6, 4, 3, 0, 2, 6, 5, 6

2. Determine the mode of the following values of a variable.

23, 15, 25, 40, 27, 25, 22, 25, 20

3. Calculate the mode of the following sizes of shoes sold by a shop on a particular day.

5, 9, 8, 6, 9, 4, 3, 9, 1, 6, 3, 9, 7, 1, 2, 5, 9

4. A cricket player scored the following runs in 12 one-day matches:

50, 30, 9, 32, 60, 50, 28, 50, 19, 50, 27, 35.

Find his modal score.

5. If the mean of the data 3, 21, 25, 17, $(x+3)$, 19, $(x-4)$ is 18, find the value of x. Using this value of x, find the mode of the data.

6. The numbers 52, 53, 54, 54, $(2x+1)$, 55, 55, 56, 57 have been arranged in an ascending order and their median is 55. Find the value of x and hence find the mode of the given data.

7. For what value of x is the mode of the data 24, 15, 40, 23, 27, 26, 22, 25, 20, $x+3$ found 25? Using this value of x, find the median.

8. The numbers 42, 43, 44, 44, $(2x+3)$, 45, 45, 46, 47 have been arranged in an ascending order and their median is 45. Find the value of x. Hence, find the mode of the above data.

ANSWERS (EXERCISE 18D)

1. 6 **2.** 25 **3.** 9 **4.** 50
5. $x = 21$, mode = 17 **6.** $x = 27$, mode = 55
7. $x = 22$, median = 24.5 **8.** $x = 21$, mode = 45

MULTIPLE-CHOICE QUESTIONS (MCQ)

Choose the correct answer in each of the following questions:

1. If the mean of five observations x, $x+2$, $x+4$, $x+6$ and $x+8$ is 11 then the value of x is

(a) 5 (b) 6 (c) 7 (d) 8

2. If the mean of x, $x+3$, $x+5$, $x+7$, $x+10$ is 9, the mean of the last three observations is

(a) $10\frac{1}{3}$ (b) $10\frac{2}{3}$ (c) $11\frac{1}{3}$ (d) $11\frac{2}{3}$

3. If $\overline{x}$ is the mean of $x_1, x_2, x_3, \ldots, x_n$ then $\sum_{i=1}^{n}(x_i - \overline{x}) = ?$

(a) –1 (b) 0 (c) 1 (d) $n-1$

4. If each observation of the data is decreased by 8 then their mean

(a) remains the same
(b) is decreased by 8
(c) is increased by 5
(d) becomes 8 times the original mean

5. The mean weight of six boys in a group is 48 kg. The individual weights of five of them are 51 kg, 45 kg, 49 kg, 46 kg and 44 kg. The weight of the 6th boy is

(a) 52 kg (b) 52.8 kg (c) 53 kg (d) 47 kg

6. The mean of the marks scored by 50 students was found to be 39. Later on it was discovered that a score of 43 was misread as 23. The correct mean is

(a) 38.6 (b) 39.4 (c) 39.8 (d) 39.2

7. The mean of 100 items was found to be 64. Later on it was discovered that two items were misread as 26 and 9 instead of 36 and 90 respectively. The correct mean is

(a) 64.86 (b) 65.31 (c) 64.91 (d) 64.61

8. The mean of 100 observations is 50. If one of the observations 50 is replaced by 150, the resulting mean will be

(a) 50.5 (b) 51 (c) 51.5 (d) 52

9. Let $\overline{x}$ be the mean of $x_1, x_2, \ldots, x_n$ and $\overline{y}$ be the mean of $y_1, y_2, \ldots, y_n$. If $\overline{z}$ is the mean of $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ then $\overline{z} = ?$

(a) $(\overline{x}+\overline{y})$ (b) $\frac{1}{2}(\overline{x}+\overline{y})$ (c) $\frac{1}{n}(\overline{x}+\overline{y})$ (d) $\frac{1}{2n}(\overline{x}+\overline{y})$

10. If $\overline{x}$ is the mean of $x_1, x_2, \ldots, x_n$ then for $a \neq 0$, the mean of $ax_1, ax_2, \ldots, ax_n, \frac{x_1}{a}, \frac{x_2}{a}, \ldots, \frac{x_n}{a}$ is

(a) $\left(a+\frac{1}{a}\right)\overline{x}$ (b) $\left(a+\frac{1}{a}\right)\frac{\overline{x}}{2}$ (c) $\left(a+\frac{1}{a}\right)\frac{\overline{x}}{n}$ (d) $\frac{\left(a+\frac{1}{a}\right)\overline{x}}{2n}$

11. If $\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n$ are the means of n groups with $n_1, n_2, \ldots, n_n$ number of observations respectively then the mean $\overline{x}$ of all the groups taken together is

(a) $\sum_{i=1}^{n} n_i\overline{x}_i$ (b) $\frac{\sum_{i=1}^{n} n_i\overline{x}_i}{n^2}$ (c) $\frac{\sum_{i=1}^{n} n_i\overline{x}_i}{\sum_{i=1}^{n} n_i}$ (d) $\frac{\sum_{i=1}^{n} n_i\overline{x}_i}{2n}$

12. The mean of the following data is 8.

x	3	5	7	9	11	13
y	6	8	15	p	8	4

Then, the value of p is

(a) 23 (b) 24 (c) 25 (d) 21

13. The runs scored by 11 members of a cricket team are

15, 34, 56, 27, 43, 29, 31, 13, 50, 20, 0.

The median score is

(a) 27 (b) 29 (c) 31 (d) 20

14. The weight of 10 students (in kg) are

55, 40, 35, 52, 60, 38, 36, 45, 31, 44.

The median weight is

(a) 40 kg (b) 41 kg (c) 42 kg (d) 44 kg

15. The median of the numbers 4, 4, 5, 7, 6, 7, 7, 12, 3 is

(a) 4 (b) 5 (c) 6 (d) 7

16. The median of the numbers 84, 78, 54, 56, 68, 22, 34, 45, 39, 54 is

(a) 45 (b) 49.5 (c) 54 (d) 56

17. Mode of the data 15, 17, 15, 19, 14, 18, 15, 14, 16, 15, 14, 20, 19, 14, 15 is

(a) 14 (b) 15 (c) 16 (d) 17

18. The median of the data arranged in ascending order 8, 9, 12, 18, $(x+2)$, $(x+4)$, 30, 31, 34, 39 is 24. The value of x is

(a) 22 (b) 21 (c) 20 (d) 24

ANSWERS (MCQ)

1. (c) **2.** (c) **3.** (b) **4.** (b) **5.** (c) **6.** (b) **7.** (c) **8.** (b)
9. (b) **10.** (b) **11.** (c) **12.** (c) **13.** (b) **14.** (c) **15.** (c) **16.** (c)
17. (b) **18.** (b)

HINTS TO SOME SELECTED QUESTIONS

2. $\dfrac{x+x+3+x+5+x+7+x+10}{5}=9 \Rightarrow 5x+25=45 \Rightarrow 5x=20 \Rightarrow x=4.$

Last 3 observations are 9, 11, 14.

Their mean $=\dfrac{9+11+14}{3}=\dfrac{34}{3}=11\dfrac{1}{3}\cdot$

3. $\sum\limits_{i=1}^{n}(x_i-\overline{x})=(x_1-\overline{x})+(x_2-\overline{x})+\ldots+(x_n-\overline{x})$

$=(x_1+x_2+\ldots+x_n)-n\overline{x}=(n\overline{x}-n\overline{x})=0.$

4. If each observation is decreased by 8 then the mean is decreased by 8.

5. Let the weight of the 6th boy be x kg. Then,

$51+45+49+46+44+x=48\times 6$

$\Rightarrow\quad 235+x=288 \Rightarrow x=288-235=53.$

$\therefore\quad$ weight of the 6th boy = 53 kg.

6. Calculated sum $=(39\times 50)=1950.$

Correct sum $=(1950+43-23)=1970.$

Correct mean $=\dfrac{1970}{50}=39.4.$

7. Calculated sum $=(64\times 100)=6400.$

Correct sum $=(6400+36+90-26-9)=6491.$

Correct mean $=\dfrac{6491}{100}=64.91.$

8. Resulting sum $= (50\times 100 - 50 + 150) = (5150 - 50) = 5100$.

Resulting mean $= \dfrac{5100}{100} = 51$.

9. $\bar{z} = \dfrac{(x_1 + x_2 + \ldots + x_n) + (y_1 + y_2 + \ldots + y_n)}{2n}$.

But, $\bar{x} = \dfrac{x_1 + x_2 + \ldots + x_n}{n} \Rightarrow (x_1 + x_2 + \ldots + x_n) = n\bar{x}$.

And, $\bar{y} = \dfrac{y_1 + y_2 + \ldots + y_n}{n} \Rightarrow (y_1 + y_2 + \ldots + y_n) = n\bar{y}$.

$\therefore \quad \bar{z} = \dfrac{n\bar{x} + n\bar{y}}{2n} = \dfrac{n(\bar{x}+\bar{y})}{2n} = \dfrac{(\bar{x}+\bar{y})}{2}$.

10. Required mean $= \dfrac{(ax_1 + ax_2 + \ldots + ax_n) + \left(\dfrac{x_1}{a} + \dfrac{x_2}{a} + \ldots + \dfrac{x_n}{a}\right)}{2n}$

$$= \frac{1}{2}\left\{\frac{a(x_1 + x_2 + \ldots + x_n)}{n} + \frac{\frac{1}{a}(x_1 + x_2 + \ldots + x_n)}{n}\right\}$$

$$= \frac{1}{2}\left\{a\bar{x} + \frac{1}{a}\bar{x}\right\} = \left(a + \frac{1}{a}\right)\frac{\bar{x}}{2}.$$

11. Sum of all terms $= n_1\bar{x}_1 + n_2\bar{x}_2 + \ldots + x_n\bar{x}_n$.

Number of terms $= n_1 + n_2 + \ldots + n_n$.

$\therefore \quad$ required mean $= \dfrac{\sum_{i=1}^{n} n_i x_i}{\sum_{i=1}^{n} n_i}$.

12. Mean $= \dfrac{\Sigma f_i \times x_i}{\Sigma f_i} = \dfrac{(3\times 6) + (5\times 8) + (7\times 15) + 9p + (11\times 8) + (13\times 4)}{(6 + 8 + 15 + p + 8 + 4)}$

$$= \frac{(18 + 40 + 105 + 9p + 88 + 52)}{41 + p} = \frac{303 + 9p}{41 + p}.$$

$\therefore \quad \dfrac{303 + 9p}{41 + p} = 8 \Rightarrow 303 + 9p = 328 + 8p \Rightarrow p = 25$.

13. Arranging the runs in an ascending order, we have

0, 13, 15, 20, 27, 29, 31, 34, 43, 50, 56.

Here $n = 11$, which is odd.

Median score = value of $\frac{1}{2}(11+1)$th term = value of 6th term = 29.

14. Arranging the weights in an ascending order, we have

31, 35, 36, 38, 40, 44, 45, 52, 55, 60.

Here, $n = 10$, which is even.

Median $= \frac{1}{2}$ [5th term + 6th term] $= \frac{1}{2}(40 + 44)$ kg = 42 kg.

15. Arranging the given numbers in an ascending order, we get

3, 4, 4, 5, 6, 7, 7, 7, 12.

Here $n = 9$, which is odd.

Median score = value of $\frac{1}{2}(9+1)$th term = 5th term = 6.

16. The given numbers in an ascending order are 22, 34, 39, 45, 54, 54, 56, 68, 78, 84.

Here $n = 10$ which is even.

Median = $\frac{1}{2}$ {5th item + 6th item} = $\frac{1}{2}(54+54) = 54$.

17. Here 14, 15, 16, 17, 18, 19, 20 respectively occur

4 times, 5 times, 1 time, 1 time, 1 time and 1 time.

$\therefore$ 15 occurs most often.

$\therefore$ mode = 15.

18. Here $n = 10$ and median = 24.

Median = value of $\frac{1}{2}$ {5th term + 6th term} = value of $\frac{1}{2}[(x+2)(x+4)]$

$\Rightarrow \quad x+3 = 24 \Rightarrow x = 21.$

SUMMARY OF FACTS AND FORMULAE

1. **MEAN** We have, mean $= \dfrac{\text{sum of observations}}{\text{number of observations}}$.

(i) **Mean of Raw Data**

Mean of n observations $x_1, x_2, x_3, \ldots, x_n$ is given by

$$\bar{x} = \frac{(x_1 + x_2 + x_3 + \ldots + x_n)}{n} = \frac{\sum_{i=1}^{n} x_i}{n}.$$

(ii) **Mean of Ungrouped Data**

If the frequencies of n observations $x_1, x_2, \ldots, x_n$ are $f_1, f_2, \ldots, f_n$ respectively then their mean is given by

$$\bar{x} = \frac{\sum_{i=1}^{n} (f_i \times x_i)}{\sum_{i=1}^{n} f_i}.$$

2. Let the mean of n observations $x_1, x_2, \ldots, x_n$ be $\bar{x}$.

(i) If each observation is increased by p then new mean $= (\bar{x}+p)$.

(ii) If each observation is decreased by p then new mean $= (\bar{x}-p)$.

(iii) If each observation is multiplied by a nonzero number p then new mean is $(p\bar{x})$.

(iv) If each observation is divided by a nonzero number p then new mean $= \left(\frac{\overline{x}}{p}\right)$.

3. MEDIAN It is the value of the middle-most observation(s).

Case 1. *When n is odd*:

Median = value of the $\left(\frac{n+1}{2}\right)$th observation.

Case 2. *When n is even*:

Median = mean of $\left(\frac{n}{2}\right)$th and $\left(\frac{n}{2}+1\right)$th observations.

4. Mode is the most frequently occurring observation.

19 Probability

INTRODUCTION

In everyday life, we come across statements such as:

(i) *Most probably* it will rain today.

(ii) *Chances* are high that the price of petrol will go up.

(iii) I *doubt* that he will win the race.

The words *'most probably'*, *'chances'*, *'doubt'*, etc., show uncertainty or probability of occurrence of an event.

Though probability started with gambling, it is now used extensively in science, commerce, biological sciences, weather forecasting, etc.

HISTORY

The origin of the subject can be traced in the correspondence between two French mathematicians Blaise Pascal (1623–62) and Pierre de Fermat (1601–65). In 1654, a French nobleman and gambler Charles Mere asked Pascal to solve certain dice problems. Pascal solved these problems in collaboration with Fermat.

Dutch scientist Huygens (1629–95) wrote the first book on probability.

In 1713, a more comprehensive book was written by J Bernoulli.

In 1748, A De Moivre published his book *'The Doctrine of Chances'*.

In 1812, Laplace published his book *'Analytic Theory of Probability'*.

In the 20th century, important contributions were made by Russian mathematicians, A A Markov and A N Kolmogorov.

These days, probability theory is extensively used in various fields.

Blaise Pascal

Pierre de Fermat

SOME TERMS RELATED TO PROBABILITY

EXPERIMENT *An operation which can produce some well-defined outcomes, is called an experiment.*

Each outcome is called an event.

RANDOM EXPERIMENT *An experiment in which all possible outcomes are known and the exact outcome cannot be predicted in advance, is called a random experiment.*

TRIAL *By a trial, we mean performing a random experiment.*

EMPIRICAL PROBABILITY

Suppose we perform an experiment and let n be the total number of trials. The empirical probability of happening of an event E is defined as

$$P(E) = \frac{\textit{number of trials in which the event happened}}{\textit{total number of trials}}.$$

REMARK In this chapter, by probability, we shall mean empirical probability.

SOME OPERATIONS AND THEIR OUTCOMES

1. Tossing A Coin

When we toss a coin, we get either heads or *tails.*

2. Tossing Two Coins

When we toss 2 *coins, we get:*

2 heads or 1 head or 0 head.

2 tails or 1 tail or 0 tail.

3. Throwing A Die

A die (whose plural is dice) is a cubical solid having six faces, marked as 1, 2, 3, 4, 5, 6 respectively.

In throwing a die, whatever number comes on the upper face, is called the outcome.

4. Choosing Various Types of Numbers Out of Given Numbers

Suppose counting numbers 1 to 10 are given.

If E_1 is the event of choosing even numbers then the favourable outcomes are 2, 4, 6, 8, 10.

If E_2 is the event of choosing prime numbers then the favourable outcomes are 2, 3, 5, 7.

SOLVED EXAMPLES

EXAMPLE 1 *A coin is tossed* 600 *times with the frequencies as:*

heads: 342 *and tails:* 258.

If a coin is tossed at random, what is the probability of getting

(i) a head? *(ii) a tail?*

SOLUTION Total number of trials = 600.

Number of heads = 342.

Number of tails = 258.

On tossing a coin, let E_1 and E_2 be the events of getting a head and of getting a tail respectively. Then,

(i) $P(\textit{getting a head}) = P(E_1)$

$$= \frac{\text{number of heads coming up}}{\text{total number of trials}}$$

$$= \frac{342}{600} = \frac{57}{100} = 0.57.$$

(ii) $P(\textit{getting a tail}) = P(E_2)$

$$= \frac{\text{number of tails coming up}}{\text{total number of trials}}$$

$$= \frac{258}{600} = \frac{43}{100} = 0.43.$$

REMARK It may be noted here that E_1 and E_2 are the only possible outcomes of each trial and $P(E_1) + P(E_2) = (0.57 + 0.43) = 1$.

EXAMPLE 2 *Two coins are tossed simultaneously* 400 *times, and we get*

two heads: 180 *times*

one head: 148 *times*

no head: 72 *times.*

If two coins are tossed at random, what is the probability of getting *(i)* 2 *heads?* *(ii)* 1 *head?* *(iii)* 0 *head?*

SOLUTION Total number of trials = 400.

Number of times 2 heads appear = 180.

Number of times 1 head appears = 148.

Number of times 0 head appears = 72.

In a random toss of two coins let E_1, E_2, E_3 be the events of getting 2 heads, 1 head and 0 head respectively. Then,

(i) $P(getting\ 2\ heads) = P(E_1)$

$$= \frac{\text{number of times 2 heads appear}}{\text{total number of trials}}$$

$$= \frac{180}{400} = \frac{9}{20} = 0.45.$$

(ii) $P(getting\ 1\ head) = P(E_2)$

$$= \frac{\text{number of times 1 head appears}}{\text{total number of trials}}$$

$$= \frac{148}{400} = \frac{37}{100} = 0.37.$$

(iii) $P(getting\ 0\ head) = P(E_3)$

$$= \frac{\text{number of times no head appears}}{\text{total number of trials}}$$

$$= \frac{72}{400} = \frac{18}{100} = 0.18.$$

REMARK In tossing 2 coins, the only possible outcomes are E_1, E_2, E_3, and $P(E_1) + P(E_2) + P(E_3) = 0.45 + 0.37 + 0.18 = 1$.

EXAMPLE 3 *A die is thrown 500 times and the outcomes are noted as given below:*

Outcome	1	2	3	4	5	6
Frequency	95	80	84	68	70	103

If a die is thrown at random, find the probability of getting
(i) 1 *(ii)* 2 *(iii)* 3 *(iv)* 4 *(v)* 5 *(vi)* 6.

SOLUTION Total number of trials = 500.

In a random throw of a die, let E_1, E_2, E_3, E_4, E_5 and E_6 be the events of getting 1, 2, 3, 4, 5 and 6 respectively. Then,

(i) $P(getting\ 1) = P(E_1) = \dfrac{\text{number of times 1 appears}}{\text{total number of trials}}$

$$= \frac{95}{500} = \frac{19}{100} = 0.19.$$

(ii) $P(getting\ 2) = P(E_2) = \dfrac{\text{number of times 2 appears}}{\text{total number of trials}}$

$$= \frac{80}{500} = \frac{16}{100} = 0.16.$$

(iii) $P(getting\ 3) = P(E_3) = \dfrac{\text{number of times 3 appears}}{\text{total number of trials}}$

$$= \frac{84}{500} = 0.168.$$

(iv) $P(getting\ 4) = P(E_4) = \dfrac{\text{number of times 4 appears}}{\text{total number of trials}}$

$= \dfrac{68}{500} = 0.136.$

(v) $P(getting\ 5) = P(E_5) = \dfrac{\text{number of times 5 appears}}{\text{total number of trials}}$

$= \dfrac{70}{500} = \dfrac{7}{50} = 0.14.$

(vi) $P(getting\ 6) = P(E_6) = \dfrac{\text{number of times 6 appears}}{\text{total number of trials}}$

$= \dfrac{103}{500} = 0.206.$

REMARK In throwing a die, all possible outcomes are $E_1, E_2, E_3, E_4, E_5, E_6$, and $P(E_1) + P(E_2) + P(E_3) + P(E_4) + P(E_5) + P(E_6)$

$= 0.19 + 0.16 + 0.168 + 0.136 + 0.14 + 0.206 = 1.$

EXAMPLE 4 1500 *families with* 2 *children each, were selected randomly and the following data were recorded.*

Number of girls in a family	2	1	0
Number of families	102	675	723

Out of these families, one family is selected at random. What is the probability that the selected family has (i) 2 *girls, (ii)* 1 *girl, (iii) no girl?*

SOLUTION Total number of families = 1500.

Let E_1, E_2, E_3 be the events that the selected family has 2 girls, 1 girl and 0 girl respectively. Then,

(i) $P(selected\ family\ has\ 2\ girls)$

$= P(E_1) = \dfrac{\text{number of families having 2 girls}}{\text{total number of families}} = \dfrac{102}{1500} = 0.068.$

(ii) $P(selected\ family\ has\ 1\ girl)$

$= P(E_2) = \dfrac{\text{number of families having 1 girl}}{\text{total number of families}} = \dfrac{675}{1500} = 0.45.$

(iii) $P(selected\ family\ has\ 0\ girl)$

$= P(E_3) = \dfrac{\text{number of families having 0 girl}}{\text{total number of families}} = \dfrac{723}{1500} = 0.482.$

REMARK In the example given above, the only possible outcomes are E_1, E_2, E_3, and $P(E_1) + P(E_2) + P(E_3) = 0.068 + 0.45 + 0.482 = 1.$

EXAMPLE 5 *On one page of a telephone directory, there were 200 telephone numbers. The frequency distribution of their unit's digits is given in the following table:*

Unit's digit	0	1	2	3	4	5	6	7	8	9
Frequency	22	26	22	22	20	10	14	28	16	20

Out of the numbers on the page, a number is chosen at random. What is the probability that the unit's digit of the chosen number is

(i) 6? *(ii) a nonzero multiple of* 3?

(iii) a nonzero even number? *(iv) an odd number?*

SOLUTION Number of all telephone numbers on the given page = 200.

(i) Let E_1 be the event of choosing a number with unit's digit 6. Number of such numbers = 14.

$\therefore$ *P*(*getting a number with unit's digit* 6)

$= P(E_1)$

$$= \frac{\text{number of times 6 appears as unit's digit}}{\text{total number of numbers on the given page}}$$

$$= \frac{14}{200} = \frac{7}{100} = 0.07.$$

(ii) Let E_2 be the event of choosing a number whose unit's digit is a nonzero multiple of 3.

Each such number has unit's digits 3, 6 or 9.

$\therefore$ *P*(*getting a number whose unit's digit is a nonzero multiple of* 3)

$= P(E_2)$

$$= \frac{\text{number of numbers with unit's digits 3, 6 or 9}}{\text{total number of numbers on the given page}}$$

$$= \frac{22 + 14 + 20}{200} = \frac{56}{200} = 0.28.$$

(iii) Let E_3 be the event of choosing a number whose unit's digit is a nonzero even number.

Each such number has unit's digits 2, 4, 6 or 8.

$\therefore$ *P*(*getting a number whose unit's digit is a nonzero even number*)

$= P(E_3)$

$$= \frac{\text{number of numbers with unit's digits 2, 4, 6 or 8}}{\text{total number of numbers on the given page}}$$

$$= \frac{22 + 20 + 14 + 16}{200} = \frac{72}{200} = \frac{36}{100} = 0.36.$$

(iv) Let E_4 be the event of choosing an odd number.

Each such number has unit's digits 1, 3, 5, 7 or 9.

$\therefore$ *P*(*getting a number whose unit's digit is an odd number*)

$= P(E_4)$

$$= \frac{\text{number of numbers with unit's digits 1, 3, 5, 7 or 9}}{\text{total number of numbers on the given page}}$$

$$= \frac{(26+22+10+28+20)}{200} = \frac{106}{200} = \frac{53}{100} = 0.53.$$

EXAMPLE 6 *Fifty seeds were selected at random from each of* 5 *bags of seeds and were kept under standardised conditions favourable to germination. After* 20 *days, the number of seeds which had germinated in each collection were counted and recorded as follows:*

Bag	1	2	3	4	5
Number of seeds germinated	40	48	42	39	41

What is the probability of germination of

(i) *more than* 40 *seeds from a bag?*

(ii) 49 *seeds from a bag?*

(iii) *more than* 35 *seeds from a bag?*

SOLUTION Total number of bags = 5.

(i) Let E_1 be the event of germination of more than 40 seeds from a bag. Then,

P(*germination of more than* 40 *seeds from a bag*)

$= P(E_1)$

$$= \frac{\text{number of bags from which more than 40 seeds germinate}}{\text{total number of bags}}$$

$= \frac{3}{5} = 0.6.$

[*There are* 3 *bags from which more than* 40 *seeds germinate.*]

(ii) Let E_2 be the event of germination of 49 seeds. Then,

P(*germination of* 49 *seeds from a bag*)

$= P(E_2)$

$$= \frac{\text{number of bags from which 49 seeds germinate}}{\text{total number of bags}}$$

$= \frac{0}{5} = 0.$

[*Clearly, seeds from none of the given bags contain* 49 *germinated seeds.*]

(iii) Let E_3 be the event of germination of more than 35 seeds from a bag. Then,

P(*germination of more than* 35 *seeds from a bag*)

$= P(E_3)$

$$= \frac{\text{number of bags from which more than 35 seeds germinate}}{\text{total number of bags}}$$

$$= \frac{5}{5} = 1.$$

[*Seeds from each of the five given bags contain more than* 35 *germinated seeds.*]

EXAMPLE 7 *A tyre manufacturing company kept a record of the distance covered before a tyre needed to be replaced. The table given below shows the results of* 1000 *cases.*

Distance in km	Less than 4000	4000 to 9000	9001 to 14000	More than 14000
Frequency	20	210	325	445

If you buy a tyre of this company, what is the probability that

(i) it will need to be replaced before it has covered 4000 km?

(ii) it will last more than 9000 km?

(iii) it will need to be replaced after it has covered somewhere between 4000 km *and* 14000 km?

SOLUTION Total number of cases = 1000.

(i) Let E_1 be the event that a tyre will need to be replaced before covering 4000 km.

Number of tyres to be replaced before covering 4000 km = 20.

$$\therefore \quad P(E_1) = \frac{20}{1000} = 0.02.$$

(ii) Let E_2 be the event that a tyre will last more than 9000 km.

Number of tyres that will last more than 9000 km

$$= 325 + 445 = 770.$$

$$\therefore \quad P(E_2) = \frac{770}{1000} = 0.77.$$

(iii) Let E_3 be the event that a tyre needs replacement between 4000 km and 14000 km.

Number of tyres which need replacement after covering between 4000 km and 14000 km = 210 + 325 = 535.

$$\therefore \quad P(E_3) = \frac{535}{1000} = 0.535.$$

EXAMPLE 8 *Bulbs are packed in cartons, each containing* 40 *bulbs.* 700 *cartons were examined for defective bulbs and the results are given in the following table:*

Number of defective bulbs	0	1	2	3	4	5	6	More than 6
Frequency (No. of cartons)	371	162	55	49	41	15	5	2

One carton is selected at random. What is the probability that it has

(i) no defective bulb?

(ii) defective bulbs less than 4*?*

(iii) defective bulbs more than 3 *but less than* 6*?*

(iv) defective bulbs 6 *or more?*

SOLUTION Total number of cartons = 700.

(i) Let E_1 be the event of choosing a carton having no defective bulb. Then,

P(*choosing a carton having no defective bulb*)

$= P(E_1)$

$$= \frac{\text{number of cartons having 0 defective bulb}}{\text{total number of cartons}}$$

$$= \frac{371}{700} = \frac{53}{100} = 0.53.$$

(ii) Let E_2 be the event of choosing a carton having defective bulbs less than 4. Then,

P(*choosing a carton having defective bulbs less than* 4)

$= P(E_2)$

$$= \frac{\text{number of cartons having defective bulbs 0, 1, 2 or 3}}{\text{total number of cartons}}$$

$$= \frac{371 + 162 + 55 + 49}{700} = \frac{637}{700} = \frac{91}{100} = 0.91.$$

(iii) Let E_3 be the event of choosing a carton having defective bulbs more than 3 but less than 6. Then,

P(*choosing a carton having defective bulbs more than* 3, *but less than* 6)

$= P(E_3)$

$$= \frac{\text{number of cartons having defective bulbs 4 or 5}}{\text{total number of cartons}}$$

$$= \frac{41+15}{700} = \frac{56}{700} = \frac{8}{100} = 0.08.$$

(iv) Let E_4 be the event of choosing a carton having defective bulbs 6 or more. Then,

P(*choosing a carton having defective bulbs* 6 *or more*)

$= P(E_4)$

$$= \frac{\text{number of cartons having defective bulbs 6 or more}}{\text{total number of cartons}}$$

$$= \frac{5+2}{700} = \frac{7}{700} = \frac{1}{100} = 0.01.$$

EXAMPLE 9 *Over the past* 200 *working days, the number of defective parts produced by a machine is given in the following table:*

Number of defective parts	0	1	2	3	4	5	6	7	8	9	10	11	12	13
Number of days	50	32	22	18	12	12	10	10	10	8	6	6	2	2

From these days, one day is chosen at random. What is the probability that on that day, the output has

(i) *no defective part?*

(ii) *at least* 1 *defective part?*

(iii) *not more than* 5 *defective parts?*

(iv) *more than* 5, *but less than* 8 *defective parts?*

(v) *more than* 13 *defective parts?*

SOLUTION Total number of working days = 200.

(i) Let E_1 be the event that the output has 0 defective part on the chosen day. Then,

P(*event of producing* 0 *defective part on the chosen day*)

$= P(E_1)$

$$= \frac{\text{number of days when the output has 0 defective part}}{\text{total number of working days}}$$

$$= \frac{50}{200} = \frac{1}{4} = 0.25.$$

(ii) Number of days on which the output has at least 1 defective part

$= 200 -$ *number of days with* 0 *defective part*

$= 200 - 50 = 150.$

Let E_2 be the event that the output has at least 1 defective part on the chosen day. Then,

$$P(E_2) = \frac{150}{200} = \frac{3}{4}.$$

(iii) Let E_3 be the event that the output has not more than 5 defective parts, i.e., 5 or less defective parts, on the chosen day. Then,

P(*event that the output has not more than* 5 *defective parts on the chosen day*)

$= P$(*event that the output has* 5 *or less defective parts on the chosen day*)

$= P(E_3)$

$$= \frac{\text{number of days when the output has 5 or less defective parts}}{\text{total number of working days}}$$

$$= \frac{50 + 32 + 22 + 18 + 12 + 12}{200} = \frac{146}{200} = \frac{73}{100} = 0.73.$$

(iv) Let E_4 be the event that the output has more than 5, but less than 8 defective parts on the chosen day. Then,

P(*event that the output has more than* 5 *but less than* 8 *defective parts on the chosen day*)

$= P(E_4)$

$$= \frac{\text{number of days when the output has 6 or 7 defective parts}}{\text{total number of working days}}$$

$$= \frac{10 + 10}{200} = \frac{20}{200} = \frac{1}{10} = 0.1.$$

(v) Let E_5 be the event that the output has more than 13 defective parts on that day. Then,

P(*event that the output has more than* 13 *defective parts on that day*)

$= P(E_5)$

$$= \frac{\text{number of days when the output has more than 13 defective parts}}{\text{total number of working days}}$$

$$= \frac{0}{200} = 0.$$

EXAMPLE 10 *The table given below shows the months of birth of 40 students of a class in a school.*

Month of birth	*Jan.*	*Feb.*	*March*	*April*	*May*	*June*	*July*	*Aug.*	*Sept.*	*Oct.*	*Nov.*	*Dec.*
No. of students	3	4	2	2	5	1	2	6	3	4	4	4

If one student is chosen at random, what is the probability that the student is born

(i) in the latter half of the year?

(ii) in a month having 31 days?

(iii) in a month having 30 days?

SOLUTION Total number of students in the class = 40.

(i) Let E_1 be the event that the chosen student is born in the latter half of the year. Then,

$$P(E_1) = \frac{\text{no. of students born in latter half of the year}}{\text{total number of students}}$$

$$= \frac{2+6+3+4+4+4}{40} = \frac{23}{40} = 0.575.$$

(ii) Let E_2 be the event that the chosen student is born in a month having 31 days. Then,

$$P(E_2) = \frac{\text{number of students born in a month having 31 days}}{\text{total number of students}}$$

$$= \frac{3+2+5+2+6+4+4}{40} = \frac{26}{40} = \frac{13}{20} = 0.65.$$

(iii) Let E_3 be the event that the chosen student is born in a month having 30 days. Then,

$$P(E_2) = \frac{\text{number of students born in a month having 30 days}}{\text{total number of students}}$$

$$= \frac{2+1+3+4}{40} = \frac{10}{40} = \frac{1}{4} = 0.25.$$

EXAMPLE 11 *According to a meteorological report for 300 consecutive days in a year, its weather forecasts were correct 180 times.*

Out of these days, one day is chosen at random.

What is the probability that the weather forecast was

(i) correct on that day? *(ii) not correct on that day?*

SOLUTION Total number of days = 300.

(i) Let E = event that the forecast was correct on the chosen day. Then,

$$P(E) = \frac{\text{no. of days for which the forecasts were correct}}{\text{total number of days}}$$

$$= \frac{180}{300} = \frac{3}{5} = 0.6.$$

(ii) Number of days on which the forecast was not correct

$= 300 - 180 = 120.$

Let F = event that the forecast was not correct on the given day.

Then, $P(F) = \dfrac{120}{300} = \dfrac{2}{5} = 0.4.$

REMARK Clearly, E and F are the possible outcomes such that $P(E) + P(F) = 0.6 + 0.4 = 1.$

EXAMPLE 12 *A survey of* 250 *girls of a school was conducted and it was found that* 105 *girls like tea while* 145 *dislike it. Out of these girls, one girl is selected at random.*

What is the probability that the selected girl (i) likes tea, (ii) does not like tea?

SOLUTION Total number of girls = 250.

Number of girls who like tea = 105.

Number of girls who dislike tea = 145.

(i) Let E_1 = event that the selected girl likes tea. Then,

$P(\textit{selected girl likes tea})$

$$= P(E_1) = \frac{\text{number of girls who like tea}}{\text{total number of girls}} = \frac{105}{250} = 0.42.$$

(ii) E_2 = event that the selected girl dislikes tea. Then,

$P(\textit{selected girl dislikes tea})$

$$= P(E_2) = \frac{\text{number of girls who dislike tea}}{\text{total number of girls}} = \frac{145}{250} = 0.58.$$

REMARK In selecting 1 girl at random, the possible outcomes are E_1, E_2, and $P(E_1) + P(E_2) = 0.42 + 0.58 = 1.$

EXAMPLE 13 *In a cricket match, a batsman hits the boundary* 5 *times out of* 40 *balls played by him. Find the probability that the boundary is not hit by the ball.*

SOLUTION Total number of balls thrown = 40.

Number of times, the boundary is hit by the ball = 5.

Number of times, the boundary is not hit by the ball = 40 − 5
= 35.

Let E be the event that the boundary is not hit by the ball. Then,

$$P(E) = \frac{\text{number of times the boundary is not hit by the ball}}{\text{total number of balls thrown}}$$

$$= \frac{35}{40} = \frac{7}{8}.$$

EXAMPLE 14 *Two dice are thrown simultaneously 500 times. Each time, the sum of the two numbers appearing on their tops is noted and recorded as given below:*

Sum	2	3	4	5	6	7	8	9	10	11	12
Frequency	22	30	48	56	64	70	64	26	53	39	28

If the two dice are thrown once more, what is the probability of getting a sum

(i) 5? *(ii) more than* 9?

(iii) less than or equal to 6? *(iv) between* 6 *and* 10?

SOLUTION Total number of times the two dice are thrown = 500.

(i) Let E_1 be the event of getting the sum 5. Then,

$$P(\textit{getting the sum } 5)$$
$$= P(E_1)$$
$$= \frac{\text{number of times the sum 5 is obtained}}{\text{total number of times the two dice are thrown}}$$
$$= \frac{56}{500} = \frac{14}{125}.$$

(ii) Let E_2 be the event of getting a sum more than 9. Then,

E_2 = event of getting a sum 10, 11 or 12.

$$\therefore \quad P(\textit{getting a sum more than } 9)$$
$$= P(E_2)$$
$$= \frac{\text{number of times a sum 10, 11 or 12 is obtained}}{\text{total number of times the two dice are thrown}}$$
$$= \frac{53 + 29 + 28}{500} = \frac{110}{500} = \frac{11}{50}.$$

(iii) Let E_3 be the event of getting a sum less than or equal to 6.

Then, E_3 = event of getting a sum 2, 3, 4, 5 or 6.

$\therefore$ *P*(*getting a sum less than or equal to* 6)

$= P(E_3)$

$= \dfrac{\text{number of times a sum 2, 3, 4, 5 or 6 is obtained}}{\text{total number of times the two dice are thrown}}$

$= \dfrac{22+30+48+56+64}{500} = \dfrac{220}{500} = \dfrac{11}{25}.$

(iv) Let E_4 be the event of getting the sum between 6 and 10.

Then, E_4 = event of getting a sum 7, 8 or 9.

$\therefore$ *P*(*getting the sum between* 6 *and* 10)

$= P(E_4)$

$= \dfrac{\text{number of times a sum 7, 8 or 9 is obtained}}{\text{total number of times the two dice are thrown}}$

$= \dfrac{70+64+26}{500} = \dfrac{160}{500} = \dfrac{8}{25}.$

EXAMPLE 15 *A recent survey shows that the ages of* 200 *workers in a factory is distributed as follows:*

Age (in years)	20–29	30–39	40–49	50–59	60 and above
Number of workers	37	28	86	46	3

If a worker is selected at random, find the probability that the selected worker is

(i) 40 *years or more*

(ii) *under* 40 *years*

(iii) *having an age from* 30 *to* 39 *years*

(iv) *under* 60 *but over* 39 *years.*

SOLUTION Total number of workers in the factory = 200.

(i) Let E_1 be the event of selecting a worker who is 40 years or more. Then,

P(*selecting a worker who is* 40 *years or more*)

$= P(E_1)$

$= \dfrac{\text{number of workers who are 40 years or more}}{\text{total number of workers}}$

$= \dfrac{86+46+3}{200} = \dfrac{135}{200} = \dfrac{27}{40}.$

(ii) Let E_2 be the event of selecting a worker who is under 40 years. Then,

P(*selecting a worker who is under* 40 *years*)

$$= P(E_2)$$

$$= \frac{\text{number of workers having an age less than 40 years}}{\text{total number of workers}}$$

$$= \frac{37+28}{200} = \frac{65}{200} = \frac{13}{40}.$$

(iii) Let E_3 be the event of selecting a worker having an age from 30 to 39 years. Then,

P(*selecting a worker having an age from* 30 *to* 39 *years*)

$$= P(E_3)$$

$$= \frac{\text{number of workers having age from 30 to 39 years}}{\text{total number of workers}}$$

$$= \frac{28}{200} = \frac{7}{50}.$$

(iv) Let E_4 be the event of selecting a worker who is under 60, but over 39 years. Then,

P(*selecting a worker who is under* 60 *but over* 39 *years*)

$$= P(E_4)$$

$$= \frac{\text{number of workers having age more than 39 years, but less than 60 years}}{\text{total number of workers}}$$

$$= \frac{86+46}{200} = \frac{132}{200} = \frac{33}{50}.$$

EXAMPLE 16 *Following frequency distribution gives the weights of* 40 *students of a class.*

Weight (in kg*)*	31–35	36–40	41–45	46–50	51–55	56–60	61–65	66–70	71–75
Number of students	10	6	14	4	1	1	2	1	1

A student from the class is chosen at random. What is the probability that the weight of the chosen student is

(i) at most 60 kg? *(ii) at least* 56 kg? *(iii) not more than* 50 kg?

SOLUTION Total number of students = 40.

(i) Let E_1 be the event of choosing a student whose weight is at most 50 kg. Then,

P(*choosing a student whose weight is at most* 60 kg)

$$= P(E_1)$$

$$= \frac{\text{number of students whose weight is 60 kg or less}}{\text{total number of students}}$$

$$= \frac{10+6+14+4+1+1}{40} = \frac{36}{40} = \frac{9}{10}.$$

(ii) Let E_2 be the event of choosing a student whose weight is at least 56 kg. Then,

P(*choosing a student whose weight is at least* 56 kg)

$$= P(E_2)$$

$$= \frac{\text{number of students whose weight is 56 kg or more}}{\text{total number of students}}$$

$$= \frac{1+2+1+1}{40} = \frac{5}{40} = \frac{1}{8}.$$

(iii) Let E_3 be the event of choosing a student whose weight is not more than 50 kg. Then,

P(*choosing a student whose weight is not more than* 50 kg)

$$= P(E_3)$$

$$= \frac{\text{number of students whose weight is 50 kg or less}}{\text{total number of students}}$$

$$= \frac{10+6+14+4}{40} = \frac{34}{40} = \frac{17}{20}.$$

EXAMPLE 17 *An insurance company selected* 2000 *drivers at random in a particular city to find a relationship between age and accidents. The data obtained are given in the following table.*

Age of drivers (in years)	*Accidents in one year*				
	0	1	2	3	Over *3*
18–29	440	160	110	60	36
30–50	505	125	60	22	18
Above 50	360	45	35	15	9

Find the probability of each of the following events for a driver chosen at random from the city:

(i) being 18–29 *years of age and having exactly* 3 *accidents in one year*

(ii) being 30–50 *years of age and having one or more accidents in one year*

(iii) having no accident in one year.

SOLUTION Total number of drivers = 2000.

(i) Let E_1 be the event of choosing drivers of age 18–29 years with exactly 3 accidents in a year. Then,

P(*choosing drivers of age* 18–29 *years with exactly* 3 *accidents in one year*)

$$= P(E_1)$$

$$= \frac{\text{number of drivers of age 18–29 years having exactly 3 accidents in 1 year}}{\text{total number of drivers}}$$

$$= \frac{60}{2000} = \frac{3}{100}.$$

(ii) Let E_2 be the event of choosing drivers of age 30–50 years and having one or more accidents in one year. Then,

P(*choosing drivers of age* 30–50 *years with* 1 *or more accidents in* 1 *year*)

$$= P(E_2)$$

$$= \frac{\text{number of drivers of age 30–50 years with 1 or more accidents in 1 year}}{\text{total number of drivers}}$$

$$= \frac{125 + 60 + 22 + 18}{2000} = \frac{225}{2000} = \frac{9}{80}.$$

(iii) Let E_3 be the event of choosing drivers having no accident in 1 year. Then,

P(*choosing drivers having no accident in* 1 *year*)

$$= P(E_3)$$

$$= \frac{\text{number of drivers having no accident in 1 year}}{\text{total number of drivers}}$$

$$= \frac{440 + 505 + 360}{2000} = \frac{1305}{2000} = \frac{261}{400}.$$

EXAMPLE 18 *The table given below shows the marks obtained by* 80 *students of a class in a test with maximum marks* 100.

Marks	0–15	15–30	30–45	45–60	60–75	Above 75
Number of students	6	13	17	24	16	4

A student of the class is selected at random.

Find the probability that he gets

(i) *less than* 15% *marks,* (ii) 60 *or more marks and*

(iii) *less than* 45 *marks.*

NOTE *Here* 15–30 *means* 15 *and more but less than* 30.

SOLUTION Total number of students = 80.

(i) P(*The student gets less than* 15% *marks*)

$$= \frac{\text{number of students getting less than 15 marks}}{\text{total number of students}}$$

$$= \frac{6}{80} = \frac{3}{40}.$$

(ii) P(*The student gets* 60 *or more marks*)

$$= \frac{\text{number of students getting 60 or more marks}}{\text{total number of students}}$$

$$= \frac{16+4}{80} = \frac{20}{80} = \frac{1}{4}.$$

(iii) P(*The student gets less than* 45 *marks*)

$$= \frac{\text{number of students getting less than 45 marks}}{\text{total number of students}}$$

$$= \frac{6+13+17}{80} = \frac{36}{80} = \frac{9}{20}.$$

EXERCISE 19

1. A coin is tossed 500 times and we get

heads: 285 times and tails: 215 times.

When a coin is tossed at random, what is the probability of getting

(i) a head? (ii) a tail?

2. Two coins are tossed 400 times and we get

two heads: 112 times; one head: 160 times; 0 head: 128 times.

When two coins are tossed at random, what is the probability of getting

(i) 2 heads? (ii) 1 head? (iii) 0 head?

3. Three coins are tossed 200 times and we get

three heads: 39 times; two heads: 58 times;

one head: 67 times; 0 head: 36 times.

When three coins are tossed at random, what is the probability of getting (i) 3 heads? (ii) 1 head? (iii) 0 head? (iv) 2 heads?

4. A die is thrown 300 times and the outcomes are noted as given below:

Outcome	1	2	3	4	5	6
Frequency	60	72	54	42	39	33

When a die is thrown at random, what is the probability of getting a

(i) 3? (ii) 6? (iii) 5? (iv) 1?

5. In a survey of 200 ladies, it was found that 142 like coffee, while 58 dislike it.

Find the probability that a lady chosen at random

(i) likes coffee, (ii) dislikes coffee.

6. The percentages of marks obtained by a student in six unit tests are given below:

Unit test	I	II	III	IV	V	VI
Percentage of marks obtained	53	72	28	46	67	59

A unit test is selected at random. What is the probability that the student gets more than 60% marks in the test?

7. On a particular day, at a crossing in a city, the various types of 240 vehicles going past during a time interval were observed as under:

Type of vehicle	Two-wheelers	Three-wheelers	Four-wheelers
Frequency	84	68	88

Out of these vehicles, one is chosen at random. What is the probability that the chosen vehicle is a two-wheeler?

8. On one page of a telephone directory, there are 200 phone numbers. The frequency distribution of their unit's digits is given below:

Unit's digit	0	1	2	3	4	5	6	7	8	9
Frequency	19	22	23	19	21	24	23	18	16	15

One of the numbers is chosen at random from the page. What is the probability that the unit's digit of the chosen number is (i) 5? (ii) 8?

9. The following table shows the blood groups of 40 students of a class.

Blood group	A	B	O	AB
Number of students	11	9	14	6

One student of the class is chosen at random. What is the probability that the chosen student has blood group (i) O? (ii) AB?

10. 12 packets of salt, each marked 2 kg, actually contained the following weights (in kg) of salt:

1.950, 2.020, 2.060, 1.980, 2.030, 1.970,
2.040, 1.990, 1.985, 2.025, 2.000, 1.980.

Out of these packets, one packet is chosen at random.

What is the probability that the chosen packet contains more than 2 kg of salt?

11. In a cricket match, a batsman hits a boundary 6 times out of 30 balls he plays. Find the probability that he did not hit a boundary.

12. An organisation selected 2400 families at random and surveyed them to determine a relationship between the income level and the number of vehicles in a family. The information gathered is listed in the table below:

Monthly income (in ₹)	Number of vehicles per family			
	0	1	2	3 or more
Less than ₹ 25000	10	160	25	0
₹ 25000– ₹ 30000	0	305	27	2
₹ 30000– ₹ 35000	1	535	29	1
₹ 35000– ₹ 40000	2	469	59	25
₹ 40000 or more	1	579	82	88

Suppose a family is chosen at random. Find the probability that the family chosen is

(i) earning ₹ 25000–₹ 30000 per month and owning exactly 2 vehicles.
(ii) earning ₹ 40000 or more per month and owning exactly 1 vehicle.
(iii) earning less than ₹ 25000 per month and not owning any vehicle.
(iv) earning ₹ 35000–₹ 40000 per month and owning 2 or more vehicles.
(v) owning not more than 1 vehicle.

13. The table given below shows the marks obtained by 30 students in a test.

Marks (Class interval)	1–10	11–20	21–30	31–40	41–50
Number of students (Frequency)	7	10	6	4	3

Out of these students, one is chosen at random. What is the probability that the marks of the chosen student

(i) are 30 or less? (ii) are 31 or more? (iii) lie in the interval 21–30?

14. The table given below shows the ages of 75 teachers in a school.

Age (in years)	18–29	30–39	40–49	50–59
Number of teachers	3	27	37	8

A teacher from this school is chosen at random. What is the probability that the selected teacher is

(i) 40 or more than 40 years old?

(ii) of an age lying between 30–39 years (including both)?

(iii) 18 years or more and 49 years or less?

(iv) 18 years or more old?

(v) above 60 years of age?

NOTE Here 18–29 means 18 or more but less than or equal to 29.

15. Following are the ages (in years) of 360 patients, getting medical treatment in a hospital:

Age (in years)	10–20	20–30	30–40	40–50	50–60	60–70
Number of patients	90	50	60	80	50	30

One of the patients is selected at random.

What is the probability that his age is

(i) 30 years or more but less than 40 years?

(ii) 50 years or more but less than 70 years?

(iii) 10 years or more but less than 40 years?

(iv) 10 years or more?

(v) less than 10 years?

16. The marks obtained by 90 students of a school in mathematics out of 100 are given as under:

Marks	0–20	20–30	30–40	40–50	50–60	60–70	70 and above
No. of students	7	8	12	25	19	10	9

From these students, a student is chosen at random.

What is the probability that the chosen student

(i) gets 20% or less marks? (ii) gets 60% or more marks?

17. It is known that a box of 800 electric bulbs contains 36 defective bulbs. One bulb is taken at random out of the box. What is the probability that the bulb chosen is nondefective?

18. Fill in the blanks.

(i) Probability of an impossible event = ...

(ii) Probability of a sure event = ...

(iii) Let E be an event. Then, $P(\text{not } E)$ = ...

(iv) $P(E) + P(\text{not } E)$ = ...

(v) ...... $\leq P(E) \leq$ ....

ANSWERS (EXERCISE 19)

1. (i) 0.57 (ii) 0.43 **2.** (i) 0.28 (ii) 0.4 (iii) 0.32

3. (i) 0.195 (ii) 0.335 (iii) 0.18 (iv) 0.29

4. (i) 0.18 (ii) 0.11 (iii) 0.13 (iv) 0.2

5. (i) 0.71 (ii) 0.29 **6.** $\frac{1}{3}$ **7.** 0.35

8. (i) 0.12 (ii) 0.08 **9.** (i) 0.35 (ii) 0.15

10. $\frac{5}{12}$ **11.** 0.8 **12.** (i) $\frac{9}{800}$ (ii) $\frac{193}{800}$ (iii) $\frac{1}{240}$ (iv) $\frac{7}{200}$ (v) $\frac{1031}{1200}$

13. (i) $\frac{23}{30}$ (ii) $\frac{7}{30}$ (iii) $\frac{1}{5}$ **14.** (i) $\frac{3}{5}$ (ii) $\frac{9}{25}$ (iii) $\frac{67}{75}$ (iv) 1 (v) 0

15. (i) $\frac{1}{6}$ (ii) $\frac{2}{9}$ (iii) $\frac{5}{9}$ (iv) 1 (v) 0 **16.** (i) $\frac{7}{90}$ (ii) $\frac{19}{90}$

17. $\frac{191}{200}$ **18.** (i) 0 (ii) 1 (iii) $1 - P(E)$ (iv) 1 (v) 0, 1

MULTIPLE-CHOICE QUESTIONS (MCQ)

Choose the correct answer in each of the following questions:

1. In a sample survey of 645 people, it was found that 516 people have a high school certificate. If a person is chosen at random, what is the probability that he/she has a high school certificate?

(a) $\frac{1}{2}$ (b) $\frac{3}{5}$ (c) $\frac{7}{10}$ (d) $\frac{4}{5}$

2. In a medical examination of students of a class, the following blood groups are recorded:

Blood group	A	B	AB	O
Number of students	11	15	8	6

From this class, a student is chosen at random. What is the probability that the chosen student has blood group AB?

(a) $\frac{13}{20}$ (b) $\frac{3}{8}$ (c) $\frac{1}{5}$ (d) $\frac{11}{40}$

3. 80 bulbs are selected at random from a lot and their lifetime in hours is recorded as under.

Lifetime (in hours)	300	500	700	900	1100
Frequency	10	12	23	25	10

One bulb is selected at random from the lot. What is the probability that its life is 1150 hours?

(a) $\frac{1}{80}$ (b) $\frac{7}{16}$ (c) 1 (d) 0

4. In a survey of 364 children aged 19–36 months, it was found that 91 liked to eat potato chips. If a child is selected at random, the probability that he/she does not like to eat potato chips is

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) $\frac{3}{4}$ (d) $\frac{4}{5}$

5. Two coins are tossed 1000 times and the outcomes are recorded as given below:

Number of heads	2	1	0
Frequency	200	550	250

Now, if two coins are tossed at random, what is the probability of getting at most one head?

(a) $\frac{3}{4}$ (b) $\frac{4}{5}$ (c) $\frac{1}{4}$ (d) $\frac{1}{5}$

6. 80 bulbs are selected at random from a lot and their lifetime in hours is recorded as under.

Lifetime (in hours)	300	500	700	900	1100
Frequency	10	12	23	25	10

One bulb is selected at random from the lot. What is the probability that the selected bulb has a life more than 500 hours?

(a) $\frac{27}{40}$ (b) $\frac{29}{40}$ (c) $\frac{5}{16}$ (d) $\frac{11}{40}$

7. To know the opinion of the students about the subject Sanskrit, a survey of 200 students was conducted. The data is recorded as under.

Opinion	like	dislike
Number of students	135	65

What is the probability that a student chosen at random does not like it?

(a) $\frac{13}{27}$ (b) $\frac{27}{40}$ (c) $\frac{13}{40}$ (d) $\frac{27}{13}$

8. A coin is tossed 60 times and the tail appears 35 times. In a random throw of a coin, what is the probability of getting a head?

(a) $\frac{7}{12}$ (b) $\frac{12}{7}$ (c) $\frac{5}{12}$ (d) $\frac{1}{25}$

9. It is given that the probability of winning a game is 0.7. What is the probability of losing the game?

(a) 0.8 (b) 0.3 (c) 0.35 (d) 0.15

10. In a cricket match, a batsman hits a boundary 6 times out of 30 balls he plays. What is the probability that in a given throw, the ball does not hit the boundary?

(a) $\frac{1}{4}$ (b) $\frac{1}{5}$ (c) $\frac{4}{5}$ (d) $\frac{3}{4}$

11. A bag contains 16 cards bearing number 1, 2, 3, ..., 16 respectively. One card is chosen at random. What is the probability that the chosen card bears a number which is divisible by 3?

(a) $\frac{3}{16}$ (b) $\frac{5}{16}$ (c) $\frac{11}{16}$ (d) $\frac{13}{16}$

12. A bag contains 5 red, 8 black and 7 white balls. One ball is chosen at random. What is the probability that the chosen ball is black?

(a) $\frac{2}{3}$ (b) $\frac{2}{5}$ (c) $\frac{3}{5}$ (d) $\frac{1}{3}$

13. In 65 throws of a die, the outcomes were noted as under:

Outcome	1	2	3	4	5	6
Number of times	8	10	12	16	9	10

A die is thrown at random. What is the probability of getting a prime number?

(a) $\frac{3}{35}$ (b) $\frac{3}{5}$ (c) $\frac{31}{65}$ (d) $\frac{36}{65}$

14. In 50 throws of a die, the outcomes were noted as under:

Outcome	1	2	3	4	5	6
Number of times	8	9	6	7	12	8

A die is thrown at random. What is the probability of getting an even number?

(a) $\frac{12}{25}$ (b) $\frac{3}{50}$ (c) $\frac{1}{8}$ (d) $\frac{1}{2}$

15. The table given below shows the months of birth of 36 students of a class:

Month of birth	Jan.	Feb.	March	April	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
No. of students	4	3	5	0	1	6	1	3	4	3	4	2

A student is chosen at random from the class. What is the probability that the chosen student was born in October?

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{1}{4}$ (d) $\frac{1}{12}$

16. Two coins are tossed simultaneously 600 times to get

2 heads: 234 times, 1 head: 206 times, 0 head: 160 times.

If two coins are tossed at random, what is the probability of getting at least one head?

(a) $\frac{103}{300}$ (b) $\frac{39}{100}$ (c) $\frac{11}{15}$ (d) $\frac{4}{15}$

ANSWERS (MCQ)

1. (d)	**2.** (c)	**3.** (d)	**4.** (c)	**5.** (b)	**6.** (b)	**7.** (c)	**8.** (c)
9. (b)	**10.** (c)	**11.** (b)	**12.** (b)	**13.** (c)	**14.** (a)	**15.** (d)	**16.** (c)

HINTS TO SOME SELECTED QUESTIONS

1. Let E be the event that the chosen person has a high school certificate. Then,

$$P(E) = \frac{\text{number of people having high school certificate}}{\text{total number of people}}$$

$$= \frac{516}{645} = \frac{4}{5}\cdot \quad \text{[HCF (516, 645) = 129.]}$$

3. Total number of bulbs = 80.

Number of times 1150 appears = 0.

Required probability $= \frac{0}{80} = 0$.

4. Total number of children = 364.

Number of children who like to eat potato chips = 91.

Number of children who do not like to eat potato chips = (364 − 91) = 273.

Required probability $= \frac{273}{364} = \frac{3}{4}\cdot$

5. Total number of throws = 1000.

Number of times at most 1 head appears

= (number of time 0 head appears) + (number of times 1 head appears)
= $(250 + 550) = 800$.

Required probability $= \frac{800}{1000} = \frac{4}{5}$.

6. Total number of bulbs = 80.

   Number of bulbs having their lifetime more than 500 hours $= (23 + 25 + 10) = 58$.

   Required probability $= \frac{58}{80} = \frac{29}{40}$.

7. Total number of students = 200.

   Number of students who do not like Sanskrit = 65.

   Required probability $= \frac{65}{200} = \frac{13}{40}$.

8. Total number of times the coin is tossed = 60.

   Number of times tails appear = 35.

   Number of times heads appear $= (60 - 35) = 25$.

   $P(\textit{getting a head}) = \frac{25}{60} = \frac{5}{12}$.

9. $P(\textit{losing the game}) + P(\textit{winning the game}) = 1$.

   $\therefore \quad P(\textit{losing the game}) + 0.7 = 1$.

   $\therefore \quad P(\textit{losing the game}) = (1 - 0.7) = 0.3$.

10. Total number of balls thrown = 30.

    Number of times the boundary is hit = 6.

    Number of times the ball does not hit the boundary $= 30 - 6 = 24$.

    Required probability $= \frac{24}{30} = \frac{4}{5}$.

11. Total number of cards = 16.

    Numbers divisible by 3 are 3, 6, 9, 12, 15.

    Number of such numbers = 5.

    Required probability $= \frac{5}{16}$.

12. Total number of balls $= 5 + 8 + 7 = 20$.

    Number of black balls = 8.

    $P(\textit{getting a black ball}) = \frac{8}{20} = \frac{2}{5}$.

13. Total number of throws = 65.

    Number of throws giving 2, 3 and 5 $= 10 + 12 + 9 = 31$.

    $P(\textit{getting a prime number}) = \frac{31}{65}$.

14. Total number of throws = 50.

    Number of times we get 2, 4 or 6 $= 9 + 7 + 8 = 24$.

    $P(\textit{getting an even number}) = \frac{24}{50} = \frac{12}{25}$.

15. Total number of students = 36.

    Number of students born in October = 3.

$P(\textit{the chosen student was born in october}) = \frac{3}{36} = \frac{1}{12}$.

16. Total number of tosses = 600.

Number of times we get 1 head or 2 heads = $(234 + 206) = 440$.

Required probability $= \frac{440}{600} = \frac{11}{15}$.

REVIEW OF FACTS AND FORMULAE

1. Uncertainty or probability can be measured numerically.
2. An operation which can produce some well-defined outcomes is called an experiment. Each outcome is called an *event*.
3. The empirical probability of an event E is defined as

$$P(E) = \frac{\text{number of trials in which event happened}}{\text{total number of trials}}.$$

4. $\theta \leq P(E) \leq 1$.
5. (i) If $P(E) = 0$ then E is called an *impossible event*.

   (ii) If $P(E) = 1$ then E is called a *sure event*.

   (iii) $P(E) + P(\text{not } E) = 1$.